*The Annals of Probability*
1981, Vol. 9, No. 5, 881–884

# BOUNDS FOR WEIGHTED EMPIRICAL DISTRIBUTION FUNCTIONS

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Let $G_n$ be the empirical distribution based on $n$ independent uniform random variables. Criteria for bounds on the supremum of weighted discrepancies between $G_n(u)$ and $u$ of the form: $|w_\nu(u)D_n(u)|$, where $D_n(u) = G_n(u) - u$, $w_\nu(u) = (u(1-u))^{-1+\nu}$ and $0 \le \nu \le 1$, are derived. Also an inequality closely related to an equality due to Daniels (1945) is given.

**1. Introduction and Preliminaries.** Let $U_1, U_2, \cdots$ be a sequence of independent uniform (0, 1) random variables and for each $n \ge 1$ let $U_{1n} \le \cdots \le U_{nn}$ be the order statistics of $U_1, \cdots, U_n$. $G_n$ will denote the empirical distribution based on $U_1, \cdots, U_n$, and let $D_n(u) = G_n(u) - u$ for $0 \le u \le 1$. $|D_n(u)|$ is usually called the discrepancy between $G_n(u)$ and $u$. We will investigate bounds on the supremum of weighted discrepancies between $G_n(u)$ and $u$ of the form:

$$\|w_\nu D_n\| \equiv \sup_{0\le u\le 1}|w_\nu(u)D_n(u)|, \tag{1.1}$$

where $w_\nu(u) = (u(1-u))^{-1+\nu}$ for $0 \le u \le 1$ and $0 \le \nu \le 1$.

When $1 \ge \nu > \frac{1}{2}$, Corollary 2 of James (1975) implies that

$$\limsup_{n\to\infty} n^{1/2}\|w_\nu D_n\|(2\ell n\,\ell n n)^{-1/2} = 4^{1/2-\nu} \text{ a.s.} \tag{1.2}$$

When $\nu = 0$ or $\frac{1}{2}$, criteria for bounds on $\|w_\nu D_n\|$ can be characterized as follows: Let $a_n$ be an increasing sequence of positive constants then

$$P(n^\nu\|w_\nu D_n\| \ge a_n \text{ i.o.}) = 0 \text{ or } 1$$

according to whether $\sum_{n=1}^\infty n^{-1}a_n^{-1/(1-\nu)}$ converges or diverges.

The case when $\nu = \frac{1}{2}$ is due to Csáki (1974) and the case when $\nu = 0$ can be easily demonstrated to be equivalent to Theorem 2 of Shorack and Wellner (1978). It will be shown that the criteria stated above for $\nu = 0$ or $\frac{1}{2}$ are true for all $\nu$ such that $0 \le \nu \le \frac{1}{2}$.

Results of this type have been recently used by Govindarajulu and Mason (1980) to prove a strong representation theorem for functions of order statistics; and should be applicable to proving strong limit theorems for other statistics which can be represented as a functional of $D_n$.

**2. The Main Result.** We will first prove an inequality which will be an essential tool in the proof of the main result of this paper. This inequality is closely related to an equality due to Daniels (1945) and is likely to have uses elsewhere.

THEOREM 1.—AN INEQUALITY. *For every $\nu$ such that $0 \le \nu < \frac{1}{2}$ there exists a constant $C_\nu > 0$ such that for all $a > 1$*

$$P(\sup_{0\le u\le a_{\nu n}} n^\nu G_n(u)u^{-1+\nu} \ge (1+\nu)a) \le C_\nu a^{-1/(1-\nu)}$$

*and*

$$P(\sup_{1-a_{\nu n}\le u\le 1} n^\nu(1-G_n(u))(1-u)^{-1+\nu} \ge (1+\nu)a) \le C_\nu a^{-1/(1-\nu)}$$

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Received March 25, 1980; revised July 22, 1980.
*AMS* 1970 *subject classifications.* Primary 60F15; secondary 60617, 62G30.
*Key words and phrases.* Weighted empirical distributions, bounds, uniform order statistics.

