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GENERALIZED POISSON SHOCK MODELS[1]

BY SHELDON M. ROSS

University of California, Berkeley

Suppose that shocks hit a device in accordance with a nonhomogeneous Poisson process with intensity function $\lambda(t)$. The i^{th} shock has a value X_i attached to it. The X_i are assumed to be independent and identically distributed positive random variables, and are also assumed independent of the counting process of shocks. Let $D(x_1, \ldots, x_n, \underline{0}) \equiv D(x_1, \ldots, x_n, 0, 0, 0, \ldots)$ denote the total damage when n shocks having values $x_1, \ldots, x_n$ have occurred. It has previously been shown that the first time that D exceeds a critical threshold value is an increasing failure rate average random variable whenever (i) $\int_0^t \lambda(s)\, ds/t$ is nondecreasing in t and (ii) $D(\underline{x}) = \sum x_i$. We extend this result to the case where $D(\underline{x})$ is a symmetric, nondecreasing function. The extension is obtained by making use of a recent closure result for increasing failure rate average stochastic processes.

1. Model and Result. We consider a unit subject to shocks which occur in accordance with a nonhomogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$. We suppose that the ith shock has a random value X_i associated with it. The X_i, $i \geq 1$, are assumed to be independent positive random variables each having distribution F. They are also assumed to be independent of the counting process of shocks. We suppose that there is a function D such that if exactly n shocks having values $x_1, \ldots, x_n$ have occurred by time t, then $D(x_1, \ldots, x_n, \underline{0})$ represents the damage to the unit at time t, where D is a nonnegative function whose domain is $\{(x_1, x_2, \ldots), x_i \geq 0, i = 1, 2, \ldots\}$, and $\underline{0} = (0, 0, \ldots)$. We suppose that the unit will fail the first time its damage exceeds some constant C, and we let T denote the failure time. We have the following theorem.

THEOREM 1. *If*

(i) $T < \infty$ *with probability* 1,
(ii) $\int_0^t \lambda(s)\, ds/t$ *is nondecreasing in* t,
(iii) $D(x_1, \ldots, x_n, \underline{0}) = D(x_{i_1}, \ldots, x_{i_n}, \underline{0})$ *whenever* $(i_1, \ldots, i_n)$ *is a permutation of* 1, 2, ..., n, *for all* n,
(iv) D *is nondecreasing in each of its arguments,*

then T has an increasing failure rate average distribution.

Before proving the above theorem we need some preliminaries.

2. Preliminaries. We start with some definitions.

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DEFINITIONS.

(i) The nonnegative continuous random variable X having failure rate function $r(t) = \left(\frac{d}{dt} P\{X \le t\}\right) \Big/ P\{X > t\}$ is said to have an increasing failure rate average distribution if $\int_0^t r(s)\, ds/t$ is nondecreasing in t.

(ii) The nondecreasing real valued stochastic process $\{X(t),\ t \ge 0\}$ is said to be an increasing failure rate average (IFRA) stochastic process if T_a has an increasing failure rate average distribution for all a, where $T_a = \inf\{t{:}X(t) > a\}$.

For an example of an IFRA stochastic process, let $\{N(t), t \ge 0\}$ be a nonhomogeneous Poisson process with intensity function $\lambda(t)$ where $\int_0^t \lambda(s)\, ds/t$ is assumed to be nondecreasing in t. Further, suppose that there is a value X_i associated with the i^{th} event. The X_i, $i \ge 1$, are assumed to be independent random variables each having the same distribution H, and they are also assumed to be independent of $\{N(t), t \ge 0\}$. Define $X(t)$ by

$$X(t) = \begin{cases} \max(X_1, \ldots, X_{N(t)}) \text{ if } N(t) \ge 1 \\ 0 \text{ if } N(t) = 0. \end{cases}$$

Then it is easy to see that the failure rate function for $T_a = \inf\{t{:}X(t) > a\}$ ($= +\infty$ if $X(t) \le a$ for all t) is given by

$$r(t) = \lambda(t)(1 - H(a))$$

and so $X(t)$ is an IFRA process. We call it a "record process with value distribution H and intensity function $\lambda(t)$, $t \ge 0$." We say that an event of the record process occurs whenever an event from the related Poisson process occurs. (Note that the value of $X(t)$ need not change when an event occurs.)

