The Annals of Probability
1981, Vol. 9, No. 3, 460–467

A LIMIT THEOREM FOR DOUBLE ARRAYS[1]

BY ANDREW ROSALSKY AND HENRY TEICHER

University of Florida and Rutgers University

The main result establishes that row sums S_n of a double array of rowwise independent, infinitesimal (or merely uniformly asymptotically constant) random variables satisfying $\limsup |S_n - M_n| \le M_0 < \infty$ a.c. (for some choice of constants M_n), obey a weak law of large numbers, i.e., $S_n - \text{med } S_n$ converges in probability to 0. No moment assumptions are imposed on the individual summands and zero-one laws are unavailable. As special cases, a new result for weighted i.i.d. random variables and a result of Kesten are obtained.

1. Introduction and main result. Let $\{X_{nj}, 1 \le j \le k_n \to \infty, n \ge 1\}$ be a double array of rowwise independent random variables with row sums $S_n = \sum_{j=1}^{k_n} X_{nj}$, $n \ge 1$. Under the standard proviso that $\{X_{nj}, 1 \le j \le k_n\}$ are uniformly asymptotically constant or equivalently that X_{nj} centered at their medians are infinitesimal (uniformly asymptotically negligible), i.e.,

$$\lim_{n\to\infty} \max_{1\le j\le k_n} P\{|X_{nj} - \text{med } X_{nj}| > \epsilon\} = 0, \qquad \text{all } \epsilon > 0, \tag{1}$$

the main result, Theorem 1, asserts that if there exist constants M_n for which $\limsup_{n\to\infty} |S_n - M_n| \le M_0 < \infty$ almost certainly (a.c.), then $S_n - \text{med } S_n \to 0$ in probability. A surprising feature is that the conclusion obtains in the absence of zero-one laws. No moment conditions are imposed on $\{X_{nj}\}$ which are constrained solely by (1), in whose absence the theorem may fail.

The case of normed sums $\frac{1}{a_n} \sum_{j=1}^{n} X_j$, $n \ge 1$ of independent random variables (rv's) $\{X_n, n \ge 1\}$ and, in particular, of weighted i.i.d. rv's $X_{nj} = \sigma_j X_j / a_n$, $\{X_j, j \ge 1\}$ i.i.d., $\sigma_n = o(a_n)$ is of special interest. In the i.i.d. case ($\sigma_j \equiv 1$), Theorem 1 is due to Kesten [5] who gave a sufficient condition as well. Apart from absence of zero-one laws, the pattern of the proof of Theorem 1 parallels that of Kesten and indeed the argument establishing (9) is virtually identical. On the other hand, the proof of (5) is completely different and considerably simpler than its counterpart in [5].

THEOREM 1. *Let* $\{X_{nj}, 1 \le j \le k_n \to \infty, n \ge 1\}$ *be a double array of rowwise independent random variables obeying* (1). *If for some constants* M_n, $n \ge 0$, *the row sums* $S_n = \sum_{j=1}^{k_n} X_{nj}$ *satisfy*

$$\limsup_{n\to\infty} |S_n - M_n| \le M_0 < \infty, \qquad \text{a.c.} \tag{2}$$

then the weak law of large numbers

$$S_n - \text{med } S_n \to_P 0 \tag{3}$$

obtains.

Received March 3, 1980.
[1] Research supported by National Science Foundation under Grant Number MCS78-00805.
AMS 1970 *subject classifications.* Primary 60F05, 60F15.
Key words and phrases. Law of the iterated logarithm, weak law of large numbers, row sums of independent infinitesimal random variables, weighted i.i.d. random variables.

PROOF. Let $S_n^* = \sum_{j=1}^{k_n} Y_{nj}$, $n \geq 1$, where $\{Y_{nj}, 1 \leq j \leq k_n\}$ are a symmetrized version of the $\{X_{nj}$'s$\}$, that is, $Y_{nj} = X_{nj} - X'_{nj}$ where $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ and $\{X'_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ are i.i.d. stochastic processes. Then $\{Y_{nj}, 1 \leq j \leq k_n \to \infty\}$ are infinitesimal rv's since (1) and the Weak Symmetrization Inequality ([6], page 245) ensure that

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P\{|Y_{nj}| > \lambda\} = 0, \qquad \text{all } \lambda > 0. \tag{4}$$

