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CONDITIONAL DISTRIBUTIONS AND ORTHOGONAL MEASURES

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It is shown that every family of mutually singular measures in a conditional probability distribution is countable or else there is a perfect set of measures which form a strongly orthogonal family. THEOREM: *Let X and Y be complete separable metric spaces and μ a conditional probability distribution on $X \times \mathscr{B}(Y)$. Then either* (1) *there is a nonempty compact perfect subset P of X and a Borel subset D of $X \times Y$ so that if x and y are distinct elements of P, then $\mu(x, D_x) = 1$, $\mu(y, D_x) = 0$, and $D_x \cap D_y = \phi$ or else* (2) *if K is a subset of X so that $\{\mu(x, \cdot): x \in K\}$ is a pairwise orthogonal family, then K is countable.*

I. Introduction. A family $\mathscr{M}$ of measures, defined on a Borel field $\mathscr{B}$ of subsets of a space X, is said to be pairwise orthogonal (mutually singular) if, given $\lambda, \mu \in \mathscr{M}$ with $\lambda \neq \mu$, there exist $H_{\lambda\mu} \in \mathscr{B}$ such that $\lambda(H_{\lambda\mu}) = 0 = \mu(X - H_{\lambda\mu})$. Such a family will be called uniformly orthogonal provided there is, for each $\lambda \in \mathscr{M}$, a set $H_\lambda \in \mathscr{B}$ such that, for each $\mu \in \mathscr{M} - \{\lambda\}$, $\mu(H_\lambda) = 0 = \lambda(X - H_\lambda)$. In [4], D. Maharam proved the following theorem (assuming the continuum hypothesis holds):

THEOREM. (*CH*) *There exists an uncountable family $\mathscr{M}$ of pairwise orthogonal Borel probability measures on the unit square I^2, such that no uncountable subset of $\mathscr{M}$ is uniformly orthogonal.*

Maharam has raised the following question:

Is there a function μ from $I \times \mathscr{B}(I^2)$ so that (1) for each x in I, $\mu(x, \cdot)$ is a probability measure on $\mathscr{B}(I^2)$, (2) for each $E \in \mathscr{B}(I^2)$, $\mu(\cdot, E)$ is Borel measurable, (3) if $x, y \in I$ and $x \neq y$, then $\mu(x, \cdot)$ and $\mu(y, \cdot)$ are pairwise orthogonal and (4) no uncountable subset of $\mathscr{M} = \{\mu(x, \cdot): x \in I\}$ is uniformly orthogonal?

In this note we give a negative solution to this problem in Theorem 1. We then strengthen Theorem 1 in Theorem 4 by demonstrating that the property of being a family of mutually singular measures in a conditional distribution is "Cantorian." However, a statistically nonnegligible version of our theorem stated later as a problem is left unsolved.

Our setting will be a follows.

Let X and Y be complete separable metric spaces. Let $\mathscr{B}(Y)$ denote the σ-algebra of Borel subsets of Y. Let μ be a conditional probability distribution on $X \times \mathscr{B}(Y)$. This means that for each x in X, $\mu(x, \cdot)$ is a probability measure on $\mathscr{B}(Y)$ and for each $E \in \mathscr{B}(Y)$, $\mu(\cdot, E)$ is Borel measurable with respect to $\mathscr{B}(X)$. If $E \subset X \times Y$ and $x \in X$, then $E_x = \{y: (x, y) \in E\}$. The set of all positive integers will be denoted by N. The set of all finite sequences of positive integers will be denoted by Seq. By $\{0, 1\}^*$ is meant the set of all finite sequences of zeros and ones. We set $J = N^N$ and if $s = (s_1, \cdots, s_n) \in$ Seq, then $J(s) = \{\sigma \in J: \sigma | n = s\}$.

