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# THE 1979 WALD MEMORIAL LECTURES

## INFINITE SYSTEMS WITH LOCALLY INTERACTING COMPONENTS[1]

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In 1963 Glauber analyzed a one dimensional model for magnetism. It was the first study of the Markovian time evolution of a system with infinitely many interacting components. In Section 1 it will be discussed in the light of recent progress in this field. The remaining Sections (2, 3, and 4) form an introduction to recent joint work with Thomas M. Liggett. They concern new types of systems where each component takes on a real value which fluctuates in a way depending linearly on the values of neighboring components. The ergodic theory for such systems with finitely many components is the subject of Section 2. The results suggest conjectures for the case of infinitely many components, stated in Section 3 and proved in the joint paper with T. Liggett (*ibid.*). Section 4 introduces another class of time evolutions whose ergodic behavior may be analyzed by similar methods.

**1. The Glauber model.** Let $\mathbb{Z}$ be the set of integers, and $\Sigma = \{-1, +1\}^{\mathbb{Z}}$ the space of *configurations*. Thus a configuration $\sigma \in \Sigma$ is a given assignment of spins $\sigma(x)$, with values $+1$ or $-1$, to each site $x \in \mathbb{Z}$. R. Glauber [5] in his original model required these spins to "flip" at random times, according to the flip rate

$$c(x, \sigma) = 1 - \alpha\sigma(x)\frac{\sigma(x-1) + \sigma(x+1)}{2}, \qquad -1 \le \alpha \le 1. \tag{1.1}$$

Here $\alpha$ is a fixed parameter, and $c(x, \sigma)\,dt$ represents the probability that $\sigma(x)$ changes to $-\sigma(x)$ in time $dt$, when the entire configuration is $\sigma$.

The simplest case is obtained when $\alpha = 0$. Then there is no interaction between components, i.e., at each $x$ the spin $\sigma_t(x)$ at time $t$ is a $\pm 1$ valued Markov process whose ergodic behavior is completely understood. Since the processes $\sigma_t(x_1), \sigma_t(x_2) \cdots \sigma_t(x_n)$ are mutually independent for distinct $x_1, x_2, \cdots, x_n$, they determine a product process $\sigma_t \in \Sigma$ whose ergodic theory is also quite transparent: the probability measure of $\sigma_t$ converges to the product measure on $\Sigma$ with probability ½ for $\pm 1$ at each site. This is independent of the starting point $\sigma_0$.

There are three basic problems in the analysis of such models:

(i) To show that the flip rate in (1.1) determines a *unique Markov process* $\sigma_t$, i.e., Feller semigroup $T_t$. Then, if $E\sigma$ denotes the expectation, given that $\sigma_0 = \sigma$,

$$T_t f(\sigma) = E^{\sigma} f(\sigma_t), \qquad f \in C(\Sigma), \tag{1.2}$$

$$(\mu T_t) f = \int_{\Sigma} (T_t f)(\sigma)\mu(d\sigma) \tag{1.3}$$

for all probability measures $\mu$ on $\Sigma$.

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(ii) To find all *equilibrium states*, i.e., all probability measures $\mu$ on $\Sigma$ such that

$$\mu T_t = \mu \qquad \text{for all} \quad t \geq 0.$$

By a compactness argument there is always at least one.

(iii) For each equilibrium state $\mu$, to find all $\nu$ such that $\nu T_t \Rightarrow \mu$ as $t \to \infty$. ($\Rightarrow$ denotes weak convergence.) We call the time evolution $\sigma_t$ *ergodic* if there is some $\mu$ such that $\nu T_t \Rightarrow \mu$ for all $\nu$.

It should be obvious that the Glauber model is ergodic when $\alpha = 0$ and has at least two equilibrium states when $\alpha = 1$ or $\alpha = -1$. We shall, however, generalize the model and only then investigate its ergodicity.

Let us replace $\mathbb{Z}$ by an arbitrary countable set $S$, so that the configuration space becomes $\Sigma = \{+1, -1\}^S$. Secondly, in the definition (1.1) of the flip rate, let us replace

$$\frac{\sigma(x-1) + \sigma(x+1)}{2} \quad \text{by} \quad P\sigma(x) = \textstyle\sum_{y \in S} P(x, y)\sigma(y),$$

where $P(x, y)$ is an irreducible Markov transition function on $S$. Then the *generalized Glauber model* is usually defined in one of two different ways:

$$(1.4) \qquad \begin{cases} \text{(A)} \quad c(x, \sigma) = 1 - \alpha\sigma(x)P\sigma(x), & 0 \leq \alpha \leq 1 \\ \text{(B)} \quad c(x, \sigma) = e^{-\beta\sigma(x)P\sigma(x)}, & 0 \leq \beta < \infty \end{cases}$$

The existence problem has been solved [10] for both models. So have the ergodicity questions (ii), (iii) to a remarkable extent. We shall make a brief detour to discuss the profound results for (B) without proof, and then return to case (A) for which it is quite easy to derive the corresponding results by standard mathematical methods.

In the model (1.4) (B) assume $S = \mathbb{Z}_d$, the $d$-dimensional integer lattice, and $P(x, y) = 1/2d$ if $|x - y| = 1$, 0 otherwise. Then for

$$d = 1: \quad \sigma_t \textit{ is ergodic for all } \beta.$$

*$d \geq 2$: There is a critical $\beta_c$ (depending on $d$) such that $\sigma_t$ is ergodic for all $\beta \leq \beta_c$, and nonergodic for $\beta > \beta_c$.*

*When $d = 2$, $\beta_c$ is the root of* $\sinh \beta = 1$

*Now let $d = 2$ and $\beta > \beta_c$. Then every equilibrium state is of the form*

$$\mu = \theta\mu^+ + (1 - \theta)\mu^-, \qquad 0 \leq \theta \leq 1,$$

*where $\mu^+(\mu^-)$ are translation invariant states whose densities*

$$\rho = E_{\mu^+}[\sigma(x)] = -E_{\mu^-}[\sigma(x)] > 0$$

*are explicitly known.* According to a famous formula of Onsager [13],

$$\rho = [1 - (\sinh \beta)^{-4}]^{1/8}, \qquad \beta > \beta_c.$$

These results are based on a combination of research both in equilibrium statistical mechanics (Aizenman [1], Higuchi [6]) and time evolutions (Holley and Stroock [8]).

*When $d \geq 3$ and $\beta$ is sufficiently large there are also nontranslation invariant equilibrium states* [2].

We now return to the generalized Glauber model defined by (1.4) (A). The treatment will require the method of *characteristic functions*. Let $X$ be the collection of finite subsets of $S$, and define

$$\chi_A(\sigma) = \textstyle\prod_{x \in A} \sigma(x), \qquad \sigma \in \Sigma, A \in X.$$

Then $\chi_A(\cdot)$ is called a *character* of $\Sigma$ (in agreement with the terminology in group theory if $\Sigma$ is interpreted as the direct product of cyclic groups of order two). If $\mu$ is a probability measure on $\Sigma$, then its *characteristic function* $\varphi$ is defined as that function from $X$ to $\mathbb{R}$

whose value at $A$ is the expectation, with respect to $\mu$, of the character $\chi_A(\cdot)$. Thus

$$\varphi(A) = E_\mu[\chi_A(\sigma)] = \int_\Sigma \chi_A(\sigma)\mu(d\sigma), \qquad A \in X.$$

*All the usual theorems for characteristic functions remain valid*, which will be very useful in investigating ergodicity.

