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WHEN IS THE CLUSTER SET OF S_n/a_n EMPTY?

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We investigate the relationship between the bounded law of the iterated logarithm and the compact law of the iterated logarithm for Banach space valued random variables through the cluster set of S_n/a_n. Some rate of escape questions are also examined.

1. Introduction Let B denote a real separable Banach space with norm $\|\cdot\|$, and throughout assume $X, X_1, X_2, \cdots$ are independent identically distributed (i.i.d.) B-valued random variables such that $E(X) = 0$ and $E\|X\|^2 < \infty$. As usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$, and we write Lx to denote the function $\max(\log x, 1)$. The function $L(Lx)$ is written LLx, B^* denotes the topological dual of B, and $a_n = (2nLLn)^{1/2}$ for $n \geq 1$. The set of all limit points of $\{x_n\}$ is denoted by $C(\{x_n\})$ and is called the cluster set of $\{x_n\}$.

If $\{\alpha_n\}$ is any sequence of nonzero constants, then it is an easy consequence of the Hewitt-Savage zero-one law that with probability one the cluster set $C(\{S_n/\alpha_n\})$ is a nonrandom set A depending on $\{\alpha_n\}$ and the law of X (see Lemma 1 below). If $\{\alpha_n\}$ is such that $\lim\sup_n \|S_n/\alpha_n\| = 0$, then we see $A = \{0\}$. However, if $0 < \lim\sup_n \|S_n/\alpha_n\|$, the nature of the almost sure cluster set A is much less obvious. The situation of interest in this paper is the case of the law of the iterated logarithm (LIL), i.e., when $\alpha_n = a_n$ for $n \geq 1$, but other normalizing constants α_n are important as well. Of course, if X is not the zero random variable, we always have $0 < \lim\sup_n \|S_n/a_n\|$, and hence A is to be determined.

Of course, A being a cluster set implies A is always closed, and since $E(X) = 0$, $E\|X\|^2 < \infty$ there is a canonical set K, depending only on the covariance function

$$T(f, g) = E(f(X)g(X)) \qquad (f, g \in B^*)$$

of X, such that we always have

$$P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) \subseteq K\right) = 1. \tag{1.1}$$

For the definition of K we refer to Lemma 2.1 of [4], and mention that since $E\|X\|^2 < \infty$ we have K compact. The proof that $A \subseteq K$ or, equivalently, that (1.1) holds, follows from [4], page 745.

In view of recent results on the LIL we say X satisfies the compact LIL if there is a compact set D such that

$$P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) = D\right) = 1 \tag{1.2}$$

and

$$P\left(\lim_n d\left(\frac{S_n}{a_n}, D\right) = 0\right) = 1. \tag{1.3}$$

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Here $d(x, D) = \inf_{y \in D} \| x - y \|$, and we call D the limit set in the compact LIL.

If X satisfies the compact LIL with limit set D, then [4], Corollary 3.1 implies $D = K$ and hence the cluster set is completely determined in this case. However, some examples of G. Pisier [7] show that a random variable X may fail to satisfy the compact LIL yet have

$$P\left(\limsup_n \left\| \frac{S_n}{a_n} \right\| < \infty\right) = 1. \tag{1.4}$$

If (1.4) holds we say X satisfies the bounded LIL, and it is the purpose of this note to examine the relationship of the compact LIL and the bounded LIL by examining the nature of the cluster set $C(\{S_n/a_n\})$ in the examples of Pisier mentioned above. This appears to be a rather modest goal, but the difficulties are substantial and one would hope that future investigations could accomplish more.

One of our results states that for the "regular examples" of Pisier [7] satisfying the bounded LIL, but not the compact LIL, the nonrandom cluster set A is empty. Combining this information we easily obtain a natural rate of escape and a natural rate of growth for these processes. For the example of [3], which satisfies the compact LIL but not the central limit theorem (CLT), we prove a rate of escape result, as well as examine the related cluster set.

Of course, a conjecture which immediately suggests itself is that if X satisfies the bounded LIL, but not the compact LIL, then $P(C(\{S_n/a_n\}) = \emptyset) = 1$. In Section 4 we provide an example which shows this conjecture to be false.

2. Some notation and the statement of results. Let c_0 denote the separable Banach space of all real sequences $x = \{x_k\}$ such that $\lim_k x_k = 0$ which is normed by

$$\| x \| = \sup_k | x_k |. \tag{2.1}$$

Let $e_j = \{\delta_{ij} : i \geq 1\}$ for $j = 1, 2, \cdots$ where $\delta_{ij} = 0$ for $i \neq j$ and 1 for $i = j$. Then we define

$$X(\omega) = \textstyle\sum_{j=1} \alpha_j \varepsilon_j(\omega) e_j \tag{2.2}$$

where $\varepsilon_1, \varepsilon_2, \cdots$ are independent random variables such that $P(\varepsilon_j = \pm 1) = ½$ for $j = 1$, and $\{\alpha_j\}$ is a sequence in c_0.

If $\alpha_j = (2Lj)^{-1/2}$ then X satisfies the compact LIL but not the central limit theorem [3], and if the α_j are chosen as in [7], page IV 8, then X satisfies the bounded LIL but not the compact LIL. It is these latter examples which we are most interested in, but before describing them in detail we need a bit of terminology.

We say a positive increasing sequence $\{c_n\} = \{c(n)\}$ *increases with logarithmic regularity* if $\lim_n c_n = \infty$ and for some $\rho > 0$ and $p \geq 1$ we have for $y > 0$ that

$$L(c(c^{-1}(y) - 1)) \geq \rho (Ly)^{1/p}. \tag{2.3}$$

Here $c^{-1}(y) = \inf\{n : c(n) \geq y\}$ is the definition of c^{-1}.

If $\{\varepsilon_j : j \geq 1\}$ is a sequence of independent random variable such that $P(\varepsilon_j = \pm 1) = ½$ for $j = 1$, we define

$$M = \sup_n \frac{|\varepsilon_1 + \cdots + \varepsilon_n|}{a_n}. \tag{2.4}$$

The examples of [3] and [7] depend on the fact tht for every $\beta > 0$ we have

$$E(\exp\{\beta M^2\}) < \infty. \tag{2.5}$$

For the results of this paper we will need to know even more about the distribution of the random variable M than is given in (2.5), and this will be established below in Lemma 2.

Lemma 2 is also of interest in that it seems to be a new and rather unexpected result about coin tossing proving that

$$\lim_{\lambda\to\infty} \log P(M > \lambda)/\lambda^2 LL\lambda = -1.$$

Let $b_n > 0$, $\sum_n b_n < \infty$. Then we say the random variable X defined in (2.2) is a "regular example" of Pisier if $\alpha_1 = 1/c_1$, $\Lambda_1 = 1$ and for $n > 1$

$$\alpha_m = \frac{1}{c_n} \quad \text{for} \quad \Lambda_1 + \cdots + \Lambda_{n-1} < m \le \Lambda_1 + \cdots + \Lambda_n \tag{2.6}$$

where $\{c_n\}$ is logarithmically regular such that for n sufficiently large

$$P(M > 2c_n) \le b_n P(M > c_n) \tag{2.7}$$

and the Λ_n's are integers such that

$$\Lambda_n - 1 \le \frac{1}{P(M > c_n)} < \Lambda_n. \tag{2.8}$$

The existence of regular examples can easily be seen from Lemma 2 by choosing, for example, $c_n = n^{1/2}$ or $c_n = e^n$ when $b_n = {}^1/_{2^n}$ for large n. Many other choices are also possible, but now we can state our theorem.

THEOREM 1. *Let X be defined as in* (2.2) *and assume X is a regular example of Pisier. Then*:

$$\begin{aligned}&\text{(i) } X \text{ satisfies the bounded LIL, but not the compact LIL,}\\ &\text{(ii) } P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) = \varnothing\right) = 1, \quad \text{and}\\ &\text{(iii) } 0 < \liminf_n \frac{\| S_n \|}{a_n} \le \limsup_n \frac{\| S_n \|}{a_n} < \infty.\end{aligned} \tag{2.9}$$

The result (2.9-iii) gives us a natural rate of escape as well as a natural rate of growth for the process $\{S_n : n \ge 1\}$ generated by a regular example of Pisier. For the example of [3] we can prove the following contrasting result. Other rate of escape results for infinite dimensional processes can be found in [1].