*where* $a_{0n} = 1$ *when* $\nu = 0$ *and* $a_{\nu n} = n^{-1}(\nu a)^{1/\nu} \Lambda 1$ *when* $0 < \nu < \frac{1}{2}$. $[x \Lambda y = \min(x, y).]$ *When* $\nu = 0$, *the conclusions hold with equality with* $C_0 = 1$.

PROOF. We will only prove the first inequality. The second inequality follows from the first by symmetry considerations.

For the case when $\nu = 0$, see Daniels (1945).

Now assume $0 < \nu < \frac{1}{2}$, and set

$$\Delta_n = \sup_{0 \le u \le a_{\nu n}} a^{-1} n^{\nu} G_n(u) u^{-1+\nu}.$$

Observe that $\Delta_n \le \Delta_{1n} + \Delta_{2n}$, where

$$\Delta_{1n} = \sup_{1 \le i \le n} n^{\nu-1} i a^{-1} U_{in}^{-1+\nu} I(0 \le U_{in} \le b_{\nu n}),$$

$$\Delta_{2n} = \sup_{1 \le i \le n} n^{\nu-1} i a^{-1} U_{in}^{-1+\nu} I(b_{\nu n} \le U_{in} \le c_{\nu n}),$$

$$b_{\nu n} = \nu^{1/\nu} a^{-1/(1-\nu)} n^{-1} \text{ and } c_{\nu n} = (\nu a)^{1/\nu} n^{-1}.$$

[For $x \le y$, $I(x \le u \le y) = 1$ if $x \le u \le y$ and 0 otherwise.]

Now $P(\Delta_n \ge 1 + \nu) \le$

$$P(\Delta_{1n} \ge 0) + P(\Delta_{2n} \ge 1 + \nu). \tag{2.1}$$

First note that $P(\Delta_{1n} > 0) = P(U_{1n} \le b_{\nu n}) = 1 - (1 - b_{\nu n})^n \le \nu^{1/\nu} a^{-1/(1-\nu)}$.

We will show that the second term in (2.1) is also bounded by $a^{-1/(1-\nu)}$ times a positive constant depending only on $\nu$.

For $u \ge 0$ set $h_n(u) = a^{-1} u^{-1+\nu} I(b_{\nu n} \le u \le c_{\nu n})$, and $g_n(u) = h_n(b_{\nu n})$ if $0 \le u < b_{\nu n}$ and equal to $h_n(u)$ otherwise. Since $g_n$ is nonincreasing on [0, 1] and $g_n \ge h_n$, it is easy to see that for each $1 \le i \le n$

$$n^{\nu-1} i h_n(U_{in}) \le n^{\nu-1} \sum_{j=1}^n g_n(U_j).$$

Thus $P(\Delta_{2n} \ge 1 + \nu) \le$

$$P(n^{\nu-1} \sum_{j=1}^n g_n(U_j) \ge 1 + \nu) \tag{2.2}$$

Let $Eg_n = Eg_n(U_1)$. Since $g_n(u) \le a^{-1} u^{-1+\nu}$ for $u \ge 0$,

$$Eg_n \le a^{-1} \int_0^{c_{\nu n}} u^{-1+\nu}\, du = n^{-\nu}.$$

Therefore the expression in (2.2) is $\le$

$$P(n^{\nu-1} \sum_{j=1}^n (g_n(U_j) - Eg_n) \ge \nu),$$

which by Chebyshev's inequality is $\le$

$$n^{2\nu-1} \nu^{-2} \operatorname{var} g_n(U_1).$$

Now

$$\operatorname{var} g_n(U_1) \le Eg_n^2(U_1) = b_{\nu n} h_n^2(b_{\nu n}) + a^{-2} \int_{b_{\nu n}}^{c_{\nu n}} u^{-2+2\nu}\, du,$$

which is easily shown to be $\le n^{1-2\nu} a^{-1/(1-\nu)} K_\nu$, where $K_\nu$ is a positive constant depending only on $\nu$. Thus the expression in (2.2) is $\le a^{-1/(1-\nu)} \nu^{-2} K_\nu$. Letting $C_\nu = \nu^{1/\nu} + \nu^{-2} K_\nu$ completes the proof. □

We will now show that the criteria for bounds on $\| w_\nu D_n \|$ stated in the introduction are true for all $0 \le \nu \le \frac{1}{2}$.