For $t \geq 0$, $A \in X$, $\sigma \in \sum$, let

$$\varphi_t(A, \sigma) = T_t\chi_A(\sigma) = E^\sigma[\chi_A(\sigma_t)]. \tag{1.5}$$

Then, by semigroup theory

$$\frac{\partial}{\partial t}\varphi_t(A; \sigma) = E^\sigma[(G\chi_A)(\sigma_t)] \tag{1.6}$$

where $G$ is the generator defined by

$$Gf(\sigma) = \textstyle\sum_x c(x, \sigma)[f(\sigma^x) - f(\sigma)], \tag{1.7}$$

with

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x. \end{cases}$$

Take $f(\cdot) = \chi_A(\cdot)$, and compute, using (1.4)(A)

$$\begin{aligned} G\chi_A(\sigma) &= \textstyle\sum_{x\in A} c(x, \sigma)\chi_{A\setminus x}(\sigma)[-2\sigma(x)] \\ &= -2|A|\chi_A(\sigma) + 2\alpha \textstyle\sum_{x\in A}\sum_{y\in S} P(x, y)\chi_{(A\setminus x)\Delta y}(\sigma). \end{aligned} \tag{1.8}$$

(Here $\Delta$ denotes symmetric difference and $\Delta y$ is really $\Delta\{y\}$.) Now set $\sigma = \sigma_t$ and take $E^\sigma$ in (1.8). By (1.6) and (1.8)

$$\frac{\partial}{\partial t}\varphi_t(A;\sigma) = -2|A|\varphi_t(A;\sigma) + 2\alpha \textstyle\sum_{x\in A}\sum_{y\in S} P(x, y)\varphi_t[(A\setminus x)\Delta y; \sigma]. \tag{1.9}$$

Suppose now that $\mu$ is an equilibrium state, and let $\varphi(\cdot)$ denote the characteristic function of $\mu$. Then, taking $E_\mu$ in (1.9) shows that $\varphi$ satisfies the equation

$$\varphi(A) = \frac{\alpha}{|A|}\textstyle\sum_{x\in A}\sum_{y\in S} P(x, y)\varphi[(A\setminus x)\Delta y], \qquad A \in X. \tag{1.10}$$

Also, as a characteristic function $\varphi$ is *bounded* and $\varphi(\varnothing) = 1$. If it turns out that (1.10) has only one solution with these properties, then—by the method of characteristic functions—we may conclude that $\sigma_t$ has a unique equilibrium state. That is exactly what we shall show in the case $-1 < \alpha < 1$.

Thus, let $\varphi$ be a bounded solution of (1.10), such that $\varphi(\varnothing) = 1$. Let us write (1.10) in the form

$$\varphi(A) = \textstyle\sum_{B:|B|=|A|} K(A, B)\varphi(B) + \psi(A),$$

where

$$\psi(A) = \frac{\alpha}{|A|}\textstyle\sum_{x\in A}\sum_{y\in A\setminus x} P(x, y)\varphi[A\setminus(x \cup y)],$$

$$K(A, B) = \begin{cases} \dfrac{\alpha}{|A|}P(x, y) & \text{if } B = (A\setminus x) \cup y, \quad x \in A, y \in (A\setminus x)^c \\ 0 & \text{otherwise.} \end{cases}$$

Then $K$ is a contraction when $-1 < \alpha < 1$, since

$$\sum_{B:|B|=|A|} |K(A, B)| \le \frac{|\alpha|}{|A|} \sum_{x\in A} \sum_{y\in S} P(x, y) = |\alpha| < 1.$$

Therefore we have

(1.11) $$\varphi(A) = \sum_{j=0}^{\infty} \sum_{B:|B|=|A|} K^j(A, B)\psi(B).$$

Suppose now that we know $\varphi(A)$ for all sets $A$ of cardinality $|A| \le 2n$. (We do if $n = 1$, since $\varphi(\emptyset) = 1$ and $\varphi(\{x\}) = 0$ for all $x$ can be derived easily from (1.10).) Then we know $\psi(B)$ for all sets $B$ of cardinality $2n + 2$, as a look at the definition of $\psi(\cdot)$ will show. Hence, by (1.11), $\varphi(A)$ is known for all $A$ with $|A| \le 2n + 2$. This induction then completes the proof of

THEOREM 1.1. *The process $\sigma_t$ with flip rate*

$$c(x, \sigma) = 1 - \alpha\sigma(x) \sum_y P(x, y)\sigma(y)$$

*has a unique equilibrium state when* $-1 < \alpha < 1$.

Now we define a discrete time *annihilating random walk* as follows: it starts with a finite number of particles, occupying the finite set $A \subset S$ (only one particle per site.). At each unit time one of the particles is selected at random, and it jumps according to $P(x, y)$. If it lands on an occupied site then both particles disappear. Let $E^A[\cdot]$ denote the expectation, and $T \le \infty$ the first time when all particles have disappeared. (Note that $T = +\infty$ when $|A|$ is odd.)

THEOREM 1.2. *The unique equilibrium state in Theorem* 1.1 *has the characteristic function*

(1.12) $$\varphi(A) = E^A[\alpha^T], \qquad A \in X.$$

PROOF. Let $\psi(A)$ denote the right hand side in (1.12). Then $\psi(\emptyset) = 1$ and $\psi(A) = 0$ when $|A| = 1$, since $|\alpha| < 1$. Furthermore $\psi(\cdot)$ is bounded. Finally $\psi(\cdot)$ is easily seen to satisfy equation (1.10). Therefore the uniqueness result for (1.10) obtained in the proof of Theorem 1.1 implies that $\psi \equiv \varphi$. □

Now assume $S$ to be a countable (or finite) Abelian group and assume

$$P(x, y) = P(0, y - x), \qquad x, y \in S.$$

Also define

$$\tilde{P}(x, y) = \tfrac{1}{2}[P(x, y) + P(y, x)].$$

THEOREM 1.3. *Let $A = \{x, y\}$, $x \neq y$. Then the unique equilibrium state of the process $\sigma_t$ in Theorem* 1.1 *satisfies*

(1.13) $$\varphi(A) = E_{\text{equil}}[\sigma(x)\sigma(y)] = \frac{\sum_{n=0}^{\infty} \alpha^n \tilde{P}^n(x, y)}{\sum_{n=0}^{\infty} \alpha^n \tilde{P}^n(0, 0)}.$$

*Proof.* Use (1.12), noting that $T$ is the hitting time of 0, of a one-particle random walk, starting at the point $x - y$, with transition function $\tilde{P}(\cdot, \cdot)$. □

Finally, to get the most explicit results possible let $S =$ the cyclic group $\{1, 2, \cdots, N\}$ under addition (mod $N$), and let $P(x, y) = \tfrac{1}{2}$ if $|x - y| = 1$ and 0 otherwise.