THEOREM 2. *Let X be defined as in* (2.2) *with* $\alpha_j = (2Lj)^{-1/2}$ *for* $j \ge 1$. *Then*:

$$\begin{aligned}&\text{(i) } X \text{ satisfies the compact LIL, but not the CLT,}\\ &\text{(ii) } P\left(\liminf_n \left\|\frac{S_n}{\sqrt{n}}\right\| = 1\right) = 1,\\ &\text{(iii) } P\left(\limsup_n \left\|\frac{S_n}{\sqrt{n}}\right\| = \infty\right) = 1\\ &\text{(iv) } P\left(C\left(\left\{\frac{S_n}{\sqrt{n}}\right\}\right) = \varnothing\right) = 1.\end{aligned} \tag{2.10}$$

REMARK. The results in Theorem 2 also contrast with the situation that occurs when X satisfies the central limit theorem. That is, if B is separable, and X satisfies the CLT, then we have

$$P\left(\liminf_n \left\|\frac{S_n}{\sqrt{n}}\right\| = 0\right) = 1, \tag{2.11}$$

as well as the more general fact that

$$P\left(C\left(\left\{\frac{S_n}{\sqrt{n}}\right\}\right) = \overline{\cup_{n=1} nK}\right) = 1. \tag{2.12}$$

Here $\bar{A}$ denotes the closure in B of any subset A of B. The proof of (2.12) (and hence of (2.11)) will be sketched in Section four following the proof of Lemma 5.

In Section four we also provide a counterexample to the conjecture mentioned at the end of Section one. Indeed, for this example, which satisfies the bounded LIL but not the compact LIL, we will show that not only is the cluster set nonempty, but as large as possible, i.e.,

$$P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K\right) = 1.$$

In summary, then, we see that the relationship between the bounded LIL and the compact LIL is rather subtle. One possible conjecture that would unify our view of these matters to some degree is that if X satisfies the bounded LIL, then $C(\{S_n/a_n\}) = \emptyset$ or K, with probability one. If this conjecture is true, then, of course, it still remains to decide when we have the cluster set empty and when it is equal to K.

3. Some lemmas and the proofs of Theorems 1 and 2. Our first result is a zero-one law for the cluster set of $\{S_n/a_n\}$. Its proof is much like that of Kesten's Theorem 1 in [2] and in the form presented below it is due to A. Neidhardt [5].

LEMMA 1. *Suppose V is a separable topological vector space (so that addition is measurable); T is a second countable topological space, $\{f_n\}$ is a sequence of measurable mappings of V into T, and $\{X_n\}$ is a sequence of* i.i.d. *V-valued random variables on the probability space $(\Omega, \mathscr{F}, P)$. Then there is a nonrandom set A, depending only on the distribution of X_1, such that with probability one*

$$\cap_{m=1}^{\infty} \overline{\{f_n(\textstyle\sum_{i=1}^{n} X_i(\omega)) : n \geq m\}} = A. \tag{3.1}$$

PROOF. Let $\mathscr{B}$ be a countable base for T. Let $S_n = \sum_{i=1}^{n} X_i$,

$$\mathscr{B}_1 = \{B \in \mathscr{B} : P(f_n(S_n) \in B \text{ i.o.}) = 1\},$$

and

$$\mathscr{B}_2 = \{B \in \mathscr{B} : P(f_n(S_n) \in B \text{ i.o.}) = 0\}.$$

Thus $\mathscr{B}_1$ and $\mathscr{B}_2$ depend only on the distribution of the process $\{S_n\}$, and hence only on the distribution of X_1. Further, $\{f_n(S_n) \in B \text{ i.o.}\}$ is a symmetric event, so the Hewitt-Savage zero one law implies $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$.

Let $U = \cup_{B \in \mathscr{B}_2} B$, $A = T - U$, so A depends only on the distribution of X_1. Let

$$\Omega_1 = \{\omega : \forall B \in \mathscr{B}_1,\ f_n(S_n(\omega)) \in B \text{ i.o. and } \forall B \in \mathscr{B}_2,\ f_n(S_n(\omega)) \in B^c \text{ eventually}\}.$$

Then Ω_1 is the countable intersection of sets of probability one so it has probability one.

Now for $\omega \in \Omega_1$ we have

$$C(\omega) \equiv \cap_{m=1}^{\infty} \overline{\{f_n(S_n(\omega)) : n \geq m\}} = A.$$

That is, for $x \in A$ and any neighborhood N of x, there is a $B \in \mathscr{B}$ with $x \in B \subseteq N$, and as $x \notin U$, $B \notin \mathscr{B}_2$ so $B \in \mathscr{B}_1$. Hence by definition of Ω_1 we have $f_n(S_n(\omega)) \in B$ i.o. Thus $x \in C(\omega)$ and hence $A \subseteq C(\omega)$ for all $\omega \in \Omega_1$. For $x \in U$, there is a $B \in \mathscr{B}_2$ with $x \in B$, so by definition of Ω_1 we have $f_n(S_n(\omega)) \in B^c$ eventually. Hence $x \notin C(\omega)$ and thus $C(\omega) \subseteq A$ so the lemma is proved.

The next lemma gives us information regarding the tail behavior of the distribution of the random variable M defined in (2.4).

LEMMA 2. *For any $\delta > 0$ there exists $\lambda(\delta)$ such that $\lambda \geq \lambda(\delta)$ implies*

$$\exp\{-(1+\delta)\lambda^2 LL\lambda\} \leq P(M > \lambda) \leq \exp\left\{-\frac{\lambda^2 LL\lambda}{1+\delta}\right\}. \tag{3.2}$$

PROOF. It suffices to prove (3.2) for $\delta < 1$ so fix $0 < \delta < 1$. Next observe that for all $l \geq 1$

$$\begin{aligned} P(M > \lambda) &\geq P(|\textstyle\sum_{j=1}^{l} \varepsilon_j/a_l| > \lambda) \\ &\geq \exp\{-(1+\delta/2)\lambda^2 LLl\} \end{aligned} \tag{3.3}$$

where the last inequality follows from [8,] page 262 with $\gamma = \delta/2$, $c = 1/\sqrt{l}$, $s_l = \sqrt{l}$ provided λ is restricted to satisfy $\sqrt{2LLl}\lambda \geq \varepsilon(\delta/2)$ and $\lambda\sqrt{2LLl/l} \leq \pi(\delta/2)$.

We now attempt to maximize the right-hand side of (3.3) in l as a function of λ subject to the restrictions

$$\lambda\sqrt{2LLl} \geq \varepsilon(\delta/2) \qquad \text{and} \quad \lambda\sqrt{\frac{2LLl}{l}} \leq \pi(\delta/2). \tag{3.4}$$

Given λ, the right-hand side of (3.4) implies $\sqrt{l/2LLl} \geq \lambda(\pi(\delta/2))$ so take $l = [(\lambda^2 4LL\lambda)/(\pi^2(\delta/2))] + 1$. Then there exists $\eta(\delta)$ such that for $\lambda \geq \eta(\delta)$ we have the left-hand side of (3.4) and since $LL\lambda/LLl \to 1$ as $\lambda \to \infty$ we have

$$\sqrt{\frac{l}{2LLl}} \geq \frac{\lambda\sqrt{2}}{\pi(\delta/2)} \sqrt{\frac{LL\lambda}{LLl}} \geq \frac{\lambda}{\pi(\delta/2)}.$$

Thus with $l = [4\lambda^2 LL\lambda/\pi^2(\delta/2)] + 1$ *and* $\lambda \geq \eta(\delta)$ we have from (3.3) that there exists $\theta(\delta)$ such that $\lambda \geq \theta(\delta)$ implies

$$\begin{aligned} P(M > \lambda) &\geq \exp\left\{-(1+\delta/2)\lambda^2 LL\left(\frac{\lambda^2 4LL\lambda}{\pi^2(\delta/2)} + 1\right)\right\} \\ &\geq \exp\{-(1+\delta)\lambda^2 LL\lambda\} \end{aligned} \tag{3.5}$$

since $\lim_{\lambda\to\infty} LL((\lambda^2 4LL\lambda)/(\pi^2(\delta/2))/LL\lambda = 1$. Thus the left-hand side of (3.2) holds for $\lambda \geq \max(\eta(\delta), \theta(\delta))$, so it suffices to prove the right-hand inequality in (3.2).