It will be shown firstly that

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} P\{|Y_{nj}| > \lambda\} = 0, \qquad \text{all } \lambda > 0 \tag{5}$$

or equivalently that

$$\max_{1\leq j\leq k_n} |Y_{nj}| \to_P 0,$$

in view of

$$1 \geq \exp\{-\textstyle\sum_1^{k_n} P\{|Y_{ni}| > \lambda\}\} \geq \prod_{i=1}^{k_n} [1 - P\{|Y_{ni}| > \lambda\}] = P\{\bigcap_1^{k_n} [|Y_{ni}| \leq \lambda]\}$$
$$\geq 1 - \textstyle\sum_{i=1}^{k_n} P\{|Y_{ni}| > \lambda\}.$$

Suppose, to the contrary, for some λ, δ in (0, 1) that

$$P\{\textstyle\bigcup_{j=1}^{k_n} A_{nj}\} > \delta \qquad \text{for infinitely many integers } n = n(i),\ i \geq 1, \tag{6}$$

where $A_{nj} = \{|Y_{nj}| > \lambda\}$. By (4), for any integer $m > 1$ there exists an integer N_m such that for all $n \geq N_m$

$$P\{A_{nj}\} \leq \frac{\delta}{2m}, \qquad 1 \leq j \leq k_n. \tag{7}$$

For $n = n(i) \geq N_m$, let $v_1 = v_1(m, i)$ be the smallest integer v such that $P\{\bigcup_{j=1}^{v} A_{nj}\} > \delta/(2m)$. In view of (7) and (6), $1 < v_1 \leq k_n$ and since

$$P\{\textstyle\bigcup_{j=1}^{v_1} A_{nj}\} \leq P\{\bigcup_{j=1}^{v_1-1} A_{nj}\} + P\{A_{nv_1}\} \leq \dfrac{\delta}{2m} + \dfrac{\delta}{2m} = \dfrac{\delta}{m},$$

(6) implies $v_1 < k_n$. Suppose, inductively, that $1 < v_1 < \cdots < v_r < k_n$ are defined for some integer r in $[1, m)$ so that $v_q = v_q(m, i)$ is the smallest integer for which (set $v_0 = 0$)

$$P\{\textstyle\bigcup_{j=v_{q-1}+1}^{v_q} A_{nj}\} > \dfrac{\delta}{2m},\ 1 \leq q \leq r, \qquad \text{and} \quad P\{\bigcup_{j=1}^{v_r} A_{nj}\} \leq \dfrac{r\delta}{m}.$$

Then (6) ensures that $v_r < k_n$ and by subadditivity

$$P\{\textstyle\bigcup_{j=v_r+1}^{k_n} A_{nj}\} \geq \delta - P\{\bigcup_{j=1}^{v_r} A_{nj}\} \geq \delta - \dfrac{r\delta}{m} = \dfrac{(m-r)\delta}{m} > \dfrac{\delta}{2m}$$

implying $P\{\bigcup_{j=v_r+1}^{v} A_{nj}\} > \delta/(2m)$ for some integer v in $[v_r + 1, k_n]$. If $v_{r+1} = v_{r+1}(m, i)$ is the smallest such integer, then $v_{r+1} > v_r + 1$ by (7) and furthermore

$$P\{\textstyle\bigcup_{j=1}^{v_{r+1}} A_{nj}\} \leq P\{\bigcup_{j=1}^{v_r} A_{nj}\} + P\{\bigcup_{j=v_r+1}^{v_{r+1}-1} A_{nj}\} + P\{A_{nv_{r+1}}\} \leq \dfrac{r\delta}{m} + \dfrac{\delta}{2m} + \dfrac{\delta}{2m} = \dfrac{(r+1)\delta}{m}.$$

Since $r + 1 \leq m$, (6) ensures that $v_{r+1} < k_n$ and, moreover, the procedure may be repeated until $r + 1 = m$. Hence, for any integer $m > 1$, the negation of (5) implies that for every choice of $n = n(i) \geq N_m$, there exist integers $0 = v_0 < v_1 < \cdots < v_m < k_n$ (which depend on m and i) such that

$$P\{\textstyle\bigcup_{j=v_{r-1}+1}^{v_r} A_{nj}\} > \dfrac{\delta}{2m}, \qquad 1 \leq r \leq m. \tag{8}$$