THEOREM 1.4. *Let $A$ be the set of cardinality* $2n$ *consisting of the points* $i_1, \cdots, i_{2n}$, *such that*

$$1 \le i_1 < i_2 < \cdots < i_{2n} \le N.$$

*Let* $a = (i_2 - i_1) + (i_4 - i_3) + \cdots + (i_{2n} - i_{2n-1})$, $b = (i_3 - i_2) + (i_5 - i_4) + \cdots + (i_1 - i_{2n} + N)$, *so that* $a + b = N$. *Then the characteristic function of the unique equilibrium state is*

$$\varphi(A) = \frac{r^a + r^b}{1 - r^{a+b}}, \qquad \textit{where} \quad r = \frac{1 - (1 - \alpha^2)^{1/2}}{\alpha} \tag{1.14}$$

*If* $S = \mathbb{Z}$, *then*

$$\varphi(A) = r^a. \tag{1.15}$$

PROOF. We use Theorem 1.3 and observe that in the course of the annihilating random walk starting with the set $A$, the pairs $(a, b)$ undergo a Markov process at each step of which $(a, b)$ changes to $(a - 1, b + 1)$ or to $(a + 1, b - 1)$, each with probability ½. The time $T$ of annihilation is then the first time that either $(0, a + b)$ or $(a + b, 0)$ is reached. Hence $\varphi(A) = E^A(\alpha^T)$ must be the generating function of the classical gambler's ruin problem ([4] Chapter XIV) which is indeed given by (1.14). Finally, (1.15) may be obtained by letting one of the spacings tend to infinity. □

EXERCISE. Identify the state on $\Sigma = \{+1, -1\}^{\mathbb{Z}}$, whose characteristic function is given by (1.15). *Answer*: It is the Markov state with transition matrix

$$M = \begin{pmatrix} P & 1 - P \\ 1 - P & P \end{pmatrix}, \qquad \text{where} \quad P = \frac{1}{2} + \frac{1 - (1 - \alpha^2)^{1/2}}{2\alpha}. \tag{1.16}$$

Thus, by studying equilibrium states of more general Glauber models, we are studying a natural generalization of the simplest Markov states given by a $2 \times 2$ transition matrix $M$.

We return to the ergodic behavior of the Glauber model. It is not known whether there is any spin-flip evolution with a unique equilibrium state which is not ergodic. However,

THEOREM 1.5. *The Glauber model with* $-1 < \alpha < 1$ (*as in Theorem* 1.1) *is ergodic.*

There are two different proofs. One, by coupling methods valid when $0 < \alpha < 1$ is in [11] Theorem 2.2.2. The other [9], which also is easiest when $0 < \alpha < 1$ uses duality, which is a method to be explained shortly.

We finally turn to the case of the flip rate in (1.4) (A), when $\alpha = +1$, which is known as the *voter model* for the following reason: if we are on $\mathbb{Z}_d$ with $P(x, y)$ the transition function of simple symmetric random walk, then the flip rate at $x$ in (1.4)(A) is proportional to the number of neighbors of $x$ where $\sigma$ has a sign opposite to $\sigma(x)$. (Opposite signs represent opposite opinions which tend to cause the voter to change his opinion.) In any case it is clear that the voter model has *at least* two equilibrium states, namely

$$\mu^+(\mu^-) \text{ concentrated on } \sigma(x) \equiv +1(\sigma(x) \equiv -1).$$

To understand *when there are other equilibrium states* (apart from the obvious convex combinations of $\mu^+$ and $\mu^-$) we shall need the following condition

(C): *The discrete time annihilating random walk of two particles ends in annihilation in time* $T < \infty$ *with probability one.*

Note that if $S = \mathbb{Z}_d$ and $P(x, y) = 1/(2d)$ for $|x - y| = 1$, and 0 otherwise, then condition (C) holds if and only if the dimension $d = 1$ or 2.

THEOREM 1.6. *If* (C) *holds then the equilibrium states of the voter model are all of the form* $\theta\mu^+ + (1 - \theta)\mu^-$. *When* (C) *fails then there is an additional family of equilibrium* *states $\mu_\rho$, which are the weak limits, as $t \to \infty$, of $\nu_\rho T_t$, where $\nu_\rho$ is Bernoulli product measure on $\Sigma$ with $\nu_\rho[\sigma(x) = +1] = \rho$, $0 < \rho < 1$.*

PROOF. We solve the diffusion equation (1.6) by the following trick of *duality*. Let $G = G^\sigma$ be the generator of the voter model. Let $G^A$ be the generator of the following multiparticle *continuous time annihilating random walk* $A_t$, $t \geq 0$: each particle jumps after a random exponential time with mean ½; upon collision two particles instantly annihilate each other. Then straightforward computation shows

$$G^\sigma \chi_A(\sigma) = G^A \chi_A(\sigma), \qquad \sigma \in \Sigma, A \in X. \tag{1.17}$$

Thus the action of the two different generators on the system of characters is exactly the same. This fact extends from the generators to the corresponding semigroups. In terms of the expectations $E^\sigma[\cdot]$ and $E^A[\cdot]$ of the processes $\sigma_t$, $A_t$, respectively, we therefore have

$$E^\sigma[\chi_A(\sigma_t)] = E^A[\chi_{A_t}(\sigma)], \qquad \sigma \in \Sigma, A \in X. \tag{1.18}$$

Thus the solution $\varphi_t(A; \sigma)$ of equation (1.6) can be represented as follows on the probability space of the stochastic process $A_t$:

$$\varphi_t(A; \sigma) = E^A[\chi_{A_t}(\sigma)]. \tag{1.19}$$

Suppose now that condition (C) holds. Then $|A| =$ even implies that $A_t = \varnothing$ after a finite time $t$, with probability one. Since $\chi_\varnothing(\sigma) \equiv 1$, we get from (1.19) by dominated convergence

$$\lim_{t \nearrow \infty} \varphi_t(A, \sigma) = 1, \tag{1.20}$$

for any $A$ such that $|A| =$ even, and for arbitrary $\sigma$. This implies that every equilibrium state $\mu$ has a characteristic function $\varphi$, such that $\varphi(A) = 1$ when $|A| = 2$. This means that $x \neq y$ implies $\sigma(x) = \sigma(y)$ with $\mu$-probability one. The only states with this property are the convex combinations of $\mu^+$ and $\mu^-$.

Suppose finally that (C) fails. Then by (1.19)

$$\begin{aligned} E_{\nu_\rho}[\varphi_t(A; \sigma)] &= E_{\nu_\rho} E^A[\textstyle\prod_{x \in A_t} \sigma(x)] \\ &= E^A \textstyle\prod_{x \in A_t} E_{\nu_\rho} \sigma(x) = E^A[(2\rho - 1)^{|A_t|}]. \end{aligned}$$

Observe that $|A_t|$ is monotonic nonincreasing. Therefore we get in the limit, as $t \to \infty$, a family of characteristic functions

$$\psi_\rho(A) = \lim_{t \nearrow \infty} E_{\nu_\rho}[\varphi_t(A); \sigma)] = E^A[(2\rho - 1)^{|A_\infty|}].$$

It is clear that these are nontrivial characteristic functions of an equilibrium state $\mu_\rho$, since $|A_\infty| > 0$ with positive probability. □

*Note.* If $\rho = ½$ we get the characteristic function

$$\varphi(A) = P^A[|A_\infty| = 0], \qquad A \in X.$$

Much more than this was shown by Holley and Liggett [7]: When $S$ is a countable Abelian group, $P(x, y) = P(0, y - x)$, and condition (C) fails, then $\nu T_t$ converges to the equilibrium state $\mu_\rho$ in Theorem 1.6 for any ergodic translation invariant state $\nu$ such that $\nu[\sigma(x) = +1] \equiv \rho$.

**2. New models—finite $|S|$.** From now on the configuration space will be $\mathbb{R}^S$ or $\mathbb{N}^S$, where $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the nonnegative integers, and $S$ a finite or countable set. The results of this section, for $|S| < \infty$, will be extended to $|S| = \infty$ in the next section. We begin by defining three different Markovian time evolutions, to be denoted $\omega_t$, $\lambda_t$, and $\nu_t$, the first two on $\mathbb{R}^S$, the last on $\mathbb{N}^S$. They differ significantly from the Glauber model of Section 1 in that their state space is *noncompact.* Their similarity to the Glauber model lies in the feature that their generators also will depend linearly on the values of the configuration at neighboring points. This fact will make differential equation methods a successful tool, just as in Section 1. Finally, duality will play a crucial role; the three processes $\omega_t$, $\lambda_t$, $\nu_t$ to be introduced will be seen to satisfy duality relations which make it possible to learn about the ergodic behavior of each process from that of the others.