Let $M_r = \sup_{n\geq r} |\sum_{j=1}^{n} \epsilon_j/a_n|$. Then $M_1 = M$ and for $c = 1 + \frac{1}{3}\delta$ we have

$$P(M_r > \lambda) \leq \textstyle\sum_{k\geq k(r)+1} P(\max_{c^{k-1}\leq n\leq c^k} |\sum_{j=1}^{n} \varepsilon_j/a_n| \geq \lambda)$$

where $k(r) = \max\{j : c^j \leq r\} = \left[\dfrac{\log r}{\log c}\right]$

$$\begin{aligned}
&\leq \textstyle\sum_{k\geq k(r)+1} P\left(\max_{c^{k-1}\leq n\leq c^k}\left|\sum_{j=1}^{n}\dfrac{\varepsilon_j}{a_{[c^{k-1}]}}\right| \geq \lambda\right) \\
&\leq 2\textstyle\sum_{k\geq k(r)+1} P\left(\left|\sum_{j=1}^{[c^k]}\dfrac{\varepsilon_j}{\sqrt{[c^k]}}\right| > \dfrac{a_{[c^{k-1}]}\lambda}{\sqrt{[c^k]}}\right) && \text{by Levy's inequality} \\
&\leq 4\textstyle\sum_{k\geq k(r)+1} \exp\{-\lambda^2[c^{k-1}]LL[c^{k-1}]/[c^k]\} && \text{by the standard exponential inequality for Rademacher random variables (see [6], Theorem 15, page 52)} \\
&\leq 4\textstyle\sum_{k\geq k(r)+1}\left(\dfrac{1}{L[c^{k-1}]}\right)^{\lambda^2/1+\delta/2} && \text{provided } k(r) \geq k_0
\end{aligned} \tag{3.6}$$

$$\text{since } \frac{[c^{k-1}]}{[c^k]} \geq \frac{1}{1 + \tfrac{1}{2}\delta} \text{ for } k \text{ sufficiently large}$$

$$\leq 4 \int_{k(r)-2}^{\infty} \frac{dx}{(x \log c)^{\lambda^2/1+(1/2)\delta}} \qquad \text{for } k(r) \geq k_1 \text{ sufficiently large}$$

$$= 4/(\log c)^{\lambda^2/1+(1/2)\delta} \left(\frac{\lambda^2}{1 + \tfrac{1}{2}\delta} + 1\right)(k(r) - 2)^{\left(\frac{\lambda^2}{1+(1/2)\delta} - 1\right)}$$

$$\text{for } \lambda > \sqrt{1 + \tfrac{1}{2}\delta}.$$

Now there exists η_1 such that $\lambda \geq \eta_1 > \sqrt{1 + \tfrac{1}{2}\delta}$ implies $k([3\lambda)) \geq \max(k_0, k_1)$, and

$$\sup_{n \leq [3\lambda]} \sqrt{\frac{n}{2LLn}} < \lambda \qquad \text{and} \qquad \left[\frac{\log[3\lambda]}{\log c}\right] \geq \frac{\log \lambda}{\log c} \qquad \text{since} \quad c < \frac{3}{2}.$$

Hence for $\lambda \geq \eta_1$ we have

$$P(M_1 > \lambda) \leq P(M_{[3\lambda]} > \lambda) + P(\sup_{n \leq [3\lambda]} |\textstyle\sum_{j=1}^{n} \varepsilon_j / a_n| > \lambda)$$

$$= P(M_{[3\lambda]} > \lambda)$$

$$(3.7) \qquad \leq 4/(\log c)^{\frac{\lambda^2}{1+(1/2)\delta}} \left(\frac{\lambda^2}{1 + \tfrac{1}{2}\delta} - 1\right)(k([3\lambda]) - 2)^{\frac{\lambda^2}{1+(1/2)\delta} - 1}$$

$$\text{where } k([3\lambda]) = \left[\frac{\log[3\lambda]}{\log c}\right] \geq \frac{\log \lambda}{\log c}$$

$$\leq \frac{4(\log \lambda - 2 \log c)}{\log c \left(\frac{\lambda^2}{1 + \tfrac{1}{2}\delta} - 1\right)(\log \lambda - 2 \log c)^{\lambda^2/1+(1/2)\delta}}.$$

Hence there is an $\eta_2 \geq \eta_1$ such that $\lambda \geq \eta_2$ implies $\log(\log \lambda - 2 \log c) > (\log \log \lambda)^{(1+(1/2)\delta/1+\delta}$ and

$$(3.8) \qquad P(M_1 > \lambda) \leq \exp\left\{\frac{-\lambda^2}{1 + \delta} LL\lambda\right\}.$$

Taking $\lambda \geq \max(\eta_2, \eta(\delta), \theta(\delta))$ we have both (3.5) and (3.8) so the lemma is proved.

LEMMA 3. *Let X be a regular example of Pisier defined as in (2.2) through the conditions in (2.3), (2.6), (2.7) and (2.8). If p is as in (2.3) and $X_1, X_2, \cdots$ are independent copies of X, then for all ε, $0 < \varepsilon^2 < 1/12p$, and integers N there exists $\gamma > 0$ and an $n(\varepsilon, N)$ such that $n \geq n(\varepsilon, N)$ implies*

$$(3.9) \qquad P\left(\left|\frac{S_n}{a_n}\right|_{\infty,N} < \varepsilon\right) \leq \exp\{-(Ln)^{\gamma}\}$$

where

$$(3.10) \qquad |\{x_k\}|_{\infty,N} = \sup_{k \geq N+1} |x_k| \qquad (\{x_k\} \in c_0).$$

PROOF. Fix $\varepsilon > 0$ such that $\varepsilon^2 < 1/12p$ and let N be given. Let

$$(3.11) \qquad \Gamma_n = \max\left\{l : \alpha_l > \varepsilon \sqrt{\frac{2LLn}{n}}\right\}.$$

Then Γ_n converges to ∞ and for $\Gamma_n > N$ we have

$$P\left(\left|\frac{S_n}{a_n}\right|_{\infty,N} \leq \varepsilon\right) = \prod_{l=N+1}^{\Gamma_n} P\left(\left|\frac{\varepsilon_1 + \cdots + \varepsilon_n}{\sqrt{n}}\right| \leq \frac{\varepsilon}{\alpha_l} \sqrt{2LLn}\right)$$

$$(3.12)\qquad = \prod_{l=N+1}^{\Gamma_n} \left\{1 - P\left(\left|\frac{\varepsilon_1 + \cdots + \varepsilon_n}{\sqrt{n}}\right| > \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right)\right\}$$
$$\le \prod_{l=u_n}^{v_n} \left\{1 - \exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2} LLn\right\}\right\}$$

by [8], page 262 with $\gamma = 1/2$ provided $(\varepsilon/\alpha_l)\sqrt{2LLn} \ge \lambda(1/2)$, $(\varepsilon/\alpha_l)\sqrt{(2LLn/n)} \le \pi(1/2) \le 1/2$, $s_n = \sqrt{n}$, and $c_n = 1/\sqrt{n}$. Hence $\alpha_l \le \varepsilon\sqrt{2LLn}/\lambda(1/2)$ and $\alpha_l \ge (\varepsilon/\pi(1/2))\cdot\sqrt{2LLn/n}$, so $u_n = \max(N+1, \alpha^{-1}(\varepsilon\sqrt{(2LLn}/\lambda(1/2))) = N+1$ for n sufficiently large, and $v_n = \min(\Gamma_n, \alpha^{-1}((\varepsilon/\pi(1/2))\sqrt{2LLn/n})) = \Gamma_n$ as $\pi(1/2) \le 1/2$ where $\alpha^{-1}(y) = \max\{l\colon \alpha_l > y\}$ for $0 < y \le \alpha_1$ and $\alpha^{-1}(y) = 0$ for $y > \alpha_1$.