Consequently, for all integers $m > 1$ and $n = n(i) \geq N_m$, it follows via Lévy's inequality that

$$P\left\{S^*_{n(i)} \geq \frac{1}{2} m\lambda\right\} \geq P\left\{\bigcap_{r=1}^{m} \left[\sum_{v_{r-1}+1}^{v_r} Y_{nj} \geq \frac{1}{2}\lambda\right] \cap \left[\sum_{v_m+1}^{k_n} Y_{nj} \geq 0\right]\right\}$$
$$\geq \frac{1}{2} \prod_{r=1}^{m} \frac{1}{2} P\left\{\left| \sum_{v_{r-1}+1}^{v_r} Y_{nj}\right| \geq \frac{1}{2}\lambda\right\}$$
$$\geq \frac{1}{2} \prod_{r=1}^{m} \frac{1}{4} P\left\{\max_{v_{r-1}<h\leq v_r} \left|\sum_{v_{r-1}+1}^{h} Y_{nj}\right| > \frac{1}{2}\lambda\right\}$$
$$\geq \frac{1}{2} \prod_{r=1}^{m} \frac{1}{4} P\{\max_{v_{r-1}<h\leq v_r} |Y_{nh}| > \lambda\} > \frac{1}{2}\left(\frac{\delta}{8m}\right)^m$$

by (8). Thus, for every integer $m > 1$

$$P\left\{\limsup_{n\to\infty} S^*_n \geq \frac{1}{2} m\lambda\right\} \geq P\left\{S^*_{n(i)} \geq \frac{1}{2}m\lambda, \text{ i.o. } (i)\right\}$$
$$\geq \limsup_{i\to\infty} P\left\{S^*_{n(i)} \geq \frac{1}{2}m\lambda\right\} \geq \frac{1}{2}\left(\frac{\delta}{8m}\right)^m$$

contradicting the fact that (2) entails $\limsup_{n\to\infty} S^*_n \leq 2M_0$, a.c. and thereby establishing (5).

Next, it will be demonstrated that

$$a_n^2 \equiv \sum_{j=1}^{k_n} EY_{nj}^2 I_{[|Y_{nj}|\leq 1]} = o(1). \tag{9}$$

If rather, $\limsup a_n = A'$ in $(0, \infty]$, then for $0 < A < A'$ and some subsequence $n(i) \to \infty$, necessarily $a_{n(i)} \geq A$, $i \geq 1$ whence setting $Z_{nj} = Y_{nj} I_{[|Y_{nj}|\leq 1]}$, it follows via (5) for any λ in $(0, 1)$ and $n = n(i)$ that

$$a_n^{-3} \sum_{j=1}^{k_n} E|Z_{nj}|^3 \leq a_n^{-3} \left(\lambda \sum_{j=1}^{k_n} EY_{nj}^2 I_{[|Y_{nj}|\leq\lambda]} + \sum_{j=1}^{k_n} P\{|Y_{nj}| > \lambda\}\right) \leq \lambda A^{-1} + o(1) \to 0$$

as first $i \to \infty$ and then $\lambda \to 0$. Hence if $T_n = \sum_{j=1}^{k_n} Z_{nj}$, noting $ET_n = 0 = EZ_{nj}$, Liapounov's theorem for double arrays (Theorem 7.1.2, [3], page 200) ensures that $T_{n(i)}/a_{n(i)} \to_d N(0, 1)$. Thus, for any $m > 0$, if Φ denotes the normal distribution function,

$$\limsup_{i\to\infty} P\{T_{n(i)} > m\} \geq \lim_{i\to\infty} P\{T_{n(i)} > m a_{n(i)}/A\} = 1 - \Phi(m/A).$$

Now, via (5), as $i \to \infty$

$$P\{T_{n(i)} > m\} \leq P\{S^*_{n(i)} > m\} + P\{\cup_{j=1}^{n(i)} [|Y_{n(i),j}| > 1]\} \leq P\{S^*_{n(i)} > m\} + o(1)$$

implying for any $m > 0$ that

$$P\{\limsup_{n\to\infty} S^*_n \geq m\} \geq P\{S^*_{n(i)} > m, \text{ i.o. } (i)\}$$
$$= \lim_{i\to\infty} P\{\cup_{h=i}^{\infty} [S^*_{n(h)} > m]\}$$
$$\geq \limsup_{i\to\infty} P\{S^*_{n(i)} > m\} \geq 1 - \Phi(m/A)$$

again contradicting $\limsup_{n\to\infty} S^*_n \leq 2M_0$, a.c. Thus, (9) is established.