The following theorem was proven by Ross in [3], for nonincreasing IFRA processes. Its proof in the nondecreasing case is similar.

THEOREM 2. *If* $\{X_i(t),\ t \ge 0\}$, $i = 1, \ldots, m$ *are independent nondecreasing* IFRA *stochastic processes and if* ϕ *is a nondecreasing function, then* $\{\phi(X_1(t), \ldots, X_m(t)),\ t \ge 0\}$ *is also an* IFRA *process.*

We are now ready for the

PROOF OF THEOREM 1. Let m be large and fixed and consider m independent record processes each having value distribution F and intensity function $\lambda(t)/m$—call them $\{X_i(t)\}$, $i = 1, \ldots, m$. Now the shock model under consideration can be generated from these record processes by saying that a shock occurs whenever an event (from any of the m record processes) occurs and by letting its damage be the value associated with the Poisson event. Let N denote the number of shocks it takes until the component fails. Now if we define ϕ by

$$\phi(x_1, \ldots, x_m) = D(x_1, \ldots, x_m, \underline{0})$$

then it follows from Theorem 2 that the first time $D(X_1(t), \ldots, X_m(t), \underline{0})$ exceeds C has an increasing failure rate average distribution. But as long as the first N shocks all come from different record processes this will be exactly the time the unit fails. Hence, as the probability that all shocks until unit failure come from different record processes can be made arbitrarily close to 1 by letting m be large, the result follows by letting m go to infinity since the limit of increasing failure rate average random variables is also increasing failure rate average.

REMARKS:

(i) The special case where $D(\underline{x}) = \sum_1^\infty x_i$ was previously considered in [1] and [2].

(ii) The symmetry condition ((iii) of Theorem 1) on D is needed in the proof because if D is not symmetric (that is, if the damage at any time depends not only on the set of shock values that have occurred by that time but also on their order of appearance) then, even if all shocks come from different record processes, the damage at time t need not be $D(X_1(t), \ldots, X_m(t), \underline{0})$ since there is no guarantee that the i^{th} shock to occur was from the i-th process, $i = 1, 2, \ldots, m$. In fact, it is easy to construct counterexamples to Theorem 1 if the symmetry condition on D is dropped. For instance, if $\lambda(t) \equiv \lambda$ and the damage only depends on the shocks numbered k, $2k$, $3k, \ldots$ (as it could if D were not assumed symmetric) then the arrival process of relevant shocks would be a gamma (k, λ) renewal process, which by appropriately choosing k and λ, could be made to approximate a deterministic renewal process. For such a process of shocks, it is clear that T could not be IFRA (as its failure rate falls to 0 between the deterministic times at which shocks occur).

(iii) An example of D which might be of practical interest is one of the form

$$D(\underline{x}) = \begin{cases} \sum x_i \text{ if } \max x_i \leq A \\ \max(101, \sum x_i) \text{ if } \max x_i > A. \end{cases}$$

By taking $c = 100$, the above would represent a model for a unit which would fail whenever the value of any single shock is greater than A or if the sum of all shock values is greater than 100. Similarly, if the unit were to fail if hit by any single shock of value greater than A_1, or, if the sum of any 2 shocks were greater than A_2, etc., then its failure time would be IFRA.

REFERENCES

[1] Esary, J. D., Marshall, A. W., and Proschan, F. (1973). Shock Models and Wear Processes. *Ann. Probability* 1 627–650.
[2] Hameed, M. S. and Proschan, F. (1973). Nonstationary Shock Models. *Stochastic Process. Applic.* 1 383–404.
[3] Ross, S. M. (1979). Multivalued State Component Systems. *Ann. Probability* 7 379–383.

Operations Research Center
University of California, Berkeley
Berkeley, CA 904720