Thus there exists $n_1(\varepsilon, N)$ such that $n \ge n_1(\varepsilon, N)$ implies

$$(3.13)\qquad P\left(\left|\frac{S_n}{a_n}\right|_{\infty,N} \le \varepsilon\right) \le \prod_{l=N+1}^{\Gamma_n}\left\{1 - \exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2}LLn\right\}\right\}$$
$$\le \exp\left\{-\sum_{l=N+1}^{\Gamma_n}\exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2}LLn\right\}\right\}$$

since $1 - x \le e^{-x}$ for $x \ge 0$.

Now for n sufficiently large

$$(3.14)\qquad \sum_{l=N+1}^{\Gamma_n}\exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2}LLn\right\} \le \Lambda_{j^*}\exp\left\{-\frac{3\varepsilon^2}{2}c_{j^*}^2 LLn\right\}$$

where

$$(3.15)\qquad j^* = \max\{k\colon \varepsilon\sqrt{2LLn}\, c_k < \sqrt{n}\}.$$

Now $\Lambda_j > 1/P(M > c_j) \ge \exp\{c_j^2 LLc_j/2\}$ by Lemma 2 for c_j sufficiently large, so for n sufficiently large (3.14) implies

$$(3.16)\qquad \sum_{l=N+1}^{\Gamma_n}\exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2}LLn\right\} \ge \exp\left\{c_{j^*}^2\left\{\frac{LLc_{j^*}}{2} - \frac{3}{2}\varepsilon^2 LLn\right\}\right\}$$
$$\ge \exp\left\{c_{j^*}^2\left\{\frac{1}{4p}LLn - \frac{3}{2}\varepsilon^2 LLn\right\}\right\}$$

provided

$$(3.17)\qquad LLc_{j^*} \ge \frac{1}{2p}LLn.$$

To see that (3.17) holds observe that

$$(3.18)\qquad c^{-1}\left(\frac{1}{\varepsilon}\sqrt{\frac{n}{2LLn}}\right) - 1 = j^* < c^{-1}\left(\frac{1}{\varepsilon}\sqrt{\frac{n}{2LLn}}\right),$$

so from (3.18) there exists $n_2(\varepsilon, N)$ sufficiently large so that $n \ge n_2(\varepsilon, N)$ implies (3.14), $c_{j^*} \ge 1$, and by (2.3) that

$$(3.19)\qquad LLc_{j^*} = LLc\left(c^{-1}\left(\frac{1}{\varepsilon}\sqrt{\frac{n}{2LLn}}\right) - 1\right)$$
$$\ge \frac{1}{2p}LLn.$$

Thus for $\frac{3}{2}\varepsilon^2 < 1/8p$ we have for $n \ge n_2(\varepsilon, N)$ that (3.16) gives

$$(3.20)\qquad \sum_{l=N+1}^{\Gamma_n}\exp\left\{-\frac{3\varepsilon^2}{2\alpha_l^2}LLn\right\} \ge \exp\{\gamma\, LLn\}$$

for $\gamma = 1/8p$. Inserting (3.20) into (3.13) we have for $n \ge \max(n_1(\varepsilon, N), n_2(\varepsilon, N))$ that (3.9) holds, and the lemma is proved.

PROOF OF THEOREM 1. If X is defined as in (2.2) *and* X *is a regular example of Pisier,* then (2.9-i) holds by the argument in [7] since $\sum_{n=1}^{\infty} b_n < \infty$ and hence w.p. 1

$$\limsup_n \left\| \frac{S_n}{a_n} \right\| < \infty.$$

Thus to complete the proof of the theorem it suffices to show (2.9-ii) since we then also have w.p. 1

$$\liminf_n \left\| \frac{S_n}{a_n} \right\| > 0,$$

or otherwise zero would be in $C(\{S_n/a_n\})$.

If (2.9-ii) does not hold, then by Lemma 1 there exists $d \in c_0$ such that w.p. 1 $d \in C(\{S_n/a_n\})$. Hence for all $\varepsilon > 0$, $\varepsilon \le 1$,

$$P\left(\left\| \frac{S_n}{a_n} - d \right\| < \varepsilon \text{ i.o. in } n \right) = 1, \tag{3.21}$$

and thus by the Borel-Cantelli lemma for all $b > 1$

$$\textstyle\sum_k P\left(\left\| \frac{S_n}{a_n} - d \right\| < \varepsilon \text{ for some } n \in [b^k, b^{k+1}) \right) = \infty. \tag{3.22}$$

Since $d \in c_0$ there exists N such that $\sup_{n \ge N+1} |d_n| < \varepsilon$ and hence if $d \in C(\{S_n/a_n\})$ we must have

$$\textstyle\sum_k P\left(\left| \frac{S_n}{a_n} \right|_{\infty,N} < 2\varepsilon \quad \text{for some} \quad n \in [b^k, b^{k+1}) \right) = \infty. \tag{3.23}$$

Let $S(t) = S_{[t]}$ and $a(t) = a_{[t]}$ for $t \ge 0$. Next define $\tau = \min\{t: |S(t)|_{\infty,N} \le \varepsilon a(t), t \ge b^k\}$. Then

$$\begin{aligned} &E\left\{ \int_{b^k}^{b^{k+2}} I\{|S(s)|_{\infty,N} \le 2\varepsilon a(s)\}\, ds \right\} \\ &\quad \ge E\left\{ \int_{\tau}^{b^{k+2}} I\{|S(s)|_{\infty,N} \le 2\varepsilon a(s)\}\, ds;\ \tau \le b^{k+1} \right\} \\ &\quad = E\left\{ E\left(\int_0^{b^{k+2}-t} I[|S(s) + y|_{\infty,N} \le 2\varepsilon a(s+t)]\, ds \right) \Big|_{y=S(\tau);\tau=t};\ \tau \le b^{k+1} \right\}. \end{aligned} \tag{3.24}$$

When $y = S(\tau)$ and $b^k \le t = \tau \le b^{k+1}$ we have

$$|S(s) + y|_{\infty,N} \le |S(s)|_{\infty,N} + \varepsilon a(t),$$

so

$$\begin{aligned} &\int_0^{b^{k+2}-t} I[|S(s) + y|_{\infty,N} \le 2\varepsilon a(s+t)]\, ds \\ &\quad \ge \int_0^{b^{k+2}-t} I[|S(s)|_{\infty,N} \le \varepsilon a(t)]\, ds \end{aligned} \tag{3.25}$$

$$\geq \int_0^{b^{k+2}-b^{k+1}} I[\,|S(s)|_{\infty,N} \leq \varepsilon a(b^k)]\, ds$$

since $a(t)$ is monotonic increasing. Hence by Fubini's theorem we have

$$\int_{b^k}^{b^{k+2}} P(\,|S(s)|_{\infty,N} \leq 2\varepsilon a(s))\, ds$$

(3.26)
$$\geq P(b^k \leq \tau \leq b^{k+1}) \int_0^{(b-1)b^{k+1}} P(\,|S(s)|_{\infty,N} \leq \varepsilon a(b^k))\, ds$$
$$= P(\,|S_n|_{\infty,N} \leq \varepsilon a_n \quad \text{for some} \quad n \in [b^k, b^{k+1}))$$
$$\cdot \int_0^{(b-1)b^{k+1}} P(\,|S(s)|_{\infty,N} \leq \varepsilon a(b^k))\, ds.$$

Thus

(3.27)
$$P\left(\left| \frac{S_n}{a_n} \right|_{\infty,N} \leq \varepsilon \quad \text{for some} \quad n \in [b^k, b^{k+1}) \right)$$
$$\leq \frac{\int_{b^k}^{b^{k+2}} P(\,|S(s)|_{\infty,N} \leq 2\varepsilon a(s))\, ds}{\int_0^{(b-1)b^{k+1}} P(\,|S(s)|_{\infty,N} \leq \varepsilon a(b^k))\, ds}.$$

Since X satisfies the bounded LIL we have a $\theta > \max(l, \varepsilon)$ such that $P(\,|S(u)|_{\infty,N} \leq \theta a(u)) \geq 1/2$ for all $u \geq 0$. Hence

(3.28)
$$\int_0^{(b-1)b^{k+1}} P(\,|S(s)|_{\infty,N} \leq \varepsilon a(b^k))\, ds$$
$$\geq \frac{1}{2} \int_J ds$$
$$\geq \frac{1}{2} \frac{\varepsilon^2 b^k}{\theta^2}$$

where

$$J = \{s\colon 0 \leq s \leq (b-1)b^{k+1} \quad \text{and} \quad \varepsilon a(b^k) \geq \theta a(s)\}$$
$$\supseteq \left\{ s\colon 0 \leq s \leq (b-1)b^{k+1} \quad \text{and} \quad s \leq \frac{\varepsilon^2 b^k}{\theta^2} \right\}$$

since $\varepsilon \leq 1$, $L \geq 1$ and $s \leq \varepsilon^2 b^k/\theta^2$ implies $sLLs \leq (\varepsilon^2 b^k/\theta^2)LLs \leq (\varepsilon^2 b^k/\theta^2)LLb^k$ as $\varepsilon^2/\theta^2 \leq 1$.