Since $EY_{nj} I_{[|Y_{nj}|\leq 1]} = 0$, it follows from (5), (9), and the Degenerate Convergence Criterion ([6], page 317) that $S^*_n \to 0$ in probability which is equivalent to (3) (see e.g., Corollary 1, [6], page 245). □

An infinitely divisible distribution F has bounded support iff F is degenerate ([1], page 413). Thus, if (1) holds and for some constants M_n, the rv's $S_n - M_n$ converge a.c. to a nondegenerate rv S, necessarily S is unbounded. This also follows directly from Theorem 1 without invoking properties of infinite divisibility.

COROLLARY 1. *Let* $S_n = \sum_{j=1}^{k_n} a_{nj}X_j$, $n \geq 1$, *where* $\{X_n, n \geq 1\}$ *are* i.i.d. *random variables and* $\{a_{nj}, 1 \leq j \leq k_n \to \infty, n \geq 1\}$ *are constants satisfying* $\max_{1\leq j\leq k_n} |a_{nj}| = o(1)$. *If* $\limsup_{n\to\infty} |S_n - M_n| \leq M_0 < \infty$, a.c. *for some constants* M_n, *then* $S_n - \text{med } S_n \to 0$ *in probability.*

COROLLARY 2. *Let* $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$, *where the independent random variables* $\{X_n\}$ *satisfy*

$$\lim_{n\to\infty} \max_{1\leq j\leq n} P\{|X_j - \text{med } X_j| > \lambda a_n\} = 0, \qquad \text{all } \lambda > 0 \tag{10}$$

for some sequence $\{a_n\}$ *of positive constants. If*

$$P\{\limsup_{n\to\infty} |S_n - c_n|/a_n < \infty\} > 0 \tag{11}$$

for some constants c_n, *then the weak law of large numbers*

$$(S_n - \text{med } S_n)/a_n \to_P 0 \tag{12}$$

obtains.

PROOF. Note that in the nontrivial case where at least one X_j is nondegenerate, (10) ensures that $a_n \to \infty$, whence by the Kolmogorov 0-1 law and (11), $\limsup_{n\to\infty} |S_n - c_n|/a_n = M$, a.c. for some constant $M < \infty$. If $X_{nj} = (X_j - n^{-1}c_n)/a_n$, $1 \leq j \leq n$, $n \geq 1$, then (10) is tantamount to (1) and so (12) follows from Theorem 1. □

2. The weighted i.i.d. case. Of particular interest is the weighted i.i.d. case consisting of sequences $\{\sigma_n Y_n, n \geq 1\}$ where $\{Y, Y_n\}$ are i.i.d. random variables and $\{\sigma_n\}$ are nonzero constants. Let $\{a_n, n \geq 1\}$ be arbitrary positive constants and set

$$X_{nj} = \frac{\sigma_j Y_j}{a_n}, \qquad 1 \leq j \leq n.$$

When $a_n = s_n(\log\log s_n^2)^{1/2}$ where $s_n^2 = \sum_1^n \sigma_j^2 \to \infty$ and $EY = 0$, $EY^2 < \infty$, conditions for the classical law of the iterated logarithm (LIL)

$$\limsup_{n\to\infty} S_n = \limsup_{n\to\infty} \frac{\sum_{j=1}^n \sigma_j Y_j}{s_n(\log\log s_n^2)^{1/2}} = (2EY^2)^{1/2}, \qquad \text{a.c.}$$

have been given by Chow and Teicher [2] and Teicher [9], [10]. When $EY^2 = \infty$, and Y is symmetric, a nonclassical LIL has been proved by Rosalsky [8].

Without any moment conditions on Y, if (*)$a_n \uparrow \infty$, $\sigma_n = o(a_n)$ then (see e.g. (24)) $\max_{1\leq j\leq n} |\sigma_j| = o(a_n)$ and so Corollary 1 guarantees that

$$\limsup_{n\to\infty} \left| \frac{1}{a_n} \sum_{j=1}^n \sigma_j Y_j - M_n \right| \leq M < \infty, \text{ a.c.} \quad \text{implies} \quad \frac{1}{a_n}\left[\sum_{j=1}^n \sigma_j Y_j - \text{med} \sum_{j=1}^n \sigma_j Y_j\right] \to_P 0.$$

The next theorem[2], which makes no initial assumptions about a_n such as (*), asserts that a weak law of large numbers prevails provided a generalized LIL holds for some choice of constants a_n. It is convenient to first treat the case of symmetric Y.