The three processes to be defined will be seen to have two ingredients in common:

(a) a system of random exponential clocks with mean one, mutually independent, one at each site $x \in S$.

(b) an irreducible transition function $P(x, y)$ on $S$. In terms of these we define

1. $\omega_t$, *the smoothing process*: when the random clock rings at $x$, then the configuration $\omega$ changes to the configuration $\omega^x$ defined by

$$\omega^x(y) = \begin{cases} \omega(y) & \text{if } y \neq x \\ \sum_{z\in S} P(x, z)\omega(z) & \text{if } y = x. \end{cases}$$

*Note.* The existence of $\omega_t$ as a Markov process causes no concern when $|S| < \infty$, while it is very hard to prove when $|S| = \infty$. Concerning the ergodic theory of this process, note that the states concentrated on $\{\omega\colon \omega(x) \equiv c\}$ for different constants $c$ are obviously equilibrium states. The question arises whether there are others.

2. $\lambda_t$, *the potlatch process*:[2] when the random clock rings at $x$, then the configuration $\lambda$ changes to $\lambda^x$ defined by

$$\lambda^x(y) = \begin{cases} \lambda(y) + \lambda(x)P(x, y), & y \neq x. \\ P(x, x)\lambda(x), & y = x. \end{cases}$$

*Note.* For existence the remarks concerning $\omega_t$ also apply here. The process started out with $\lambda_0(x) \equiv$ constant is clearly not in equilibrium. The total "mass" $\sum \lambda_0(x)$ will be preserved however (if it is finite as in the case $|S| < \infty$). Also if, as we hope to show, the process starting with $\lambda_0(x) \equiv c$ converges to an equilibrium state $\mu_c$, then the different $\mu_c$ can clearly differ from $\mu_1$ only by trivial rescaling. Thus here too there is hope of a simple but interesting ergodic theory.

3. $\nu_t$, *the coupled random walk*: This process has values in $\mathbb{N}^S$ and $\nu_t(x)$ should be thought of as the number of particles at the site $x$. When the random clock rings at $x$, then all the particles present at the site $x$ must jump, simultaneously and independently, according to the transition function $P(x, y)$.

*Note.* When $|S| < \infty$ then this is a finite state space continuous time Markov chain (since $\sum_x \nu_t(x) = N$ is independent of $t$). Therefore the ergodic theory of this process is that of Theorem 2.1 below. For ease of comparison we state at the same time the ergodic theorems for $\omega_t$ (in Theorem 2.2) and for $\lambda_t$ (in Theorem 2.3) in each case under the important restriction that $|S| < \infty$.

THEOREM 2.1. *When $\sum_x \nu_0(x) = N$, then $\nu_t$ converges to an equilibrium state $\mu_N$, and every equilibrium state is a convex combination of these.*

THEOREM 2.2. *For every $\omega_0 \in \mathbb{R}^S$, $\omega_t$ converges with probability one to a random vector $\xi \in \mathbb{R}^S$, such that $\xi(x)$ is independent of $x$ with probability one and*

$$E\xi(x) = \sum_{t\in S} \pi(t)\omega_0(t).$$

[2] This name was suggested by Benjamin Weiss. See Encyclopedia Britannica, 15th ed., Vol. VIII, for a brief description of potlatch.

*Here $\pi$ is the (unique) invariant probability measure of the kernel $P(x, y)$ ($\pi = \pi P$).*

THEOREM 2.3. *For every $\lambda_0 \in \mathbb{R}^S$, $\lambda_t$ converges weakly to an equilibrium state which depends only on $|S|^{-1} \sum_x \lambda_0(x)$. Thus there is a one-parameter family of equilibrium states which differ from each other only by scale change. Every equilibrium state is a convex combination of these.*

We shall now develop certain duality relations between the processes $\omega_t$, $\lambda_t$, and $\nu_t$. These are not strictly necessary for the proofs of Theorems 2.2 and 2.3 but they will simplify the proofs and moreover they give valuable insight into the nature of the processes themselves (which has suggested the study of the processes to be introduced in Section 4).

*The duality of $\lambda_t$ and $\omega_t$.* One first checks from the definition of $\lambda^x$ and $\omega^x$ that

$$\sum_y \lambda^x(y)\omega(y) = \sum_y \lambda(y)\omega^x(y), \qquad x \in S. \tag{2.1}$$

This in turn implies the generator duality

$$G^\lambda \varphi(\lambda, \omega) = G^\omega \varphi(\lambda, \omega) \tag{2.2}$$

if $G^\lambda$, $G^\omega$ are the generators of $\lambda_t$ and $\omega_t$ respectively defined by

$$G^\lambda f(\lambda) = \sum_x [f(\lambda^x) - f(\lambda)],$$

$$G^\omega f(\lambda) = \sum_x [f(\omega^x) - f(\omega)],$$

and $\varphi(\lambda, \omega)$ is one of the family of functions

$$\varphi(\lambda, \omega) = \exp[i\alpha \textstyle\sum_{x\in S} \omega(x)\lambda(x)], \qquad \alpha \in \mathbb{R}. \tag{2.3}$$

Finally, going from generators to their semigroups, let $E^{\omega_0}$ and $E^{\lambda_0}$ denote the expectations for the processes $\omega_t$ and $\lambda_t$ with initial state $\omega_0$ and $\lambda_0$ respectively. Then

$$E^{\omega_0}\exp[i\alpha \textstyle\sum \omega_t(x)\lambda_0(x)] = E^{\lambda_0}\exp[i\alpha \sum \lambda_t(x)\omega_0(x)], \tag{2.4}$$

for all $\alpha \in \mathbb{R}$, $\lambda_0$, $\omega_0 \in \mathbb{R}^S$. Finally since characteristic functions determine probability measures we have

PROPOSITION 2.1. *For all $\alpha$, $\beta \in \mathbb{R}^S$ the probability distribution of $\sum_x \omega_t(x)\alpha(x)$ conditioned on $\omega_0(\cdot) = \beta(\cdot)$ is the same as that of $\sum_x \lambda_t(x)\beta(x)$ conditioned on $\lambda_0(\cdot) = \alpha(\cdot)$.*

There is a somewhat different *duality relation between the processes* $\omega_t$ and $\nu_t$. It is based on the system of functions

$$\varphi(\omega, \nu) = \textstyle\prod_{x\in S} [1 + \alpha\omega(x)]^{\nu(x)}, \qquad \alpha \in \mathbb{C}. \tag{2.5}$$

If $G^\omega$ and $G^\nu$ are the respective generators then one easily verifies that

$$G^\omega \varphi(\omega, \nu) = G^\nu \varphi(\omega, \nu), \qquad \omega \in \mathbb{R}^S, \nu \in \mathbb{N}^S. \tag{2.6}$$

This in turn gives, in terms of expectations,

PROPOSITION 2.2. *For every $\omega_0 \in \mathbb{R}^S$, $\nu_0 \in \mathbb{N}^S$, and $\alpha \in \mathbb{C}$,*

$$E^{\omega_0} \textstyle\prod_{x\in S} [1 + \alpha\omega_t(x)]^{\nu_0(x)} = E^{\nu_0} \prod_{x\in S} [1 + \alpha\omega_0(x)]^{\nu_t(x)}. \tag{2.7}$$

The process $\omega_t$ is associated with an interesting *martingale*. Let $\pi$ be the unique invariant probability measure of the transition function $P(x, y)$. This exists since $|S| < \infty$ and $P(\cdot, \cdot)$ is irreducible. Then