Therefore from (3.27) and (3.28) we get

(3.29)
$$P\left(\left| \frac{S_n}{a_n} \right|_{\infty,N} \leq \varepsilon \text{ for some } n \in [b^k, b^{k+1}) \right)$$
$$\leq \frac{2\theta^2}{\varepsilon^2 b^k} \textstyle\sum_{j=b^k}^{b^{k+2}} P(\,|S_j|_{\infty,N} \leq 2\varepsilon a_j)$$

$$\leq \frac{2\theta^2}{\varepsilon^2 b^k} b^k(b^2-1)\exp\{-k^\gamma(\log b)^\gamma\}$$

where the second inequality follows from Lemma 3 with $\gamma > 0$ provided n is sufficiently large. Hence

$$\sum_k P\left(\left|\frac{S_n}{a_n}\right|_{\infty,N} \leq \varepsilon \quad \text{for some} \quad n \in [b^k, b^{k+1})\right) \leq \frac{2\theta^2}{\varepsilon^2}(b^2-1)\sum_k \exp\{-k^\gamma(\log b)^\gamma\} < \infty \tag{3.30}$$

and the theorem is proved because $A \neq \varnothing$ implies (3.23), and $0 < \varepsilon \leq 1$ was arbitrary.

PROOF OF THEOREM 2. The result (2.10-i) follows from [3], so we will first prove that for all $\varepsilon > 0$, $\varepsilon < 1$,

$$\liminf_n \left\|\frac{S_n}{\sqrt{n}}\right\| > 1-\varepsilon, \tag{3.31}$$

and (3.31) will hold if for all $\varepsilon > 0$, $\varepsilon < 1$,

$$\sum_{n=1}^\infty P(\|S_n\| \leq \sqrt{n}(1-\varepsilon)) < \infty. \tag{3.32}$$

Now

$$\begin{aligned} P(\|S_n\| \leq \sqrt{n}(1-\varepsilon)) &= \prod_{k=1}^{\Gamma_n} P\left(\left|\sum_{j=1}^n \frac{\varepsilon_j}{\sqrt{n}}\right| \leq \sqrt{2Lk}(1-\varepsilon)\right) \\ &\qquad \text{where } \Gamma_n = \inf\{j: \sqrt{2Lj}(1-\varepsilon) \geq \sqrt{n}\} \\ &\leq \prod_{k=1}^{\Gamma_n}\left[1 - P\left(\sum_{j=1}^n \frac{\varepsilon_j}{\sqrt{n}} > \sqrt{2Lk}(1-\varepsilon)\right)\right] \\ &\leq \prod_{u_n \leq k \leq v_n}[1-\exp\{-Lk(1-\varepsilon)^2(1+\varepsilon)\}] \end{aligned} \tag{3.33}$$

by [8], page 262, with $\gamma = \varepsilon$, $c_n = \dfrac{1}{\sqrt{n}}$, $s_n = \sqrt{n}$ provided

$$\frac{\sqrt{2Lk}(1-\varepsilon)}{\sqrt{n}} \leq \Pi(\varepsilon) \leq 1$$

and $\sqrt{2Lk}(1-\varepsilon) \geq \lambda(\varepsilon) \geq 2$. Hence in (3.33) if $\sqrt{2Lk}(1-\varepsilon) \geq \lambda(\varepsilon) \geq 2$ we must have

$$Lk \geq \frac{\lambda^2(\varepsilon)}{2(1-\varepsilon)^2} \qquad \text{so} \quad u_n = \exp\left\{\frac{\lambda^2(\varepsilon)}{2(1-\varepsilon)^2}\right\}.$$

If $\sqrt{2Lk}(1-\varepsilon)/\sqrt{n} \leq \pi(\varepsilon) \leq 1$ we must have

$$Lk \leq \frac{n\Pi^2(\varepsilon)}{2(1-\varepsilon)^2} \qquad \text{so} \quad v_n = \min\left(\Gamma_n, \exp\left\{\frac{n\Pi^2(\varepsilon)}{2(1-\varepsilon)^2}\right\}\right) = \exp\left\{\frac{n\Pi^2(\varepsilon)}{2(1-\varepsilon)^2}\right\}$$

since $\Pi^2(\varepsilon) \leq 1$.

Hence there is a $c(\varepsilon) < \infty$ such that

$$P(\|S_n\| \leq \sqrt{n}(1-\varepsilon)) \leq \prod_{u_n \leq k \leq v_n}^n \left[1 - \frac{1}{k^{(1-\varepsilon)^2(1+\varepsilon)}}\right]$$

$$\leq \exp\left\{-\sum_{u_n \leq k \leq v_n}^{n} \frac{1}{k^{(1-\varepsilon)^2(1+\varepsilon)}}\right\}$$

(3.34)
$$\leq \exp\left\{\frac{-v_n^{1-(1-\varepsilon)^2(1+\varepsilon)} + u_n^{1-(1-\varepsilon)^2(1+\varepsilon)}}{1-(1-\varepsilon)^2(1+\varepsilon)}\right\}$$

$$\leq c(\varepsilon)\exp\{-v_n^{1-(1-\varepsilon)^2(1+\varepsilon)}\}$$

$$= c(\varepsilon)\exp\left\{-\exp\left(\frac{n\pi^2(\varepsilon)(1-(1-\varepsilon)^2(1+\varepsilon))}{2(1-\varepsilon)^2}\right)\right\}$$

and hence (3.32) holds.

The proof of (2.10-ii) will be completed by showing that for all $\varepsilon > 0$

(3.35)
$$\liminf_n \frac{\|S_n\|}{\sqrt{n}} \leq 1 + \varepsilon.$$

To prove (3.35) we define the independent events

$$A_k = \left\{\frac{\|S_{n_k} - S_{n_{k-1}}\|}{\sqrt{n_k}} \leq 1 + \varepsilon\right\}$$

where $n_k = k^k$. Then $\sum_k P(A_k) = \infty$ and since X satisfies the LIL by [3] we have

$$\lim_k \frac{\|S_{n_{k-1}}\|}{\sqrt{n_k}} = 0,$$

so (3.35) follows from the Borel-Cantelli lemma.

To show $\sum_k P(A_k) = \infty$ we observe that

$$P(A_k) = \prod_{l=1}^{\Gamma_k} P\left(\left|\sum_{n_{k-1}<j\leq n_k} \frac{\varepsilon_j}{\sqrt{n^k}}\right| \leq (1+\varepsilon)\sqrt{2Ll}\right.$$

(3.36)
$$\text{where } \Gamma_k = \inf\{l: \sqrt{2Ll}\,(1+\varepsilon) \geq \sqrt{n^k}\}$$

$$= \prod_{l=1}^{\Gamma_k}\left[1 - P\left(\left|\sum_{n_{k-1}<j\leq n_k} \frac{\varepsilon_j}{\sqrt{n}}\right| \geq (1+\varepsilon)\sqrt{2Ll}\right)\right]$$

$$= \prod_{l=1}^{\Gamma_k}\left[1 - 2\exp\left\{-\frac{(1+\varepsilon)^2 2Ll}{2}\cdot\frac{n_k}{n_k - n_{k-1}}\right\}\right].$$

The inequality in (3.36) follows immediately from the well-known exponential inequality ([6], page 52) implying

(3.37)
$$P\left(\left|\sum_{j=1}^{n} \frac{\varepsilon_j}{\sqrt{n}}\right| > \lambda\right) \leq 2\exp\left\{-\frac{\lambda^2}{2}\right\}.$$

Hence there is a $c > 0$ such that for k large

$$P(A_k) \geq c\prod_{l=3}^{\infty}\left[1 - \frac{2}{l^{1+\delta}}\right].$$

for some $\delta > 0$. Thus $P(A_k) \geq \gamma > 0$ for k sufficiently large, and hence (2.10-ii) is proved.