We shall strengthen Theorem 1 by demonstrating that the property of being a family of mutually singular measures in a conditional distribution is "Cantorian" in the following sense:

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THEOREM 4. *Let μ be a conditional probability distribution on $X \times \mathscr{B}(Y)$. Then either* (1) *there is a nonempty compact perfect subset P of X and a Borel subset D of $X \times Y$ so that if x and y are distinct elements of P, then $\mu(x, D_x) = 1$, $\mu(y, D_x) = 0$ and $D_x \cap D_y = \phi$ or else* (2) *if K is a subset of X so that $\{\mu(x, \cdot): x \in K\}$ is a pairwise orthogonal family, then K is countable.*

II. Results. We now turn to the theorems and proofs.

THEOREM 1. *Let μ be a conditional probability distribution on $X \times \mathscr{B}(Y)$ such that if x and y are distinct points of X, then $\mu(x, \cdot)$ and $\mu(y, \cdot)$ are orthogonal. Then there is an F_σ subset B of $X \times Y$ such that $T = \pi_X(B)$ is a homeomorph of the Cantor set and if x and y are distinct elements of T, then $\mu(y, B_x) = 0$, $\mu(x, B_x) = 1$ and $B_x \cap B_y = \phi$.*

PROOF. Let $\{U(n)\}_{n=1}^{\infty}$ be a base for the topology of Y. For each $s = \langle s_1, \cdots, s_n \rangle \in$ Seq, let $V(s) = \cup\{U(s_i): i = 1, \cdots, n\}$. For each s, $g_s = \mu(\cdot, V(s))$ and $f_s = \mu(\cdot, \overline{V(s)})$ are Borel measurable maps from X into $[0, 1]$. Let P be a nonempty compact perfect subset of X so that $g_s | P$ and $f_s | P$ are continuous for all s.

We suspend the proof of the theorem in order to prove the following lemma.

LEMMA. *For each $e = \langle e_1, \cdots, e_k \rangle \in \{0, 1\}^*$ there is a nonempty open subset $T(e)$ of P and a finite sequence, $\tau(e)$, of positive integers such that if e, e' are distinct elements of $\{0, 1\}^n$, then*

1) *if $x \in \overline{T(e)}$, $\mu(x, V(\tau(e))) > 1 - 2^{-n}$*
2) $\overline{T(e)} \cap \overline{T(e')} = \phi, = \overline{V(\tau(e))} \cap \overline{V(\tau(e'))}$
3) *if $y \in T(e')$, then $\mu(y, \overline{V(\tau(e))}) < 2^{-n}$*
4) $\overline{T(e*i)} \subset T(e)$, *for $i = 0, 1$.*
5) diam $(T(e)) \leq 2^{-n}$

PROOF OF LEMMA. Let $x_0 = x(\langle 0 \rangle)$ and $x_1 = x(\langle 1 \rangle)$ be distinct elements of P. Since $\mu(x_0, \cdot)$ and $\mu(x_1, \cdot)$ are mutually singular, there are pairwise disjoint compact sets K_0 and K_1 so that $\mu(x_0, K_0) > ½$, $\mu(x_1, K_1) > ½$, $\mu(x_0, K_1) < ½$, $\mu(x_1, K_0) < ½$. Let $\tau(\langle 0 \rangle)$ and $\tau(\langle 1;8)$ be finite sequences of positive integers such that $\overline{V(\tau\langle 0 \rangle)} \cap \overline{V(\tau\langle 1 \rangle)} = \phi$, $V(\tau\langle i \rangle) \supset K_i$, and $\mu(x_i, \overline{V(\tau\langle i' \rangle)}) < 2^{-1}$, where $i' = i - 1 \pmod 2$. Let $T(i)$ be open sets relative to P such that for $i = 0, 1$:

1) $x(\langle i \rangle) \in T(\langle i \rangle)$
2) diam $T(\langle i \rangle) < 2^{-1}$
3) $\overline{T(\langle 0 \rangle)} \cap \overline{T(\langle 1 \rangle)} = \phi$
4) if $x \in T(\langle i \rangle)$, $\mu(x, V(\tau(\langle i \rangle))) > 2^{-1}$ and $\mu(x, \overline{V(\tau\langle i' \rangle)}) < 2^{-1}$.