THEOREM 2. *Let* $S_n = \sum_{j=1}^n \sigma_j Y_j$ *and* $s_n^2 = \sum_1^n \sigma_j^2$, $n \geq 1$ *where* $\{\sigma_n, n \geq 1\}$ *are nonzero constants and* $\{Y, Y_n, n \geq 1\}$ *are symmetric* i.i.d. *random variables such that either* Y *is*

---

[2] This is one of several theorems in the Ph.D. thesis of Rosalsky (see [8]).

unbounded or $\sigma_n^2 = o(s_n^2)$. *If for some sequence of positive constants* a_n,

(13) $$\limsup_{n\to\infty} S_n/a_n = 1, \qquad \text{a.c.}$$

then the weak law of large numbers

(14) $$S_n/a_n \to_P 0$$

obtains.

PROOF. It will be shown firstly that if $EY^2 < \infty$, then

(15) $$s_n^2 \to \infty.$$

Now, if (15) fails, (13) and symmetry guarantee Y nondegenerate with $EY = 0$. Since $0 < \sum_{n=1}^{\infty} E(\sigma_n Y_n)^2 < \infty$, the Khintchine-Kolmogorov Convergence Theorem ensures that S_n converges a.c. and in quadratic mean to a nondegenerate random variable S with $ES = 0$ which contradicts (13) since $P\{S < 0\} > 0$.

Next, it will be shown that

(16) $$a_n \to \infty.$$

Otherwise, $a_{n(k)} \le M$ for some M in $(0, \infty)$ and some subsequence $n(k) \uparrow \infty$. Then via (13)

$$\begin{aligned} 0 &= P\{\limsup_{n\to\infty} S_n/a_n \ge 1 + M^{-1}\} \\ &\ge \lim_{k\to\infty} P\{\cup_{i=k}^{\infty} [S_{n(i)} \ge M + 1]\} \\ &\ge \limsup_{k\to\infty} P\{S_{n(k)} \ge M + 1\} \ge 0 \end{aligned}$$

and consequently

(17) $$\lim_{k\to\infty} P\{S_{n(k)} \ge M + 1\} = 0.$$

However, if Y is unbounded,

$$\begin{aligned} P\{S_{n(k)} \ge M + 1\} &\ge P\{\sigma_1 Y_1 \ge M + 1\} \cdot P\{\textstyle\sum_{j=2}^{n(k)} \sigma_j Y_j \ge 0\} \\ &\ge \tfrac{1}{4} P\{|\sigma_1 Y_1| \ge M + 1\} > 0, \end{aligned}$$

contradicting (17). On the other hand, if Y is bounded, then via (15)

$$\sum_{j=1}^{\infty} |\sigma_j| \ge (\sum_{j=1}^{\infty} \sigma_j^2)^{1/2} = \infty$$

and thus K may be chosen large enough so that

$$\sum_{j=1}^{n(K)} |\sigma_j| \ge (M + 1)/c$$

where c is a positive number satisfying $P\{Y > c\} > 0$. Hence, if $A_j = \{Y_j \operatorname{sign}(\sigma_j) \ge c\}$, $j \ge 1$ and $k > K$,

$$\begin{aligned} P\{S_{n(k)} \ge M + 1\} &\ge P\{S_{n(K)} \ge M + 1\} \cdot P\{\textstyle\sum_{n(K)+1}^{n(k)} \sigma_j Y_j \ge 0\} \\ &\ge \tfrac{1}{2} P\{S_{n(K)} \ge \textstyle\sum_{j=1}^{n(K)} |\sigma_j| c\} \ge \tfrac{1}{2} P\{\cap_{j=1}^{n(K)} A_j\} \\ &= \tfrac{1}{2}(P\{A_1\})^{n(K)} > 0 \end{aligned}$$

again contradicting (17) and thereby proving (16).