THEOREM 2.—THE MAIN RESULT. *Let $a_n$ be an increasing sequence of positive constants, then for every $0 \le \nu \le \frac{1}{2}$*

$$P(n^\nu \| w_\nu D_n \| \ge a_n \text{ i.o.}) = 0 \text{ or } 1$$

*according as* $\sum_{n=1}^\infty n^{-1} a_n^{-1/(1-\nu)}$ *converges or diverges.*

PROOF. For the case when $\nu = \frac{1}{2}$ see Theorem 3.1 of Csáki (1974). When $\nu = 0$, the statement of the theorem can be shown to be equivalent to Theorem 2 of Shorack and Wellner (1978). So we will only consider the case when $0 < \nu < \frac{1}{2}$.

First we will assume $(C)$: $\sum_{n=1}^\infty n^{-1} a_n^{-1/(1-\nu)} < \infty$, and without loss of generality that $a_n^{1/\nu} n^{-1} \downarrow 0$. Let

$$A = \{n^\nu \| w_\nu D_n \| \ge a_n \text{ i.o.}\},$$

$$B = \{\sup_{0 \le u \le c_\nu a_n^{1/\nu} n^{-1}} n^\nu | w_\nu(u) D_n(u) | \ge a_n \text{ i.o.}\},$$

$$C = \{\sup_{1 - c_\nu a_n^{1/\nu} n^{-1} \le u \le 1} n^\nu | w_\nu(u) D_n(u) | \ge a_n \text{ i.o.}\}, \text{ and}$$

$$D = \{\sup_{c_\nu a_n^{1/\nu} n^{-1} \le u \le 1 - c_\nu a_n^{1/\nu} n^{-1}} n^\nu | w_\nu(u) D_n(u) | \ge a_n \text{ i.o.}\};$$

where $c_\nu = 2^{-1}(\nu(4(1+\nu))^{-1})^{1/\nu}$.

Observe that $P(A) \le$

$$P(B) + P(C) + P(D). \tag{2.3}$$

We will first show that $P(B) = 0$.

Note that since $n^\nu u^\nu \le 4^{-1} a_n$ for $0 \le u \le c_\nu a_n^{1/\nu} n^{-1}$, and $a_n^{1/\nu} n^{-1} \downarrow 0$, $B \subset B'$; where

$$B' = \{\sup_{0 \le u \le c_\nu a_n^{1/\nu} n^{-1}} n^\nu G_n(u) u^{-1+\nu} \ge 2^{-1} a_n \text{ i.o.}\}.$$

Let $n_r = 2^r$ for integers $r \ge 1$. Note that $(C)$implies $\sum_{r=1}^\infty a_{n_r}^{-1/(1-\nu)} < \infty$. Now

$$P(B') \le P(\max_{n_r \le n \le n_{r+1}} \sup_{0 \le u \le c_\nu a_n^{1/\nu} n^{-1}} n^\nu G_n(u) u^{-1+\nu} \ge 2^{-1} a_{n_r} \text{ i.o.}),$$

which, since $a_n \uparrow$, $n^{-1} a_n^{1/\nu} \downarrow$, and $nG_n(u)$ is nondecreasing as a function of $n$, is

$$\le P(\sup_{0 \le u \le c_\nu a_{n_r}^{1/\nu} n_r^{-1}} 2 n_{r+1}^\nu G_{n_{r+1}}(u) u^{-1+\nu} \ge 2^{-1} a_{n_r} \text{ i.o.}).$$

Let $a'_{n_r} = a_{n_r}(4(1+\nu))^{-1}$. Observe that

$$(\nu a'_{n_r})^{1/\nu} n_{r+1}^{-1} = c_\nu (a_{n_r})^{1/\nu} n_r^{-1}$$

Application of Theorem 1 now gives

$$\sum_{r=1}^\infty P(\sup_{0 \le u \le (\nu a'_{n_r})^{1/2} n_{r+1}^{-1}} n_{r+1}^\nu G_{n_{r+1}}(u) u^{-1+\nu} \ge a'_{n_{r+1}}(1+\nu)) \le C_\nu \sum_{r=1}^\infty (a'_{n_r})^{-1/(1-\nu)}. \tag{2.4}$$

The series in (2.4) is easily seen to be convergent, so the Borel-Cantelli lemma implies that $P(B) = 0$. By an analogous argument $P(C) = 0$.