We continue the induction process one more level for illustrative purposes. Choose distinct elements $x(\langle 0, 0 \rangle)$ and $x(\langle 0, 1 \rangle)$ of $T(\langle 0 \rangle)$ and distinct elements $x(\langle 1, 0 \rangle)$ and $x(\langle 1, 1 \rangle)$ of $T(\langle 1 \rangle)$. Choose pairwise disjoint compact sets $K(e)$ containing $x(e)$, for $e \in \{0, 1\}^2$ so that $\mu(x(e), K(e)) > 1 - 2^{-2}$ and if $e \neq e'$, $\mu(x(e), K(e')) < 2^{-2}$. Next, let $\tau(e)$, for $e \in \{0, 1\}^2$ be such that $V(\tau(e)) \supset K(e)$ and if $e' \neq e$, $\mu(x(e'), \overline{V(\tau(e))}) < 2^{-2}$ and $\overline{V(\tau(e))} \cap \overline{V(\tau(e'))} = \phi$. Let $T(e)$, for $e \in \{0, 1\}^2$, be open sets relative to P such that

1) $x(e) \in T(e) \subseteq \overline{T(e)} \subset T(e \mid 1)$
2) diam $(T(e)) < 2^{-2}$
3) $\overline{T(e)} \cap \overline{T(e')} = \phi$, if $e \neq e'$
4) if $x \in T(e)$ and $e \neq e'$, $\mu(x, V(\tau(e))) > 1 - 2^{-2}$ and $\mu(x, \overline{V(\tau(e'))}) < 2^{-2}$.

We now complete the proof of the theorem.

For each n, let $B_n = \cup \{T(e) \times V(\tau(e)) : e \in \{0, 1\}^n\}$. Let

$$B = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} B_m.$$

Clearly, B is an F_σ subset of $X \times Y$ and $\pi_X(B) = T = \cap_{k=1}^{\infty} \cup\{\tau(e) : e \in \{0, 1\}^n\}$.

If $x \in T$, then there is a unique sequence $\kappa_x \in \{0, 1\}^N$ so that for each n, $x \in T(\kappa_X | n)$. This of course defines a homeomorphism of $\{0, 1\}^N$ onto T. Also, for each positive integer p, $\mu(x, B_{px}) = \mu(x, V(\tau(\kappa_x | p))) \geq \mu(x, V(\tau(\kappa_x | p))) > 1 - 2^{-p}$. So, for each n,

$$\mu(x, (\cap_{m=n}^{\infty} B_m)_x) \geq 1 - 2^{-(n-1)}.$$

Thus,

$$\mu(x, B_x) = 1, \quad \text{for each} \quad x \in T.$$

If x and y are distinct elements of T, then there is a positive integer k so that if $p > k$, $\kappa_x | p \neq \kappa_y | p$. This means $\mu(y, B_{px}) < 2^{-p}$ and $B_{px} \cap B_{py} = \phi$, for all $p > k$. It follows from this that $B_x \cap B_y = \phi$ and $\mu(y, B_x) = 0$. □

Let us note that in case $X = Y = I$ and λ is Lebesgue measure on I, the perfect set P given by the preceding theorem may be statistically negligible. We pose the following problem.

PROBLEM. Let ν be an atomless probability measure on $\mathscr{B}(I)$ and μ a conditional probability distribution on $I \times \mathscr{B}(I)$ of pairwise orthogonal measures. Is there a subset B of $I \times I$ and a perfect subset T of I satisfying the conclusion of our theorem so that $\nu(T) > 0$? *(Added in proof.* This problem has been solved negatively by R. J. Gardner [7].)

The goal of the remainder of this paper is to obtain a strengthening of Theorem 1, Theorem 4. Theorem 4 states that the property of mutually singular measures is "Cantorian." In order to prove this result, we must first determine the descriptive character of the set of mutually singular measure. This will lead to an apparently new result in descriptive set theory, Theorem 3.

First, let us make the following convention: if X is a set and $Z \subseteq X$, then $Z*Z = (Z \times Z) - \Delta$, where $\Delta = \{(x, x) : x \in X\}$.

THEOREM 2. *Let X be a compact metric space and S the space of all probability measures provided with the weak topology. Let $R = \{(\mu, \nu) \in S \times S$: μ and ν are mutually singular$\}$. Then R is a G_δ subset of $S*S$.*

PROOF. Let $\{U(n)\}_{n=1}^{\infty}$ be a base for the topology of X. For each $s = (s_1, \ldots, s_n) \in \text{Seq}$, let $V(s) = \cup \{U(s_i): i = 1, \ldots, n\}$. For each $(n, s) \in N \times \text{Seq}$, let $T(n, s) = \{\mu \in S: \mu(V(s)) > 1 - 2^{-n}\}$. Since each set $T(n, s)$ is open in S, it suffices to show that

$$(*) \qquad R = \cap_n \cup_{s,t \in \text{Seq}} \{T(n, s) \times T(n, t) : V(s) \cap V(t) = \phi\}.$$

If $(\mu, \nu) \in R$, then for each n, there are disjoint compact sets K and L such that $\mu(K) > 1 - 2^{-n}$ and $\nu(L) > 1 - 2^{-n}$.