It will now be shown that

(18) $$\limsup_{n\to\infty} S_n/a_n^* = 1, \qquad \text{a.c.}$$

where

(19) $$a_n^* \equiv \inf_{j \ge n} a_j \uparrow \infty.$$

Clearly $a_n^* \uparrow \infty$, $a_n \ge a_n^* > 0$, $n \ge 1$, and $\limsup_{n\to\infty} S_n/a_n^*$ is degenerate by the Kolmogorov 0-1 law. Consequently,

(20) $$1 = \limsup_{n\to\infty} S_n/a_n \le \limsup_{n\to\infty} S_n/a_n^* \equiv L \le \infty, \qquad \text{a.c.}$$

If $L > 1$, choose λ in $(1, L)$. For each $n \ge 1$, $a_n^* = a_{j(n)}$ for some $j(n) \ge n$. Now for all $k \ge n \ge 1$, interpreting $\sum_{i=k+1}^{k} \sigma_i Y_i$ as 0, the event $\{\sum_{i=k+1}^{j(k)} \sigma_i Y_i \ge 0\}$ and the class of events $\{\{S_r > \lambda a_{j(k)}\} : r = n, \cdots, k\}$ are independent. Moreover, $P\{\sum_{k+1}^{j(k)} \sigma_i Y_i \ge 0\} \ge \frac{1}{2}$. Thus, by the Lemma for Events ([6], page 246),

$$P\{\cup_{k=n}^{\infty} [S_{j(k)} > \lambda a_{j(k)}]\} \ge P\{\cup_{k=n}^{\infty} ([S_k > \lambda a_{j(k)}] \cap [\textstyle\sum_{i=k+1}^{j(k)} \sigma_i Y_i \ge 0])\}$$
$$\ge \tfrac{1}{2} P\{\cup_{k=n}^{\infty} [S_k > \lambda a_{j(k)}]\},$$

whence via (20), the choice of λ, and $j(k) \ge k$

$$\tfrac{1}{2} = \tfrac{1}{2} P\{S_n/a_n^* > \lambda \text{ i.o.}(n)\} = \tfrac{1}{2} \lim_{n\to\infty} P\{\cup_{k=n}^{\infty} [S_k > \lambda a_{j(k)}]\}$$
$$\le \lim_{n\to\infty} P\{\cup_{k=n}^{\infty} [S_{j(k)} > \lambda a_{j(k)}]\} \le \lim_{n\to\infty} P\{\cup_{j=n}^{\infty} [S_j > \lambda a_j]\}$$
$$\le P\{\limsup_{n\to\infty} S_n/a_n \ge \lambda\}$$

contradicting (13) and thereby establishing (18).

Hence by symmetry,

(21) $$\limsup_{n\to\infty} |S_n|/a_n = \limsup_{n\to\infty} |S_n|/a_n^* = 1, \qquad \text{a.c.}$$

Thus, $|\sigma_n Y_n|/a_n^* \le |S_n|/a_n^* + |S_{n-1}|/a_{n-1}^*$, entails $\limsup_{n\to\infty} |\sigma_n Y_n|/a_n^* \le 2$, a.c. and consequently via the Borel-Cantelli lemma

(22) $$\sum_{n=1}^{\infty} P\{|Y| > \lambda a_n^*/|\sigma_n|\} < \infty, \qquad \text{all } \lambda > 2.$$

Next, it will be demonstrated that

(23) $$\sigma_n = o(a_n^*).$$

If Y is unbounded, it follows from (22) that $P\{|Y| > 3a_n^*/|\sigma_n|\} = o(1)$ implying (23). If rather, Y is bounded, the hypothesis of the theorem asserts that $\sigma_n^2 = o(s_n^2)$ and moreover, $s_n^2 \to \infty$ by (15). It is well-known that $EY = 0$, $0 < EY^2 < \infty$, $s_n^2 \to \infty$, $\sigma_n^2 = o(s_n^2)$ entail the classical Lindeberg condition for asymptotic normality of $S_n/(s_n^2 EY^2)^{1/2}$ and, *a fortiori*, $S_{n(k)}/(s_{n(k)}^2 EY^2)^{1/2} \to_d N(0, 1)$ as $k \to \infty$ for any subsequence $n(k) \to \infty$. Now, $a_n^* \ge s_n$ for all large n since otherwise $a_{n(k)}^* < s_{n(k)}$ for some subsequence $n(k) \to \infty$ implying

$$0 < \lim_{k\to\infty} P\{S_{n(k)} > 2s_{n(k)}\} \le \lim_{k\to\infty} P\{\cup_{i=k}^{\infty} [S_{n(i)} > 2s_{n(i)}]\}$$
$$\le P\{\limsup_{n\to\infty} S_n/a_n^* \ge 2\}$$

which contradicts (18). Thus, (23) holds recalling that $\sigma_n = o(s_n)$.