PROPOSITION 2.3. *The process* $\sum_x \pi(x)\omega_t(x)$ *is a martingale.*

PROOF. For every $f\colon \mathbb{R}^S \to \mathbb{R}$ one knows that $f(\omega_t)$ is a martingale provided that $Gf(\omega) = 0$ for all $\omega \in \mathbb{R}^S$. Now the function $f\colon \omega \to \sum \pi(x)\omega(x)$ satisfies

$$Gf(\omega) = \sum_x \sum_y \pi(x)[\omega^y(x) - \omega(x)] \tag{2.8}$$
$$= \sum_x \pi(x)[\omega^x(x) - \omega(x)] = \sum_x \pi(x) \sum_y P(x, y)[\omega(y) - \omega(x)] = 0. \square$$

PROOF OF THEOREM 2.2. Since

$$|\omega_t(x)| \le \max_{x\in s} |\omega_0(x)|, \qquad t \ge 0,$$

the martingale $y_t = \sum_x \pi(x)\omega_t(x)$ is bounded and hence converges with probability one to a limiting random variable whose expectation is $\sum_x \pi(x)\omega_0(x)$. To conclude that each $\omega_t(x)$ converges with probability one it would suffice to show that

$$z_t = \max_{x,y\in S} |\omega_t(x) - \omega_t(y)|$$

tends to zero a.e., as $t \to \infty$. But $z_t$ is monotone in $t$. Therefore the proof of Theorem 2.2 will be complete if we show that

$$E^{\omega_0}[\omega_t(x) - \omega_t(y)]^2 \to 0, \qquad \text{as} \quad t \to \infty \tag{2.9}$$

for all $\omega_0 \in \mathbb{R}^S$, $x \ne y$ in $S$. But by Theorem 2.1 the limit as $t \to \infty$ of the right hand side in (2.7) is the same for every $\nu_0 \in \mathbb{N}^S$ such that $\sum_x \nu_0(x) = 2$. Then Proposition 2.2 implies that

$$E^{\omega_0}[1 + \alpha\omega_t(x)]^2, \qquad E^{\omega_0}[1 + \alpha\omega_t(y)]^2, \qquad E^{\omega_0}[1 + \alpha\omega_t(x)][1 + \alpha\omega_t(y)]$$

all three have the same limit as $t \to \infty$. Add now the first two of these limits and subtract twice the third. Looking at the coefficient of $\alpha^2$ in the result shows that (2.9) has been proved, and hence Theorem 2.2. $\square$

PROOF OF THEOREM 2.3. We have proved almost sure (and hence weak) convergence of $\omega_t$ for any $\omega_0$. Therefore weak convergence of $\lambda_t$ for any $\lambda_0$ follows immediately from the $\lambda \leftrightarrow \omega$ duality in Proposition 2.1. To show that the limiting state depends only on $|S|^{-1} \sum \lambda_0(x)$, for any $\lambda_0$, let us write

$$\lambda_0(x) = \frac{1}{|S|} \sum_x \lambda_0(x) + \lambda_0'(x).$$

Then $\sum_x \lambda_0'(x) = 0$, and so it suffices to show that $\lambda_t$ converges weakly to the state $\lambda \equiv 0$ when $\sum_x \lambda_0(x) \equiv 0$. But by duality this is equivalent to the fact that

$$\sum_x \lambda_0(x)\omega_t(x) \to 0 \qquad \text{when} \quad \sum \lambda_0(x) = 0,$$

which follows from Theorem 2.2. $\square$

In the rest of this section we assume that $S$ is a finite Abelian group, and that the transition function reflects this structure, i.e., $P(x, y) = P(0, y - x)$, $x, y \in S$, where 0 is of course the identity element. For simplicity we shall also assume symmetry of $P$, i.e., $P(x, y) = P(y, x)$, $x, y \in S$. Under these hypotheses we shall be able to obtain the *first two moments of the equilibrium states of the processes* $\lambda_t$ *and* $\nu_t$. (In [12] this is done without the assumption that $P(\cdot, \cdot)$ is symmetric, but the results are then much more complicated.) We now state the most fundamental results as

THEOREM 2.4. *Let* $\lambda_\infty(\cdot)$ *denote the coordinate random variables of the equilibrium state approached by* $\lambda_t$, *when initially* $\lambda_0(x) \equiv 1$. *Similarly, let* $\nu_\infty(\cdot)$ *denote the coordinate* *random variables of the equilibrium state approached by $\nu_t$ when initially the random variables $\nu_0(x)$ are independent Poisson random variables with mean $\rho$. Then the first and second moments are*

$$E\lambda_\infty(x) \equiv 1 \tag{2.10}$$

$$\mathrm{Cov}[\lambda_\infty(x), \lambda_\infty(y)] = \frac{\delta(x, y) - P(x, y)}{1 + P(0, 0)} \tag{2.11}$$

$$E\nu_\infty(x) \equiv \rho \tag{2.12}$$

$$E\nu_\infty^2(x) = \rho + \rho^2 \frac{2}{1 + P(0, 0)} \tag{2.13}$$

$$E\nu_\infty(x)\nu_\infty(y) = \rho^2 \frac{1 + P(0, 0) - P(x, y)}{1 + P(0, 0)} \tag{2.14}$$

*Note.* The most remarkable feature of these results is their *independence of* $|S|$, which suggests that they might be equally valid when $S$ is an infinite Abelian group.

For the proof we first note that (2.10) and (2.12) are obvious. Next we note that (2.11) follows from the pair (2.13), (2.14) and vice versa. This comes from duality: if the $\nu_0(x)$ are independent Poisson with mean $\rho$, then by duality (first Proposition 2.2, then Proposition 2.1)

$$\begin{aligned} E_{\text{Poisson}}E^{\nu_0} \textstyle\prod_{x\in S} [1 + \alpha\omega_0(x)]^{\nu_t(x)} &= E_{\text{Poisson}}E^{\omega_0} \textstyle\prod_{x\in S} [1 + \alpha\omega_t(x)]^{\nu_0(x)} \\ &= E^{\omega_0}\exp\{\alpha\rho \textstyle\sum_x \omega_t(x)\} \\ &= E^{\lambda_0 \equiv \rho}\exp\{\alpha \textstyle\sum_x \omega_0(x)\lambda_t(x)\} \\ &= E^{\lambda_0 \equiv 1}\exp\{\alpha\rho \textstyle\sum_x \omega_0(x)\lambda_t(x)\}. \end{aligned} \tag{2.15}$$

Letting $t \to \infty$ we get

$$E \textstyle\prod_{x\in S} [1 + \alpha\omega_0(x)]^{\nu_\infty(x)} = E \exp\{\alpha\rho \sum \omega_0(x)\lambda_\infty(x)\}.$$

If $\omega_0(x) = \delta_0(x)$, one obtains, comparing coefficients of $\alpha^2$,

$$E\nu_\infty(x)[\nu_\infty(x) - 1] = \rho^2 E\lambda_\infty^2(x),$$

and a similar argument with $\omega_0(x) = s$, $\omega_0(y) = t$, and $\omega_0(z) = 0$ otherwise, completes the proof that it suffices to find the second moments of one of the processes.