Thus, there are elements s and t of Seq so that $K \subset V(s)$, $L \subset V(t)$ and $V(s) \cap V(t) = \phi$. This implies R is a subset of the right hand side of equation (*).

Suppose (μ, ν) is an element of the left hand side of (*). For each n, let s_n and t_n be elements of Seq so that $V(s_n) \cap V(t_n) = \phi$, $\mu(V(s_n)) > 1 - 2^{-n}$ and $\nu(V(t_n)) > 1 - 2^{-n}$. Let $A = \cap_n [\cup_{m \geq n} V(s_n)]$. It is easy to check that $\mu(A) = 1$ and $\nu(A) = 0$. □

We note that Theorem 2 remains true if X is only assumed to be a complete separable metric space (= Polish space).

Let us recall that a Suslin space is a topological space which is a continuous image of a G_δ subset of the Cantor set [6].

THEOREM 3. *Let X be a Suslin space and R a G_δ subset of $X*X$. If there is an uncountable subset Y of X so that $Y*Y \subset R$, then there is a nonempty compact perfect subset Q of X so that $Q*Q \subset R$.*

We shall give two proofs for this theorem. First, we reduce the problem.

Let G be a G_δ subset of 2^N and f a continuous map of G onto X. Let $S = (f \times f)^{-1}(R)$. Then S is a G_δ subset of $2^N \times 2^N$ and clearly there is an uncountable subset Z of 2^N so that $Z*Z \subset S$. Notice that is now suffices to demonstrate the existence of a nonempty compact perfect subset P of G so that $P*P \subset S$, since $Q = f(P)$ satisfies the conclusion of the theorem (f is one-to-one on Q).

PROOF. I Let $A = \{M \subseteq 2^N : M \text{ is closed}, M \subseteq G \text{ and } M*M \subset S\}$. It is easy to check that A is an analytic subset of the space of closed subsets of 2^N. For each countable ordinal α, we can find a compact set M lying in Z whose Cantor-Bendixson derived set order is at least α. Thus, some closed uncountable (and therefore some perfect) subset of 2^N must be an element of A [3]. □

PROOF II. Fix a complete metric p on 2^N so that the diameter of $2^N = 1$. Let $\mathscr{U}$ be a countable field of clopen sets forming a basis for 2^N. For nonempty $U, V, W \in \mathscr{U}$, let $(V, W) < U$ denote that: (i) $V \cap W = \phi$; (ii) $V \cup W \subseteq U$; and (iii) $\max(p\text{-diam } V, p\text{-diam } W) < \frac{1}{2}(p\text{-diam } U)$. Represent $S = \cap S_n$ with each S_n open. Discarding a countable set if need be, Z may be assumed dense-in-itself. Pick distinct $z_0, z_1 \in Z$. We can find $(U_0, U_1) < 2^N$ with $z_i \in U_i$ and $U_i \times U_j \subseteq S_1$ for $0 \le i, j \le 1$ with $i \ne j$. Pick distinct $z_{i0}, z_{i1} \in Z \cap U_i$. We can find $(U_{i0}, U_{i1}) < U_i$ with $z_{ij} \in U_{ij}$ and $U_{ij} \times U_{k\ell} \subseteq S_1 \cap S_2$ for $0 \le i, j, k, \ell \le 1$ with $(i, j) \ne (k, \ell)$. Iterate. It suffices to set $P = \cap_{m \in N} \cup_{s \in 2^m} U_s$. □

THEOREM 4. *Let μ be a conditional probability distribution on $X \times \mathscr{B}(Y)$. Then either* (1) *there is a nonempty compact perfect subset C of X and a Borel subset D of $X \times Y$ so that if x and y are distinct elements of C, then $\mu(x, D_x) = 1$, $\mu(y, D_x) = 0$ and $D_x \cap D_y = \phi$ or else* (2) *if K is a subset of X so that $\{\mu(x, \cdot) : x \in K\}$ is a pairwise orthogonal family, then K is countable.*

PROOF. First, note that Theorem 4 holds if either X or Y is countable. Second, if X and Y are both uncountable, then let f and g be Borel isomorphisms of 2^N onto X and Y respectively. Let $\nu(x, B) = \mu(f(x), g(B))$, for $(x, B) \in 2^N \times \mathscr{B}(2^N)$. Then ν is a conditional probability distribution on $2^N \times \mathscr{B}(2^N)$.