Finally, (19) and (23) ensure

(24) $$\max_{1\le j\le n} |\sigma_j|/a_n \le \max_{1\le j<m} |\sigma_j|/a_n^* + \max_{m\le j\le n} |\sigma_j|/a_j^* \overset{n\to\infty}{\longrightarrow} \sup_{j\ge m} |\sigma_j|/a_j^* \overset{m\to\infty}{\longrightarrow} 0.$$

But then for all $\lambda > 0$,

$$\max_{1\le j\le n} P\{|\sigma_j Y_j| > \lambda a_n\} = P\{|Y| > \lambda a_n/\max_{1\le j\le n} |\sigma_j|\} = o(1),$$

whence (10) obtains and so (14) follows via (21) and Corollary 2. □

REMARK 1. It follows from (24) and (9) that $s_n = o(a_n)$. Moreover, (22) guarantees that $\limsup_{n\to\infty} a_n/(n\sigma_n^2)^{1/2} = \infty$ if $EY^2 = \infty$.

REMARK 2. Under the hypotheses of Theorem 2, the distribution of Y belongs to the domain of partial attraction of the normal law, i.e.,

$$\liminf_{y\to\infty} y^2 P\{|Y| > y\}/H(y) = 0 \tag{25}$$

where $H(y) = EY^2 I_{[|Y|\le y]}$, $y \ge 0$. This generalizes the finding of Heyde [4] and Rogozin [7] for $\sigma_n \equiv 1$. It has been shown by Kesten [5] in the i.i.d. case that (25) is sufficient for the existence of a sequence $0 < b_n \uparrow \infty$ for which $\limsup_{n\to\infty} b_n^{-1} \sum_{j=1}^{n} Y_j = 1$, a.c. The proof is similar to that of Heyde.

PROOF OF (25). If (25) is invalid, then there exists $\delta > 0$ such that for all large y

$$y^2 P\{|Y| > y\} \ge \delta H(y). \tag{26}$$

Let $Y_n' = Y_n I_{[|Y_n|\le 3a_n^*/|\sigma_n|]}$, $n \ge 1$, where a_n^* is as in (19). Now (23) and (26) ensure that for all large n, $E(\sigma_n Y_n'/a_n^*)^2 \le 9\delta^{-1} P\{|Y| > 3a_n^*/|\sigma_n|\}$ and so $\sum_{n=1}^{\infty} E(\sigma_n Y_n'/a_n^*)^2 < \infty$ by (22). Then by the Khintchine-Kolmogorov Convergence Theorem and Kronecker's lemma $\sum_{j=1}^{n} \sigma_j Y_j'/a_n^* \to 0$, a.c. But the Borel-Cantelli lemma guarantees that $P\{\sigma_n Y_n \ne \sigma_n Y_n'$ i.o.$(n)\} = 0$, whence $S_n/a_n^* \to 0$, a.c. contradicting (18). □

The next corollary eliminates the symmetry assumption of Theorem 2. The proviso $a_n \to \infty$ cannot be dispensed with according to the ensuing Example 2.

COROLLARY 3. *Let* $S_n = \sum_{j=1}^{n} \sigma_j Y_j$, $s_n^2 = \sum_{j=1}^{n} \sigma_j^2$, $n \ge 1$, *where* $\{\sigma_n\}$ *are nonzero constants and let* $\{Y, Y_n\}$ be i.i.d. *random variables such that either* Y *is unbounded or* $\sigma_n^2 = o(s_n^2)$. *If for* $a_n \to \infty$ *and some constants* c_n

$$\limsup_{n\to\infty} |S_n - c_n|/a_n = 1, \qquad \text{a.c.} \tag{27}$$

then the weak law of large numbers

$$(S_n - \operatorname{med} S_n)/a_n \to_P 0 \tag{28}$$

obtains.

PROOF. Let $S_n^* \sum_{j=1}^{n} \sigma_j(Y_j - Y_j')$, $n \ge 1$, be a symmetrized version of $\{S_n\}$. Then by (27), $\limsup_{n\to\infty} |S_n^*|/a_n \le 2$, a.c. implying via $a_n \to \infty$, the Kolmogorov 0-1 law, and symmetry that for some constant C, $0 \le C \le 2$,

$$\limsup_{n\to\infty} S_n^*/a_n = \limsup_{n\to\infty} |S_n^*|/a_n = C, \qquad \text{a.c.}$$

If $C > 0$, then Theorem 2 (with a_n replaced by Ca_n) ensures that

$$S_n^*/a_n \to_P 0 \tag{29}$$

and this holds trivially if $C = 0$. But (29) is equivalent to (28) (Corollary 1, [6], page 245). □