Now (2.10-iii) follows immediately from (2.10-i) since K contains nonzero elements, and hence we turn to (2.10 − iv).

If $d = \{d_j\} \in c_0$ and $d \in C(\{S_n/\sqrt{n}\})$ with probability one, then for all $\varepsilon > 0$ we must have

(3.38)
$$\sum_n P\left(\left\|\frac{S_n}{\sqrt{n}} - d\right\| \leq \varepsilon\right) = \infty,$$

and this is a contradiction. That is, fix $0 < \varepsilon < 1/3$ and choose N such that $\sup_{j\geq N} |d_j| < \varepsilon$. Then

$$P\left(\left\|\frac{S_n}{\sqrt{n}} - d\right\| \leq \varepsilon\right) \leq \prod_{k=N}^{\infty} P\left(\left|\sum_{j=1}^{n} \frac{\varepsilon_j}{\sqrt{n}}\right| \leq 2\varepsilon\sqrt{2LK}\right). \tag{3.39}$$

Since $0 < \varepsilon < 1/3$ the argument in (3.34) applied to the right-hand side of (3.39) gives a $c(\varepsilon) > 0$ and $\rho(\varepsilon) > 0$ such that

$$P\left(\left\|\frac{S_n}{\sqrt{n}} - d\right\| \leq \varepsilon\right) \leq c(\varepsilon)\exp\{-\exp\{n\rho(\varepsilon)\}\}. \tag{3.40}$$

Thus (3.38) fails for $0 < \varepsilon < 1/3$ and the theorem is proved.

4. A counterexample to the conjecture of Section one. In view of Theorem 1 it seems natural to guess that if X satisfies the bounded LIL, but not the compact LIL, then $C(\{S_n/a_n\}) = \varnothing$ with probability one. The example we present here, however, shows that this is not always the case.

The example X is again defined as in (2.2) through the conditions (2.6), (2.7) and (2.8) with $c_k = c(k)$ where $c(k)$ is such that $\log_k c(k) = k$ and $\log_k x$ is the log function iterated k times. Recall the definition $c^{-1}(y) = \inf\{n : c(n) \geq y\}$ and set $I_k = [(c(k))^{1/2}, c(k)]$ for $k \geq 1$. Then for $y \in I_k$, $k \geq 2$, we have

$$c^{-1}(y) - 1 = k - 1, \tag{4.1}$$

and we see that $\{c_k\}$ is not regularly increasing since (2.3) does not hold.

We now have the following

THEOREM 3. *If X is defined as indicated above, then:*

(i) *X satisifes the bounded* LIL, *but not the compact* LIL;

(4.2) (ii) $P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K\right) = 1$, *and with probability one*

(iii) $0 = \liminf_n \left\|\frac{S_n}{a_n}\right\| < \limsup_n \left\|\frac{S_n}{a_n}\right\| < \infty.$

To show (4.2-ii) we need some lemmas. The first gives us a way to check if $b \in C(\{S_n/a_n\})$ and partially follows [2], pages 1176–1178.

LEMMA 4. *Let $X_1, X_2, \cdots$ be independent identically distributed B-valued random variables. Then $b \in C(\{S_n/a_n\})$ with probability one* iff *for every $\varepsilon > 0$*

$$\sum_k P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon \quad \text{for some} \quad n \in [2^k, 2^{k+1})\right) = \infty. \tag{4.3}$$

PROOF. If $b \in C(\{S_n/a_n\})$ with probability one, then for every $\varepsilon > 0$ (4.3) must hold by the Borel-Cantelli lemma. Hence assume (4.3) holds for some $b \in B$ and all $\varepsilon > 0$. Now fix $\varepsilon > 0$ and choose an integer $s \geq 6$ such that $(\|b\| + 1)/\sqrt{2^{s-2}} \leq \varepsilon/4$.

Define the stopping times

$$\lambda = \min\left\{n : n \geq 2^k, \quad \left\|\frac{S_n}{a_n} - b\right\| \leq \varepsilon\right\}.$$

where we take $\lambda = \infty$ if $\|(S_n/a_n) - b\| > \varepsilon$ for all $n \geq 2^k$. Then $P(\lambda < \infty) > 0$ since for any $j > k$ we have $P(\lambda < \infty) \geq P(\|(S_n/a_n) - b\| \leq \varepsilon$ for some $n \in [2^j, 2^{j+1}))$ and the latter probabilities must be positive for infinitely many j because the series in (4.3) diverges. On $\{\lambda < \infty\}$ we have $\|(S_\lambda/a_\lambda) - b\| \leq \varepsilon$ so

$$D_k \equiv \left\{ \left\| \frac{S_n}{a_n} - b \right\| > \varepsilon \quad \text{for all} \quad n \geq 2^{k+s}, \quad \lambda \in [2^k, 2^{k+1}) \right\}$$

$$\supseteq \left\{ \left\| \left(\frac{S_{n+\lambda}}{a_{n+\lambda}} - b \right) - \left(\frac{S_\lambda}{a_\lambda} - b \right) \right\| \geq 2\varepsilon \quad \text{for all} \quad n \geq 2^{k+s} - \lambda, \quad \lambda \in [2^k, 2^{k+1}) \right\} \tag{4.4}$$

$$= \left\{ \left\| \left(\frac{S_{n+\lambda} - S_\lambda}{a_{n+\lambda}} - b \right) + S_\lambda \left(\frac{1}{a_{a+\lambda}} - \frac{1}{a_\lambda} \right) + b \right\| \geq 2\varepsilon \quad \text{for all} \quad n \geq 2^{k+s} - \lambda, \quad \lambda \in [2^k, 2^{k+1}) \right\}$$

$$\supseteq \left\{ \left\| \frac{S_{n+\lambda} - S_\lambda}{a_{n+\lambda}} - b \right\| \geq 4\varepsilon \quad \text{for all} \quad n \geq 2^{k+s} - \lambda, \quad \lambda \in [2^k, 2^{k+1}) \right\}$$

since

$$\left\| S_\lambda \left(\frac{1}{a_{n+\lambda}} - \frac{1}{a_\lambda} \right) + b \right\| \leq \varepsilon + \left\| \frac{S_\lambda}{a_{n+\lambda}} \right\|$$

$$\leq \varepsilon + \frac{a_\lambda}{a_{n+\lambda}} \left\| \frac{S_\lambda}{a_\lambda} \right\|$$

$$\leq \varepsilon + \frac{\| b \| + \varepsilon}{\sqrt{2^{s-1}}} \leq \frac{3}{2}\varepsilon.$$

Now $\| ((S_{n+\lambda} - S_\lambda)/a_n) - b \| > 6\varepsilon$ implies

$$\left\| \frac{S_{n+\lambda} - S_\lambda}{a_{n+\lambda}} - b \right\| > 6\varepsilon \frac{a_n}{a_{n+\lambda}} - \| b \| \left| 1 - \frac{a_n}{a_{n+\lambda}} \right| \tag{4.5}$$

and since $\lambda \in [2^k, 2^{k+1})$ and $n \geq 2^{k+s} - \lambda$ with $s \geq 6$ we have

$$\frac{a_n^2}{a_{n+\lambda}^2} = \frac{nLLn}{(n+\lambda)LL(n+\lambda)}$$

$$\geq \frac{1}{1 + \dfrac{2^{k+1}}{n}} \cdot \frac{LLn}{LL(n+\lambda)}$$

$$\geq \frac{1}{1 + \dfrac{2^{k+1}}{n}} \cdot \frac{LLn}{LLn + \dfrac{\lambda}{nLn}}$$

where the last inequality follows from the mean value theorem. That is, if $f(t) = LLt$ then for $t \geq e^e$ we have $f'(t) = 1/(tLt)$ so the mean value theorem implies $LL(n+\lambda) \leq LLn + \lambda/(nLn)$. Hence we have

$$\frac{a_n^2}{a_{n+\lambda}^2} \geq \frac{1}{1 + \dfrac{2^{k+1}}{n}} \cdot \frac{1}{1 + \dfrac{2^{k+1}}{nLnLLn}}$$

$$\geq \frac{1}{\left(1 + \dfrac{2^{k+1}}{n}\right)^2}$$

$$\geq \frac{1}{\left(1 + \frac{2^{k+1}}{2^{k+s} - 2^{k+1}}\right)^2},$$

so $a_n/a_{n+\lambda} \geq 1 - \frac{1}{2}^{s-1}$ when $n \geq 2^{k+s} - \lambda$ and $s \geq 6$. Inserting this estimate into (4.5) we see $s \geq 6$, $(\| b \| + 1)/\sqrt{2^{s-2}} \leq \varepsilon/4$, and $\| ((S_{n+\lambda} - S_\lambda)/a_n) - b \| \geq 6\varepsilon$ imply that

$$\left\| \frac{S_{n+\lambda} - S_\lambda}{a_{n+\lambda}} - b \right\| \geq 4\varepsilon.$$

