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A NOTE ON REGULAR CONDITIONAL PROBABILITIES IN DOOB'S SENSE

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It is pointed out that an example due to J. Pachl shows the nonexistence of regular conditional probabilities in Doob's sense when the underlying space is a perfect probability space. This answers a question of Sazonov.

Let $(X, \mathscr{A}, P)$ be a probability space and let $\mathscr{A}_1, \mathscr{A}_2$ be two sub σ-algebras of $\mathscr{A}$. A regular conditional probability (r.c.p. for short) on $\mathscr{A}_1$ given $\mathscr{A}_2$ is a function on $X \times \mathscr{A}_1$ satisfying the following properties:

(CP1) $Q(x, \cdot)$ is a probability on $\mathscr{A}_1$ for every x in X,
(CP2) $Q(\cdot, A_1)$ is $\mathscr{A}_2$-measurable for every $A_1 \in \mathscr{A}_1$ and
(CP3) $\int_{A_2} Q(\cdot, A_1) dP = P(A_1 \cap A_2)$ for every $A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2$.

If $Q(x, A_1)$ satisfies (CP1) and (CP3) and

(CP2*) $Q(\cdot, A_1)$ is $\mathscr{A}_2$-measurable a.s. $[P|_{\mathscr{A}_2}]$ for every $A_1 \in \mathscr{A}_1$

then $Q(x, A_1)$ is called an r.c.p. in Doob's sense given $\mathscr{A}_2$ (see [1], page 26).

The existence of r.c.p.'s has useful consequences, especially in perfect probability spaces (see [4], [5]). For definition and properties of perfect measures and r.c.p.'s we refer the reader to [3] and [6]. It is known that if $(X, \mathscr{A}, P)$ is a perfect probability space and if $\mathscr{A}_1$ is a countably generated sub σ-algebra of $\mathscr{A}$, then for any sub σ-algebra $\mathscr{A}_2$ of $\mathscr{A}$, an r.c.p. $Q(x, A_1)$ given $\mathscr{A}_2$ exists such that $Q(x, \cdot)$ is perfect for every x in X (Theorem 7 in [6]). That an r.c.p. need not exist in a perfect probability space when $\mathscr{A}_1$ is not countably generated can be seen by considering $(I, \mathscr{L}, \lambda)$ where I = the unit interval, $\mathscr{L}$ = Lebesgue σ-algebra, λ = Lebesgue measure and taking $\mathscr{A}_1 = \mathscr{L}$ and $\mathscr{A}_2$ = Borel σ-algebra on I. But in this case $Q(x, L) = 1_L(x)$ is an r.c.p. in Doob's sense such that $Q(x, \cdot)$ is perfect for every x in I. In [6] Sazonov states that he does not know of an example of a perfect probability space $(X, \mathscr{A}, P)$ and two sub σ-algebras $\mathscr{A}_1, \mathscr{A}_2$ of $\mathscr{A}$ such that there does not exist on $\mathscr{A}_1$ an r.c.p. in Doob's sense given $\mathscr{A}_2$ (see Q3 in [3] and page 242 of [6]). In this note we point out that an example constructed by J. Pachl [2] to show the nonexistence of disintegrations of a certain type also provides an example of the kind needed by Sazonov. Further, when an r.c.p. $Q(x, A)$ exists it is useful to know whether $Q(x, \cdot)$ can be chosen to be perfect for every x in X (see, for instance, Theorem 2 of [5]). We use Pachl's example to show the existence of a perfect probability space $(X, \mathscr{A}, P)$ and a sub σ-algebra $\mathscr{A}_2$ of $\mathscr{A}$ such that

(a) Under the continuum hypothesis (CH) there does not exist an r.c.p. on $\mathscr{A}$ in Doob's sense given $\mathscr{A}_2$, and

(b) there does not exist an r.c.p. $Q(x, A)$ on $\mathscr{A}$ in Doob's sense given $\mathscr{A}_2$ such that $Q(x, \cdot)$ is perfect for every x in X.

EXAMPLE. Let I = the unit interval, $\mathscr{L}$ = Lebesgue σ-algebra on I, λ = Lebesgue measure on $\mathscr{L}$, $\mathscr{M}$ Lebesgue = σ-algebra on $I \times I$ and let μ = Lebesgue measure on $\mathscr{M}$. Set $X = I \times I \times I = \{(x_1, x_2, x_3): x_i \in I \text{ for } i = 1, 2, 3\}$, $\mathscr{A} = \mathscr{M} \otimes \mathscr{L}$ and define P on $\mathscr{A}$ by

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$$P(A) = \mu(\{(x_1, x_2): (x_1, x_2, x_3) \in A\}), \qquad A \in \mathscr{A}.$$

Then P is a probability on $\mathscr{A}$ with marginals μ and λ on $\mathscr{M}$ and $\mathscr{L}$ respectively which are perfect. Hence $(X, \mathscr{A}, P)$ is a perfect probability space (Theorem 6 in [6]). Let $\mathscr{A}_2 = I \times I \otimes \mathscr{L} =$ cylinder sets with base in $\mathscr{L}$.

Suppose there exists an r.c.p. $Q((x_1, x_2, x_3), A)$ on $\mathscr{A}$ in Doob's sense given $\mathscr{A}_2$. Let $\mathscr{B}$ = Borel σ-algebra on $I \times I$. It is routine to check that

$$Q'((x_1, x_2, x_3), B \times I) = \lambda(B_{x_3}), \qquad B \in \mathscr{B}$$

defines an r.c.p. on $\mathscr{B} \otimes I$ given $\mathscr{A}_2$. Since $\mathscr{B} \otimes I$ is countably generated and since both $Q((x_1, x_2, x_3), B \times I)$ and $Q'((x_1, x_2, x_3), B \times I)$ are two versions of r.c.p. on $\mathscr{B} \otimes I$ in Doob's sense given $\mathscr{A}_2$, by standard techniques, we can find a set $N \in \mathscr{A}_2$ with $P(N) = 0$ such that for all $(x_1, x_2, x_3) \notin N$ and for all $B \in \mathscr{B}$,

$$Q((x_1, x_2, x_3), B \times I) = Q'((x_1, x_2, x_3), B \times I) \cdots (*)$$

Fix $(x_1^*, x_2^*, x_3^*) \notin N$. For any subset C of I define

$$Q_1(C) = Q((x_1^*, x_2^*, x_3^*), C \times \{x_3^*\} \times I)$$

In view of (*), Q_1 defines a continuous probability measure on the class of all subsets of I which is impossible under CH (see [7], page 107) and hence (a) follows. If we assume that $Q((x_1, x_2, x_3), \cdot)$ is perfect for every (x_1, x_2, x_3) in X then Q_1 defined above is a continuous perfect measure (since $Q((x_1^*, x_2^*, x_3^*\}, I \times \{x_3^*\} \times I) = 1$) on the class of all subsets of I which is impossible by a result of Sazonov (see Lemma 4, in [6]) proving (b).

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