3. Two counterexamples.

EXAMPLE 1. The following example of weighted i.i.d. random variables shows that Theorem 1 (resp., Theorem 2) can fail if (1) is violated (resp., if $\sigma_n^2 \ne o(s_n^2)$ and Y is bounded). Let $\sigma_n = cb^n$, $n \ge 1$, $c > 0$, $b > 1$ and let Y be symmetric with ess sup $Y = (b-1)^{1/2}/(b+1)^{1/2}$. It follows from Theorem 2 of [8] that $\limsup_{n\to\infty} \sum_{j=1}^{n} \sigma_j Y_j/s_n = 1$, a.c. and so (2) and (13) obtain with $M_n = 0$, $n \ge 1$ and $a_n = s_n$, $n \ge 1$. Since $s_n/\sigma_n \to b/(b^2-1)^{1/2} < \infty$,

$$\begin{aligned}\liminf_{n\to\infty} P\{|\textstyle\sum_{j=1}^{n} \sigma_j Y_j/s_n| > \lambda\} &\ge \liminf_{n\to\infty} P\{[\sigma_n Y_n > \lambda s_n] \cap [\textstyle\sum_{j=1}^{n-1} \sigma_j Y_j \ge 0]\} \\ &\ge \liminf_{n\to\infty} \tfrac14 P\{|Y| > \lambda s_n/\sigma_n\} > 0\end{aligned}$$

provided $\lambda > 0$ is sufficiently small, and consequently both (1) and (3) fail.

EXAMPLE 2. Apropos of Corollary 3, this example shows that $a_n \to \infty$ cannot be replaced by $a_n > 0$. If $P\{Y = 0\} = P\{Y = 1\} = ½$ and $\sigma_n = 2^{-n}$, $n \geq 1$, then $S_n = \sum_{j=1}^{n} \sigma_j Y_j$ converges a.c. to a random variable which is uniformly distributed on [0, 1] and, moreover, med $S_n = ½$ for all $n \geq 1$. If $c_n = a_n = n$ for n even while for n odd, $c_n = 0$ and $a_n = 1$, then (27) obtains but for $0 < \lambda < ½$ and n odd

$$P\{|S_n - ½|/a_n > \lambda\} = P\{|S_n - ½| > \lambda\} \to 1 - 2\lambda > 0$$

and so (28) fails. This example is, of course, rather extreme in the sense that S_n converges a.c. whereas infinitely often c_n lies well beyond the support of the distribution of S_n. It is not clear whether a more natural example can be constructed wherein S_n diverges a.c. or $c_n = \text{med } S_n$, $n \geq 1$.

REFERENCES

[1] CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory*: *Independence, Interchangeability, Martingales.* Springer, Berlin.

[2] CHOW, Y. S. and TEICHER, H. (1973). Iterated logarithm laws for weighted averages. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **26** 87–94.

[3] CHUNG, K. L. (1974). *A Course in Probability Theory*, 2nd ed. Academic, New York.

[4] HEYDE, C. C. (1969). A note concerning behaviour of iterated logarithm type. *Proc. Amer. Math. Soc.* **23** 85–90.

[5] KESTEN, H. (1972). Sums of independent random variables—without moment conditions. *Ann. Math. Statist.* **43** 701–732.

[6] LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand-Reinhold, New York.

[7] ROGOZIN, B. A. (1968). On the existence of exact upper sequences. *Teor. Verojatnost. i Primenen* **13** 701–707. (English translation in *Theor. Probability Appl.* **13** (1968) 667–672.)

[8] ROSALSKY, A. J. (1978). On generalizing the law of the iterated logarithm for weighted sums of i.i.d. random variables with infinite variance. Ph.D. Thesis, Statist. Dept., Rutgers Univ.

[9] TEICHER, H. (1974). On the law of the iterated logarithm. *Ann. Probability* **2** 714–728.

[10] TEICHER, H. (1975). A necessary condition for the iterated logarithm law. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **31** 343–349.

DEPARTMENT OF STATISTICS
NUCLEAR SCIENCES BUILDING
UNIVERSITY OF FLORIDA
GAINESVILLE, FLORIDA 32611

DEPARTMENT OF STATISTICS
HILL CENTER FOR THE MATHEMATICAL SCIENCES
BUSCH CAMPUS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903