Hence for ≥ 6 and $(\| b \| + 1)/\sqrt{2^{s-2}} \leq \varepsilon/4$ we have from (4.4) that

$$D_k \supseteq \left\{ \left\| \frac{S_{n+\lambda} - S_\lambda}{a_n} - b \right\| \geq 6\varepsilon \quad \text{for all} \quad n \geq 2^{k+s} - \lambda, \quad \lambda \in [2^k, 2^{k+1}) \right\}. \tag{4.6}$$

From (4.6) we see that

$$P(D_k) \geq P\left(\left\| \frac{S_{n+\lambda} - S_\lambda}{a_n} - b \right\| \geq 6\varepsilon \quad \text{for all} \quad n \geq s^{k+s} - \lambda, \quad \lambda \in [2^k, 2^{k+1}) \right)$$

$$\geq P\left(\left\| \frac{S_n}{a_n} - b \right\| \geq 6\varepsilon \quad \text{for all} \quad n \geq 2^{s-1} \right) P(\lambda \in [2^k, s^{k+1})).$$

Now from the definition of D_k we easily see that at most s of the events D_k can occur at a single time, so

$$\infty > s \geq E(\# D_k \quad \text{which occur})$$

$$= \textstyle\sum_k P(D_k)$$

$$\geq \textstyle\sum_k P\left(\left\| \frac{S_n}{a_n} - b \right\| \geq 6\varepsilon \quad \text{for all} \quad n \geq 2^{s-1} \right) P(\lambda \in [2^k, s^{k+1}))$$

$$\geq P\left(\left\| \frac{S_n}{a_n} - b \right\| \geq 6\varepsilon \quad \text{for all} \quad n \geq 2^{s-1} \right) \cdot \infty.$$

Thus $P(\| (S_n/a_n) - b \| < 6\varepsilon \text{ i.o.}) = 1$ and since $\varepsilon > 0$ is arbitrary we have $b \in C(\{S_n/a_n\})$ and the lemma is proved.

Lemma 5. *Let $X_1, X_2, \cdots$ be independent identically distributed B-valued random variables. Then $b \in C(\{S_n/a_n\})$ with probability one if for every $\varepsilon > 0$,*

$$\textstyle\sum_n P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon \right) \Big/ n = \infty. \tag{4.7}$$

Proof. If (4.7) holds for all $\varepsilon > 0$, then

$$\textstyle\sum_{2^k \leq n < 2^{k+1}} P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon \right) \Big/ n$$

$$\leq \frac{2^{k+1} - 2^k}{2^k} P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon \quad \text{for some} \quad n \in [2^k, 2^{k+1}) \right)$$

and hence (4.3) holds so $b \in C(\{S_n/a_n\})$ with probability one.

Proof of (2.11) and (2.12). Since X satisfies the CLT we have that $S_n/\sqrt{n}$ converges in distribution to a mean zero Gaussian measure μ on B determined by the covariance of X. Furthermore, it is well known that the support of μ is $\overline{\bigcup_{n=1}^\infty nK}$ and hence $\liminf_n P(\| (S_n/\sqrt{n}) - b \| < \varepsilon) \geq \mu(x : \| x - b \| < \varepsilon) > 0$ for all $b \in \overline{\bigcup_{n=1}^\infty nK}$. Hence for all $\varepsilon > 0$

$\sum_k P(\| (S_n/\sqrt{n}) - b \| < \varepsilon$ for some $n \in [2^k, 2^{k+1})) = \infty$, and if we follow the proof of Lemma 4 we see that this implies $b \in C(\{S_n/\sqrt{n}\})$. Since b was an arbitrary pointof $\overline{\cup_{n=1}^{\infty} nK}$ this gives (2.12) and hence (2.11).

PROOF OF THEOREM 3. That X satisfies the bounded LIL, but not the compact LIL, is given by the argument of Pisier in [7]. Hence (4.1-i) holds and (4.2-iii) will then follow if we verify (4.2-ii).

To verify (4.2-ii) we first will show that for all $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$ satisfying

$$\frac{1}{\varepsilon}\sqrt{\frac{n}{2LLn}} \in I_{k(n)}, \tag{4.8}$$

we have a $\gamma > 0$ such that

$$P\left(\left\|\frac{S_n}{a_n}\right\| \leq \varepsilon\right) \geq \gamma. \tag{4.9}$$

Hence fix $\varepsilon > 0$ and observe that for infinitely many n we have $k(n)$ such that (4.8) holds, i.e., to see this just observe the growth of the left-hand side of (4.8) in comparison to the growth of $\{c(k)\}$. In fact, if

$$j^* = \max\left\{k : c(k) < \frac{1}{\varepsilon}\sqrt{\frac{n}{2LLn}}\right\}, \tag{4.10}$$

then (4.8) implies that $I_{k(n)} = I_{j^*+1}$ so $k(n) = j^* + 1$.

To verify (4.9) we first choose $n_1(\varepsilon)$ such that $n \geq n_1(\varepsilon)$ implies

$$4\exp\left\{-\frac{\varepsilon^2 LLn}{\alpha_l^2}\right\} \leq 4\exp\left\{-\frac{\varepsilon^2 LLn}{\alpha_1^2}\right\} < \log 2. \tag{4.11}$$

then for $n \geq n_1(\varepsilon)$ we have

$$\begin{aligned} P\left(\left\|\frac{S_n}{a_n}\right\| \leq \varepsilon\right) &= \prod_{l=1}^{\Gamma_n} P\left(\left|\sum_{j=1}^{n}\frac{\varepsilon_j}{n^{1/2}}\right| \leq \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right) \\ &\qquad \text{where } \Gamma_n = \max\left\{l : \alpha_l > \varepsilon\sqrt{\frac{2LLn}{n}}\right\} \\ &\geq \prod_{l=1}^{\Gamma_n}\left\{1 - 2\exp\left\{-\frac{\varepsilon^2}{\alpha_l^2}LLn\right\}\right\} \end{aligned} \tag{4.12}$$

where the inequality follows from the standard exponential inequality (3.37). Hence, since $1 - (x/2) \geq e^{-x}$ for $0 \leq x < \log 2$ and (4.11) holds for $n \geq n_1(\varepsilon)$, we have from (4.12) that

$$P(\| S_n \| \leq \varepsilon a_n) \geq \exp\{-4Z_n\} \tag{4.13}$$

where

$$Z_n = \sum_{l=1}^{\Gamma_n}\exp\left\{-\frac{\varepsilon^2}{\alpha_l^2}LLn\right\}.$$

Recalling j^* from (4.10) and Γ_n as in (4.12) we see from (2.6), (2.7) and (2.8) that

$$Z_n = \sum_{j=1}^{j^*}\Lambda_j\exp\{-\varepsilon^2 c_j^2\, LLn\}. \tag{4.14}$$

Taking $\delta = 1$ in Lemma 2 we thus have an $H < \infty$ such that

$$\Lambda_j \leq \frac{1}{P(M > c_j)} + 1 \leq H\exp\{2c_j^2 LLc_j\}) \tag{4.15}$$

for all $j \geq 1$. Combining (4.14) and (4.15) we have since the $\{c_j\}$ are increasing that

$$Z_n \le H \textstyle\sum_{j=1}^{j^*} \exp\left\{2c_j^2\left\{LLc_{j^*} - \frac{\varepsilon^2}{2} LLn\right\}\right\}. \tag{4.16}$$