We will now show that $P(D) = 0$.

Observe that $(C)$ and $a_n$ increasing implies that

$$a_n^{-1} (\ell n n)^{1-\nu} \to 0.$$

Therefore we see that $D \subset D'$, where

$$D' = \{\sup_{c_n^{1/\nu} n^{-1} \le u \le 1 - c_n^{1/\nu} n^{-1}} n^\nu | w_\nu(u) D_n(u) | \ge c_n \text{ i.o.}\},$$

and $c_n = (\ell n n)^{1-\nu}$. Notice that

$$D' \subset \{\limsup_{n\to\infty} c_n^{-1} \sup_{c_n^{1/\nu} n^{-1} \le u \le 1 - c_n^{1/\nu} n^{-1}} n^{1/2} | w_{1/2}(u) D_n(u) | \ge 1\};$$

but Theorem 3.1 of either Csáki (1974) or (1977) implies that

$$\limsup_{n\to\infty} \sup_{c_n^{1/\nu}n^{-1}\le u\le 1-c_n^{1/\nu}n^{-1}} c_n^{-1}n^{1/2}\,|w_{1/2}(u)D_n(u)| = 0 \text{ a.s..}$$

Hence $P(D) = 0$. Thus we have shown that $P(A) = 0$.

Now assume $(D)$: $\sum_{n=1}^{\infty} n^{-1}a_n^{-1/(1-\nu)} = \infty$, and without loss of generality that $a_n \ge 1$ for all $n \ge 1$.

Note that

$$\begin{aligned} P(A) &\ge P(\sup_{0\le u\le a_n^{-1/(1-\nu)}n^{-1}} n^{\nu}\,|w_\nu(u)D_n(u)| \ge a_n \text{ i.o.}) \\ &\ge P(\sup_{0\le u\le a_n^{-1/(1-\nu)}n^{-1}} n^{\nu}G_n(u)u^{-1+\nu} \ge 2a_n \text{ i.o.}) \\ &\ge P(U_{1n} \le (2a_n)^{-1/(1-\nu)}n^{-1} \text{ i.o.}). \end{aligned} \tag{2.5}$$

But $(D)$ implies that the expression in (2.5) is equal to 1 (See Theorem 1 of Kiefer (1972)). Hence $P(A) = 1$. □

The following corollaries are immediate from Theorem 2.

COROLLARY 1. *For all* $0 < \nu \le \frac{1}{2}$ *and* $0 \le \beta < \nu$

$$n^{\beta}\,\|w_\nu D_n\| \to 0 \text{ a.s..}$$

PROOF. Obvious. □

Gregory (1977) showed that the convergence in Corollary 1 is true in probability. See his Lemma 7.3.

Finally we extend Corollary 3.2 of Csáki (1974).

COROLLARY 2. *For all* $\nu$ *such that* $0 \le \nu \le \frac{1}{2}$,

$$\limsup_{n\to\infty} (n^{\nu}\,\|w_\nu D_n\|)^{1/(\ell n\,\ell n n)} = e^{1-\nu} \text{ a.s..}$$

PROOF. Observe that for all $\epsilon > 0$ $P(n^{\nu}\|w_\nu D_n\| \ge a_n \text{ i.o.}) = 1$ and $P(n^{\nu}\|w_\nu D_n\| \ge b_n \text{ i.o.}) = 0$, where $a_n = (\ell nn)^{1-\nu}$ and $b_n = (\ell nn)^{1-\nu+\epsilon}$. □

**Acknowledgement.** The author would like to thank the referee for pointing out reference [3].

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