Let us assume (2) does not hold. Let $\{U_n\}_{n=1}^{\infty}$ be a countable field of compact open sets forming a basis for 2^N. Let $W = [0, 1]^N$. Define a Borel measurable map $d: 2^N \to W$ by letting $d(x) = (\nu(x, U_1), \nu(x, U_2), \cdots)$. Define a G_δ binary relation S on W by letting $(w_1, w_2) \in S$ if and only if $\forall\, k \in N\, \exists\, n \in N[w_1(n) > 1 - 2^{-k} \text{ and } w_2(n) < 2^{-k}]$. Notice that the measures $\nu(x, \cdot)$ and $\nu(y, \cdot)$ are orthogonal if and only if $(d(x), d(y)) \in S$.

Now, let $X = d(s^N)$, let $R = S \cap (X \times X)$ and $Y = d(H)$. Applying Theorem 3, we obtain a nonempty compact perfect set Q so that $Q*Q \subset R$. Let $B = d^{-1}(Q)$. Then B is an uncountable Borel subset of 2^N so that if x and y are elements of B, then either $\nu(x, \cdot)$ and $\nu(y, \cdot)$ are identical or orthogonal.

Now, let X_1 be the graph of d restricted to B so that $X_1 \subseteq 2^N \times W$. Let $((x, w), (y, z)) \in R_1 \subset X_1*X_1$ if and only if $w \ne z$. Let Y be a subset of B so that $d(z) = Q$ and d is one-to-one on Z. Applying Theorem 3, we obtain a perfect set $Q_1 \subseteq X$ so that $Q_1*Q_1 \subset R_1$. Let P be the projection of Q_1 into 2^N. Now, apply Theorem 1. □

Finally, we would like to point out that Theorem 3 is the best possible in the following sense.

EXAMPLE. There is an F_σ subset R of $2^\omega * 2^\omega$ and an uncountable subset Y of 2^ω so that $Y * Y \subset R$ and yet there is no nonempty perfect subset P of 2^ω so that $P * P \subset R$.

This may be seen as follows. Define the binary relation $\leq_T$ by $x \leq_T y$ provided "x is recursive in y." Abbreviate

$$\begin{array}{ll} x \equiv_T y & \text{for } x \leq_T y \text{ and } y \leq_T x \\ x <_T y & \text{for } x \leq_T y \text{ and not } y \leq_T x \\ x\, R_T\, y & \text{for } x \neq y \text{ and } (x \leq_T y \text{ or } y \leq_T x). \end{array}$$

It is known that R_T is an F_σ subset of $2^\omega \times 2^\omega$. It is also known that there is an uncountable sequence $Y = \{y_\alpha : \alpha < \omega_1\}$ such that $y_\alpha <_T y_\beta$ whenever $\alpha < \beta$. (This follows from the facts that (i) $\forall\, x\, \exists\, y[x <_T y]$; and (ii) $\forall$ sequence $\{x_n : n \in \omega\}\ \exists\, y\, \forall n\, [x_n <_T y]$.)

But there cannot exist a nonempty perfect set P such that $P * P \subset R$. For otherwise a theorem of Galvin [2, 3] would tell us that there is a nonempty perfect $Q \subset P$ for which one of the following holds (where $\leq$ is the lexicographic order on 2^ω):

$$\begin{array}{c} \text{(i) } \forall\, x, y \in Q[x \equiv_T y] \\ \text{(ii) } \forall\, x, y \in Q[x < y \to x <_T y] \\ \text{(iii) } \forall\, x, y \in Q[x < y \to y <_T x]. \end{array}$$

All three are impossible, since for any x, $\{y : y \leq_T x\}$ is countable.

A proof of Galvin's theorem is given in [1].

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