Now $j^* = c^{-1}((1/\varepsilon)\sqrt{n/2LLn}) - 1$ and since n satisfies (4.8) we have $k(n) = j^* + 1$ so

$$\frac{1}{\varepsilon^2} n \ge \frac{n}{2\varepsilon^2 LLn} \ge c(j^* + 1). \tag{4.17}$$

Furthermore, letting $e_j(x)$ denote the inverse of $\log_j x$ we have $c_j = c(j) = e_j(j)$ and hence for $j \ge 3$

$$LLc_j = e_{j-2}(j). \tag{4.18}$$

Using (4.17) and (4.18) we next take $n_0(\varepsilon) \ge n_1(\varepsilon)$ such that $n \ge n_0(\varepsilon)$ implies $j^* \ge 3$ and

$$\begin{aligned} \varepsilon^2 LLn &\ge \varepsilon^2 LL\varepsilon^2 c(j^* + 1) \\ &\ge \frac{\varepsilon^2}{2} LLc(j^* + 1) \\ &= \frac{\varepsilon^2}{2} e_{j^*-1}(j^* + 1) \\ &\ge 4e_{j^*-2}(j^*) \\ &= 4LLc_{j^*} \ge 4. \end{aligned} \tag{4.19}$$

Hence for $n \ge n_0(\varepsilon)$ and satisfying (4.8) we have by combining (4.16) and (4.19) that

$$\begin{aligned} Z_n &\le H \textstyle\sum_{j=1}^{j^*} \exp\{2c_j^2(LLc_{j^*} - 2LLc_{j^*})\} \\ &\le H \textstyle\sum_{j=1}^{\infty} \exp\{-2c_j^2\} \\ &\equiv H_1 < \infty. \end{aligned} \tag{4.20}$$

Thus for $n \ge n_0(\varepsilon)$ and n satisfying (4.8) we have

$$P(\| S_n \| \le \varepsilon a_n) \ge \exp\{-4H_1\} \ge \gamma, \tag{4.21}$$

and hence (4.9) holds as claimed.

Given X as in the theorem Lemma 2.1 of [4] easily implies

$$K = \left\{\{b_j\} \in c_0 : \textstyle\sum_{j=1}^{\infty} \frac{b_j^2}{\alpha_j^2} \le 1\right\}.$$

Since $C(\{S_n/a_n\})$ is closed and with probability one contained in K, ([4], page 745), we thus have (4.2-ii) if we show that all $b \in K$ of the form

$$b = (b_1, \cdots, b_N, 0, 0, \cdots) \tag{4.22}$$

with $\rho = \sum_{j=1}^{N} (b_j^2/\alpha_j^2) < 1$ are in $C(\{S_n/a_n\})$.

To do this we apply Lemma 5. That is, fix $\varepsilon > 0$ and take b as in (4.22). Then for all n sufficiently large we have

$$\begin{aligned} P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) &= \textstyle\prod_{l=1}^{N} P\left(\left|\sum_{j=1}^{n} \frac{\varepsilon_j}{\sqrt{n}} - \frac{b_l}{\alpha_l}\sqrt{2LLn}\right| < \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right) \\ &\quad \cdot \textstyle\prod_{l=N+1}^{\infty} P\left(\left|\sum_{j=1}^{n} \frac{\varepsilon_j}{a_n}\right| < \frac{\varepsilon}{\alpha_j}\right) \\ &\ge \textstyle\prod_{l=1}^{N} P\left(\left|\sum_{j=1}^{n} \frac{\varepsilon_j}{\sqrt{n}} - \frac{b_l}{\alpha_l}\sqrt{2LLn}\right| < \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right) P\left(\left\|\frac{S_n}{a_n}\right\| < \varepsilon\right). \end{aligned} \tag{4.23}$$

Combining (4.21) and (4.23) we see there exists an $n_2(\varepsilon)$ such that $n \geq n_2(\varepsilon)$ and n satisfying (4.8) implies

$$P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) \geq \prod_{l=1}^N P\left(\left|\sum_{j=1}^n \frac{\varepsilon_j}{\sqrt{n}} - \frac{b_l}{\alpha_l}\sqrt{2LLn}\right| < \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right)\gamma. \tag{4.24}$$

Now N is fixed so the Berry-Esseen estimate ([6], Theorem 4, page 111) implies

$$\prod_{l=1}^N P\left(\left|\sum_{j=1}^n \frac{\varepsilon_j}{\sqrt{n}} - \frac{b_l}{\alpha_l}\sqrt{2LLn}\right| \leq \frac{\varepsilon}{\alpha_l}\sqrt{2LLn}\right)$$

$$\geq \prod_{l=1}^N\left\{\int_{\frac{|b_l|}{\alpha_l}\sqrt{2LLn}}^{\frac{(|b_l|+\varepsilon)}{\alpha_l}\sqrt{2LLn}} \frac{e^{-s^2/2}}{\sqrt{2\pi}}\,ds + O\left(\frac{1}{n^{1/2}}\right)\right\} \tag{4.25}$$

$$\geq c_1 \prod_{l=1}^N \frac{\exp\left\{-\frac{b_l^2}{\alpha_l^2}LLn\right\}}{\varepsilon\sqrt{2LLn}} + O\left(\frac{1}{n^{1/2}}\right)$$

for all n sufficiently large since

$$\int_a^b \frac{e^{-s^2/2}}{\sqrt{2\pi}}\,ds \geq \frac{1}{b}e^{-a^2/2}\left[1 - e^{\frac{-(b^2-a^2)}{2}}\right]$$

when $0 \leq a < b$, and c_1 is some positive constant

$$= c_1\frac{\exp\{-\rho LLn\}}{(\varepsilon\sqrt{2LLn})^N} + O\left(\frac{1}{n^{1/2}}\right) \geq \frac{\beta}{(Ln)^\rho(LLn)^N}$$

for all n sufficiently large with $\rho = \sum_{l=1}^N b_l^2/\alpha_l^2 < 1$ and some $\beta > 0$.

Combining (4.24) and (4.25) we have for each α satisfying $0 \leq \rho < \alpha < 1$ an $n_3(\varepsilon)$ such that for $n \geq n_3(\varepsilon)$ and n satisfying (4.8)

$$P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) \geq \frac{1}{(Ln)^\alpha}. \tag{4.26}$$

To finish the proof let $J = \{n : n \text{ satisfies } (4.8)\}$. Then Lemma 5 implies $b \in C(\{S_n/a_n\})$ with probability one since

$$\sum_{n\in J} \frac{1}{n(Ln)^\alpha} = \infty. \tag{4.27}$$

To verify (4.27) let $J_k = \{n : (1/\varepsilon)\sqrt{n/2LLn} \in I_k\}$ and recall $I_k = [(c(k))^{1/2}, c(k)]$. Then $J = \cup_k J_k$ and for large k

$$J_k = \left\{n : \frac{n}{LLn} \in [2\varepsilon^2 c(k), 2\varepsilon^2 c^2(k)\right\}$$

$$\supseteq \{n : n \in [2\varepsilon^2 c(k)LLc^2(k), \varepsilon^2 c^2(k)LLc^2(k)]\} \tag{4.28}$$

$$\equiv \{n : n \in [x_k, y_k]\}.$$

Furthermore, for k large (4.28) implies

$$\sum_{n\in J_k} \frac{1}{n(Ln)^\alpha} \geq \sum_{n\in[x_k, y_k]} \frac{1}{n(Ln)^\alpha} \tag{4.29}$$

$$\geq \frac{1}{2(1-\alpha)}\{(\log y_k)^{1-\alpha} - (\log x_k)^{1-\alpha}\}.$$

Using the mean value theorem on $f(x) = x^{1-\alpha}$ we see that for large k

$$\sum_{n\in J_k} \frac{1}{n(Ln)^\alpha} \geq \frac{1}{2}\frac{(\log y_k - \log x_k)}{(\log y_k)^\alpha}$$

$$\geq \frac{1}{4}(\log c(k))^{1-\alpha}.$$

Hence $J = \cup_k J_k$ implies (4.27) and the theorem is proved.

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