We shall find those of $\lambda_t$. First consider the coupled random walk $\nu_t$ which consists of only two particles, i.e., $\sum_x \nu_t(x) = 2$ at $t = 0$ and hence for all $t$. It is left to the reader to verify that this unique equilibrium state (invariant measure) of this Markov process is given by

$$P[\nu_\infty(x) = 2] = 2|S|^{-2}[1 + P(0, 0)]^{-1}, \qquad x \in S \tag{2.16}$$

$$P[\nu_\infty(x) = \nu_\infty(y) = 1] = 2|S|^{-2}\left[1 - \frac{P(x, y)}{1 + P(0, 0)}\right], \qquad x \neq y. \tag{2.17}$$

From (2.16) and (2.17) it is now possible to calculate the second moments of $\lambda_\infty(\cdot)$ and hence to complete the proof of Theorem 2.4. We use duality as in the proof of (2.15). By $\lambda \leftrightarrow \omega$ duality, one has for $x \neq y$

$$\begin{aligned} E^{\lambda_0 \equiv 1}[u\lambda_\infty(x) + v\lambda_\infty(y)]^2 &= \lim_{t\nearrow\infty} E^{\omega_0(\cdot) = u\delta_x(\cdot) + v\delta_y(\cdot)}[\textstyle\sum_{x\in S} \omega_t(x)]^2 \\ &= \lim_{t\nearrow\infty} |S|^2 E^{\omega_0(\cdot) = u\delta_x(\cdot) + v\delta_y(\cdot)}\omega_t^2(0). \end{aligned} \tag{2.18}$$

If we assume that $\sum_x \nu_0(x) = 2$, then we can write (2.18) as

$$(2.19) \qquad E^{\lambda_0=1}[u\lambda_\infty(x) + v\lambda_\infty(y)]^2 = |S|^2 \lim_{t\nearrow\infty} E^{\omega_0=u\delta_x+v\delta_y} \prod_{x\in S} [\omega_t(x)]^{\nu_0(x)}.$$

By a slight modification of Proposition 2.2, namely

$$E^{\omega_0} \prod_x [\omega_t(x)]^{\nu_0(x)} = E^{\nu_0} \prod_x [\omega_0(x)]^{\nu_t(x)}$$

the right hand side of (2.19) becomes

$$|S|^2 \lim_{t\nearrow\infty} E^{\nu_0}[u^{\nu_t(x)}v^{\nu_t(y)};\ \nu_t(\cdot) = 0 \text{ elsewhere}]$$

$$= |S|^2 \lim_{t\nearrow\infty} \{u^2P^{\nu_0}[\nu_t(x) = 2] + uvP^{\nu_0}[\nu_t(x) = \nu_t(y) = 1] + v^2P^{\nu_0}[\nu_t(y) = 2]\}$$

$$= |S|^2\{u^2P[\nu_\infty(x) = 2] + uvP[\nu_\infty(x) = \nu_\infty(y) = 1] + v^2P[\nu_\infty(y) = 2]\},$$

where the $P[\ ]$ probabilities are those given in (2.16) and (2.17). It follows from (2.16) and (2.19) that the coefficient of $u^2$ in (2.19) is

$$E^{\lambda_0=1}[\lambda_\infty^2(x)] = |S|^2P[\nu_\infty(x) = 2] = \frac{2}{1 + P(0, 0)}.$$

Similarly the coefficient of $2uv$ is

$$E^{\lambda_0=1}[\lambda_\infty(x)\lambda_\infty(y)] = \frac{|S|^2}{2} P[\nu_\infty(x) = \nu_\infty(y) = 1]$$

$$= 1 - \frac{P(x, y)}{1 + P(0, 0)}. \qquad \square$$

**3. The case when $S = \mathbb{Z}_d$.** Theorems 2.1 through 2.3 suggest conjectures for the case $|S| = \infty$, and Theorem 2.4 for the case when $S$ is an infinite Abelian group. But until now only the group invariant case has been studied since calculations like those in Theorem 2.4 play a crucial role in the theory. To understand this point assume that the invariant measure is constant—as it will be in the case when $S$ is an infinite group and $P(\cdot, \cdot)$ is group invariant ($P(x, y) = P(0, y - x)$). Then the process $\sum_x \omega_t(x)$ is a nonnegative martingale, if $\omega_0(x) \geq 0$ and $\sum \omega_0(x) < \infty$. By the martingale theorem it converges with probability one. By the $\omega \leftrightarrow \lambda$ duality (there is no difficulty in establishing that even though $|S| = \infty$) the process $\lambda_t$ will converge weakly. But $\lambda_t$ *might converge to* 0 (i.e., the state concentrated at $\lambda(x) \equiv 0$). This could happen if the martingale $\sum\omega_t(x)$ is not uniformly integrable in which case it may happen that

$$(3.1) \qquad \lim_{t\to\infty} \sum_{x\in S} \omega_t(x) = 0.$$

Intuitively, imagine that $\lambda_t(x)$ represents the amount of gasoline at the site $x$ at time $t$. Then (3.1), which implies that $\lambda_t(x) \to 0$ in measure for each $x$, as $t \to \infty$, would imply that more and more gasoline would accumulate at fewer and fewer sites, the rich getting richer, and the poor poorer and more numerous. To rule out this unpleasant possibility one needs estimates such as those in Theorem 2.4. If it were possible to prove that when $\lambda_0(x) \equiv 1$

$$(3.2) \qquad E[\lambda_t^2(0)] = E^{\omega_0=\delta_0}[\sum \omega_t(x)]^2 \leq M < \infty$$

where $M$ is independent of $t$, then the martingale $\sum_x \omega_t(x)$ would be uniformly integrable and weak convergence of $\lambda_t$ to a nontrivial equilibrium state assured. This has indeed been done jointly with T. M. Liggett [12] whom we thank for his permission to give the present introduction to our joint work.

The results in [12] include the proof of existence of the processes $\omega_t$, $\lambda_t$, $\nu_t$ defined by their flip rate just as here in Section 2. The ergodic theorems (Theorems 3.1, 3.2, 3.3, and Theorem 3.4 concerning the moments in equilibrium are proved in [12] under the hypothesis that $S$ is a countable Abelian group, and $P$ an irreducible, group invariant transition function, not necessarily symmetric.

In the theorems below, a state is called integrable if its coordinate random variables have finite expectation. It is called ergodic, if all events invariant under group translation have probability zero and one. Convergence of states to a limiting state will be meant in the sense of convergence of finite dimensional distributions.

THEOREM 3.1. *The extremal integrable translation invariant equilibrium states of $\omega_t$ are the states $\mu_m$ concentrated on $\{\sigma: \sigma(x) \equiv m\}$, $-\infty < m < \infty$. If $\omega_0$ is an ergodic translation invariant state with mean m, then $\omega_t$ converges to $\mu_m$.*

THEOREM 3.2. *The extremal integrable translation invariant equilibrium states of $\lambda_t$ form a one parameter family $\tilde{\mu}_m$, $-\infty < m < \infty$. $\tilde{\mu}_m$ is obtained from $\tilde{\mu}_1$ by the obvious change of scale. If $\lambda_0$ is an ergodic translation invariant state with mean m, then $\lambda_t$ converges to $\tilde{\mu}_m$.*

THEOREM 3.3. *The extremal integrable translation invariant equilibrium states of $\nu_t$ form a one-parameter family $\hat{\mu}_\rho$, $0 \le \rho < \infty$. If $\nu_0$ is an ergodic translation invariant state on $\mathbb{N}^S$ with density $\rho$, then $\nu_t$ converges to $\hat{\mu}_\rho$.*

THEOREM 3.4. *The moments of order one and two of the equilibrium states $\tilde{\mu}_m$ and $\hat{\mu}_\rho$ in the last two theorems are exactly those listed in Theorem 2.4, provided $P(x, y) = P(y, x)$, $x, y \in S$. For the nonsymmetric case they are given in* [12].

**4. Another new class of evolutions on $\mathbb{R}^S$ and $\mathbb{N}^S$.** We shall now define another triple of time evolutions, denoted $\rho_t$, $\eta_t$, $\gamma_t$. The order is important, for when we compare the ordered triple $\rho_t$, $\eta_t$, $\gamma_t$ to the ordered triple $\omega_t$, $\lambda_t$, $\nu_t$ of the past two sections, we shall find remarkable similarities. Here are a few: both $\rho_t$ and $\omega_t$ are in equilibrium when their coordinate variables are all the same; both $\Sigma\, \rho_t(x)$ and $\Sigma\, \omega_t(x)$ are martingales (when the invariant measure is constant); both $\eta_t$ and $\lambda_t$ preserve mass; the duality relation between $\lambda_t$ and $\omega_t$ is exactly the same as that between $\eta_t$ and $\rho_t$; finally both $\nu_t$ and $\gamma_t$ take place on $\mathbb{N}^S$ and are particle jump processes. The duality between $\gamma_t$ and $\rho_t$ will be exactly the same as between $\nu_t$ and $\omega_t$.

The definitions of the three processes will have the same common ingredients:

(a) a set of independent exponential mean one clocks, one at each site $x \in S$;

(b) an irreducible transition function $P(x, y)$ on $S$;

(c) a parameter $p$, $0 \le p \le 1$;

(d) the following procedure: when the random clock rings at a site $x \in S$, then another site $y$ is instantly chosen according to the probabilities $P(x, y)$, $y \in S$. At that instant

1: in the process $\rho_t$, the *generalized voter model*, the configuration $\rho$ changes into the configuration $\rho^{xy}$ defined by

$$\rho^{xy}(z) = \begin{cases} \rho(z) & \text{if } z \neq x \\ p\rho(x) + (1-p)\rho(y) & \text{if } z = x. \end{cases} \tag{4.1}$$

2: in the process $\eta_t$, called *the coalescing process*, the configuration $\eta$ changes to $\eta^{xy}$, defined by

$$\eta^{xy}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ p\eta(x) & \text{if } z = x \\ \eta(y) + (1-p)\eta(x) & \text{if } z = y. \end{cases} \tag{4.2}$$

3: in the process $\gamma_t$, called the *strongly coupled random walk*, each of the $\gamma(x)$ particles present at $x$ tosses a coin with probability $p$ of heads. If heads occurs then the particle remains at the site $x$. Otherwise it jumps to the site $y$.

It is not difficult to verify from these definitions that *we get exactly the duality and martingale properties stated in* Propositions 2.1, 2.2, and 2.3; we just have to replace $\omega_t$ by $\rho_t$, $\lambda_t$ by $\eta_t$ and $\nu_t$ by $\gamma_t$.

The ergodic theory also offers no surprises in the case when $0 < p < 1$. Let us therefore first discuss the degenerate cases $p = 1$ and $p = 0$.

*Case* $p = 1$. This case is trivial, $\rho_t$, $\eta_t$ and $\gamma_t$ are then constant (independent of $t$).

*Case* $p = 0$. In this case $\rho_t$ is a genuine voter model, and $\eta_t$, $\nu_t$ may be thought of as coalescing random walks. When $|S| < \infty$, then $\rho_t$ will converge almost surely to a state with $\rho(x)$ random but independent of $x$. On the other hand, $\gamma_t$ and $\eta_t$ converge to states where all the mass is located at one single point which moves around and is distributed according to the invariant measure $\pi$ of $P$. When $|S| = \infty$, even when $S = \mathbb{Z}_d$ there are many interesting open problems.

*Example.* Let $S = \mathbb{Z}_d$, $P(x, y) = 1/2d$ when $|x - y| = 1$, 0 otherwise. Let $p = 0$ and consider the processes $\eta_t$ and $\rho_t$ with initial configurations $\eta_0(x) \equiv 1$ and $\rho_0(x) = \delta(0, x)$, $x \in S$. Then $\eta_t$ is a coalescing random walk starting with a particle at each site, while $\rho_t$ is a voter model (0, 1 valued) starting with a single 1 (yes vote) at the origin. By duality we have

$$E \exp[i\alpha \textstyle\sum_x \rho_t(x)] = E \exp[i\alpha\eta_t(0)]. \tag{4.3}$$

What is the asymptotic behavior as $t \to \infty$ of the probability

$$p(t) = P[\eta_t(0) > 0]?$$

By the above duality

$$p(t) = P[\textstyle\sum_x \rho_t(x) > 0],$$

i.e., $p(t)$ is the probability that some one still votes yes at time $t$. When $d = 1$ the set of sites $x$ where $\rho_t(x) = 1$ must form an interval which can only grow or decrease by one point at a time with equal chance per unit time. Thus $p(t)$ is the probability that a symmetric continuous time random walk starting at +1 has not yet visited 0 by time $t$, which gives

$$p(t) \sim (\pi t)^{-1/2} \quad \text{as} \quad t \to \infty. \tag{4.4}$$

The far more difficult solution to this problem in dimension $d \geq 2$ has been obtained by Bramson and Griffeath [3], who show that

$$p(t) \sim \begin{cases} \dfrac{\log t}{\pi t} & \text{if } d = 2 \\ c_d \cdot \dfrac{1}{t} & \text{if } d \geq 3. \end{cases} \tag{4.5}$$

*Case* $0 < p < 1$. In this case we obtain *exactly the same ergodic theory* for $\rho_t$, $\eta_t$, $\gamma_t$ as for $\omega_t$, $\lambda_t$, $\nu_t$. Thus *Theorems* 2.1, 2.2, 2.3 *hold with* $\omega_t$, $\lambda_t$, $\nu_t$ *replaced by* $\rho_t$, $\eta_t$, $\gamma_t$, which is elementary, and *so do Theorems* 3.1, 3.2, and 3.3, which will be discussed in [12]. These last three theorems depend on moment calculations like those in Theorems 2.4 and 3.4. The results of these are, however *not quite the same, but extremely interesting*. Here is the main result from [12].

THEOREM 4.1. *Let $S$ be an infinite Abelian group, $P(x, y) = P(0, y - x)$, $0 < p < 1$. Let*

*$\eta_\infty(\cdot)$ be the coordinates of the equilibrium state approached by $\eta_t$ starting with $\eta_0 \equiv 1$. Let $\gamma_\infty$ be the equilibrium state approached by $\gamma_t$ when the $\gamma_0(x)$ are independent Poisson random variables with mean $\rho$. Then*

$$E\ \eta_\infty(x) \equiv 1.$$

$$E[\eta_\infty(x)\eta_\infty(y)] = \begin{cases} \frac{1}{p} & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases} \tag{4.6}$$

$$E\ \gamma_\infty(x) \equiv \rho$$

$$E[\gamma_\infty(x)\gamma_\infty(y)] = \begin{cases} \rho + \frac{\rho^2}{p} & \text{if } x = y \\ \rho^2 & \text{if } x \neq y. \end{cases} \tag{4.7}$$

REMARKS. 1. These moments are completely *independent of the choice of transition function $P(x, y)$.*

2. The coordinate variables in equilibrium are all *uncorrelated.* It can be shown, however, that they are *not independent.*

PARTS OF A PROOF FOR $\eta_t$. Let

$$\psi_t(x) = E[\eta_t(0)\eta_t(x)],\ \tilde{P}(x, y) = \frac{1}{2}[P(x, y) + P(y, x)].$$

By careful use of the generator, just as the diffusion equation (1.9) was derived in Section 1, one obtains

$$\frac{\partial\psi_t(x)}{\partial t} = 2(1 - p)(\tilde{P} - I)\psi_t(x) + u_t(x), \tag{4.8}$$

where

$$u_t(x) = 2(1 - p)^2[\delta(0, x) - \tilde{P}(0, x)]\psi_t(0). \tag{4.9}$$

Now it is possible to solve (4.8) by a variant of the Feynman-Kac formula: if $\psi_t$ is a bounded solution of

$$\frac{\partial\psi_t(x)}{\partial t} = \mathscr{L}\psi_t(x) + u_t(x)$$

with initial condition $\psi_0(x) = g(x)$, then

$$\psi_t(x) = E^x[g(x_t)] + E^x\int_0^t u_{t-s}(x_s)\, ds, \tag{4.10}$$

where $E^x[\cdot]$ is the expectation for the Markov process $x_t$ which has the generator $\mathscr{L}$. In our case the initial condition is $g(x) \equiv 1$. Therefore (4.10) applied to (4.8) gives

$$\begin{aligned} \psi_t(x) &= 1 + E^x\int_0^t u_{t-s}(x_s)\, ds \\ &= 1 + \int_0^t \textstyle\sum_{y\in S} e^{2(1-p)s(\tilde{P}-I)}(x, y)u_{t-s}(y)\, ds \\ &= 1 + \int_0^t \psi_{t-s}(0)h_s(x)\, ds, \end{aligned} \tag{4.11}$$

where

$$h_s(x) = 2(1-p)^2 e^{2(1-p)s(\tilde{P}-I)}(I-\tilde{P})(0,x). \tag{4.12}$$

Next one proves (using the fact that $|S| = \infty$)

$$\int_0^\infty h_s(x)\,ds = (1-p)\delta(x,0), \qquad x \in S. \tag{4.13}$$

Now setting $x = 0$ in the integral equation (4.11) gives

$$\psi_t(0) = 1 + \int_0^t \psi_{t-s}(0)h_s(0)\,ds. \tag{4.14}$$

This is a renewal equation. In view of (4.13) it has the solution

$$\psi_t(0) = 1(t) + 1{*}f(t) + 1{*}f{*}f(t) + \cdots,$$

where $f(t) = h_t(0)$. Hence, letting $t \nearrow \infty$ and using (4.13) with $x = 0$,

$$\lim_{t\nearrow\infty} \psi_t(0) = 1 + (1-p) + (1-p)^2 + \cdots = \frac{1}{1-(1-p)} = \frac{1}{p}. \tag{4.15}$$

This almost proves the first part of (4.6), but not quite since we have not shown that

$$E[\eta_\infty^2(x)] = \lim_{t\nearrow\infty} E[\eta_t^2(x)].$$

This requires coupling and truncation arguments to be given in [12]. Similarly, going back to (4.11) one obtains

$$\begin{aligned} \lim_{t\nearrow\infty} E[\eta_t(x)\eta_t(0)] &= \lim_{t\nearrow\infty} \psi_t(x) \\ &= 1 + \lim_{t\nearrow\infty} \int_0^t \psi_{t-s}(0)h_s(x)\,ds \\ &= 1 + \lim_{t\nearrow\infty} \psi_t(0) \int_0^\infty h_s(x)\,ds = 1, \end{aligned} \tag{4.16}$$

by use of (4.13) with $x \neq 0$.

*Another heuristic proof of Theorem* 4.1 can be given along the lines of the proof of Theorem 2.4. Let $S$ be a *finite* Abelian group, and consider the process $\gamma_t$ with parameter $0 < p < 1$ and group invariant transition function. Start $\gamma_t$ off with two particles. Then it is easy to verify the surprising result that, in equilibrium,

$$P[\gamma_\infty(x) = 2] = \frac{1}{|S|\,[1-p+p\,|S|]}, \qquad x \in S \tag{4.17}$$

while

$$P[\gamma_\infty(x) = \gamma_\infty(y) = 1] = \frac{p}{1-p+p\,|S|}\cdot\frac{2}{|S|}, \qquad x \neq y,\ x, y \in S. \tag{4.18}$$

These are the equations analogous to (2.16) and (2.17). Then, since $\gamma_t$ is in the same relation of duality to $\eta_t$ as $\nu_t$ is to $\lambda_t$, we can copy the proof of Theorem 2.4 verbatim to conclude that

$$E^{\eta_0\equiv 1}[\eta_\infty^2(x)] = \frac{|S|}{1-p+p\,|S|}, \qquad x \in S \tag{4.19}$$

$$E^{\eta_0 \equiv 1}[\eta_\infty(x)\eta_\infty(y) = \frac{p|S|}{1-p+p|S|}, \qquad x \neq y,\ x, y \in S. \tag{4.20}$$

It remains only to let $|S| \to \infty$ to obtain (formally) the result of (4.6).

*Example.* Let $S = \mathbb{Z}, p = \frac{1}{2}$, $P(x, x-1) = 1$ while $P(x, y) = 0$ in all other cases. Let $\eta_0(\cdot)$ be ergodic and translation invariant with mean 1 (e.g., $\eta_0(x) \equiv 1$, or $\eta_0(x)$ independent, exponential with mean 1). This process can be reinterpreted as a model for the time evolution of a point process (or of vehicle traffic to be concrete): we take $\eta_t(k)$, $k \in \mathbb{Z}$ to be the spacings between successive points on $\mathbb{Z}$; then the evolution of $\eta_t$ is exactly the same as if each point (vehicle) has a random exponential clock such that when it rings, *the vehicle jumps to the midpoint of the vacant interval in front* of it. Our results imply that such a point process converges to an equilibrium state in which the spacings are *uncorrelated* random variables with mean 1 and (by (4.6)) variance 1, but are *not independent.*

Note added in proof. This example is contained in a recent study of the process $\nu_t$ by Roussignol [14].

## REFERENCES


[1] Aizenman, M. (1979). Translation invariance and instability of phase coexistence in the two dimensional Ising system. *Phys. Rev. Lett.*, **43**, 407–409.

[2] van Beijeren, H. (1975). Interface sharpness in the Ising system. *Comm. Math. Phys.* **40** 1–6.

[3] Bramson, M. and D. Griffeath (1980). Asymptotics for interacting particle systems on $\mathbb{Z}^d$. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **53** 183–196.

[4] Feller, W. (1968). *Introduction to Probability Theory.* Vol. 1. 3rd ed. *Wiley*, New York.

[5] Glauber, R. J. (1963). Time dependent statistics of the Ising model. *J. Math. Phys.* 4 294–307.

[6] Higuchi, Y. (1979). On the absence of non-translationally invariant Gibbs states for the two dimensional Ising model. *Proc. Int. Coll. Random Fields, Esztergom* 1979. To appear in 1981.

[7] Holley, R. and T. Liggett (1975). Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probability* **3** 643–663.

[8] Holley, R. and D. Stroock (1977). In one and two dimension, every stationary measure for a stochastic Ising model is a Gibbs state. *Comm. Math. Phys.* **55** 37–45.

[9] Holley, R. and D. Stroock (1979). Dual processes and their application to infinite interacting systems. *Advances in Math.* **32** 149–174.

[10] Liggett, T. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.*, **165** 471–481.

[11] Liggett, T. (1977). The stochastic evolution of infinite systems of interacting particles. *Lecture Notes in Math.* **598** 188–248. Springer, Berlin.

[12] Liggett, T. and F. Spitzer (1981). Ergodic theorems for coupled random walks and other systems with locally interacting components. To appear in *Z. Wahrscheinlichkeitstheorie und verw. Gebiete.*

[13] Onsager, L. (1944). Crystal Statistics I, A two dimensional model with an order-disorder transition. *Phys. Rev.*, **65** 117–149.

[14] Roussignol, M. (1980). Un processus de saut sur $\mathbb{R}$ a une infinité de particules. *Ann. Inst. Henri Poincaré*, **16** 101–108.



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