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LIMIT THEOREMS ON ORDER STATISTICS

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Let F belong to the domain of attraction of a stable law with parameters α and p. Let $X_1, X_2, \cdots$ be a sample from F. Put $|\tilde{X}_1| \le |\tilde{X}_2| \le \cdots \le |\tilde{X}_n|$. We consider the asymptotic properties as $n \to \infty$ (and $k \to \infty$) of the ratio of order statistics $(\tilde{X}_1 + \cdots + \tilde{X}_{n-k})/|\tilde{X}_{n-k+1}|$.

1. Preliminary results. Let $X_1, X_2, \cdots, X_n$ be a sample of size n from a distribution F on $(-\infty, +\infty)$ which belongs to the domain of attraction of a stable law, i.e. for $x \to \infty$

$$\begin{cases} 1 - F(x) \sim p\{xL(x)\}^{-\alpha} \\ F(-x) \sim q\{xL(x)\}^{-\alpha} \end{cases} \tag{1}$$

where $0 < \alpha < 2$, $0 < p \le 1$, $p + q = 1$ and L is slowly varying (s.v.) at infinity. We exclude the case $\alpha = 1$.

To unify the statements of forthcoming theorems, we introduce

$$\nu = \begin{cases} 0 & \text{if} \quad 0 < \alpha < 1 \\ \mu & \text{if} \quad 1 < \alpha < 2 \end{cases} \tag{2}$$

where μ is the mean of F.

If $S_n = X_1 + X_2 + \cdots + X_n$ then (1) implies that for a sequence $a_n > 0$, $a_n \to \infty$, $a_n^{-1}\{S_n - n\nu\}$ converges weakly to a stable law with parameters α and p [4, page 574], [5, page 46]. We can in fact take a_n such that as $n \to \infty$

$$n\{1 - F(a_n)\} \to p.$$

Solving for a_n we obtain

$$a_n \sim n^{1/\alpha} L^*(n^{1/\alpha}) \tag{3}$$

where L^* is the s.v. function conjugated to L [6, page 25] [3].

LEMMA 1. *Assume* (1) *and* (3) *hold. Then*

(i) $$\begin{cases} n\{1 - F(a_n x)\} \to p x^{-\alpha}, & n \to \infty, \quad x > 0, \\ nF(a_n x) \to q|x|^{-\alpha}, & n \to \infty, \quad x < 0; \end{cases}$$

(ii) $$\int_0^1 \frac{1 - F(vy)}{1 - F(v)}\, dy \to \frac{1}{1 - \alpha}, \qquad 0 < \alpha < 1, \quad v \to \infty$$

(iii) $$\int_1^\infty \frac{1 - F(vy)}{1 - F(v)}\, dy \to \frac{1}{\alpha - 1}, \qquad 1 < \alpha < 2, \quad v \to \infty$$

(iv) $$\int_0^1 y \frac{1 - F(vy) + F(-vy)}{1 - F(v) + F(-v)}\, dy \to \frac{1}{2 - \alpha}, \qquad 0 < \alpha < 2, \quad v \to \infty.$$

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A crucial role will be played by a truncated part of the characteristic function of F. To abbreviate the writing we introduce the notation ($t \in \mathbb{R}$, $y > 0$).

$$E_\alpha(t, y) = \begin{cases} y^{-\alpha} + \int_0^y (e^{itx} - 1)\, dx^{-\alpha} & 0 < \alpha < 1, \\ y^{-\alpha} + \int_0^y (e^{itx} - 1 - itx)\, dx^{-\alpha} & 1 < \alpha < 2. \end{cases} \tag{4}$$

LEMMA 2. *Assume* (1) *and* (3) *hold. Then as* $n \to \infty$ *and* $u < 0 < v$

$$n\left\{\int_u^v e^{itw} dF(a_n w) - 1 - \frac{itv}{a_n}\right\} \to -K_\alpha(u, v, t)$$

where

$$K_\alpha(u, v, t) = \begin{cases} pE_\alpha(t, v) + qE_\alpha(-t, -u) & 0 < \alpha < 1, \\ pE_\alpha(t, v) + qE_\alpha(-t, -u) + \dfrac{\alpha it}{\alpha - 1}[pv^{1-\alpha} - q(-u)^{1-\alpha}] & 1 < \alpha < 2. \end{cases}$$

PROOF.
(i) $0 < \alpha < 1$. Let $I_n \equiv \int_{0+}^v e^{itw}\, dF(a_n w)$. Then

$$I_n = -\int_0^v (e^{itw} - 1)\, d\{1 - F(a_n w)\} - \{1 - F(a_n v)\} + \{1 - F(0)\}.$$

Similarly let

$$II_n \equiv \int_u^0 e^{itw}\, dF(a_n w).$$

Then

$$II_n = \int_u^0 (e^{itw} - 1)\, dF(a_n w) + F(0) - F(a_n u).$$

Hence by Lemma 1 (*i*)

$$n\{1 - I_n - II_n\} \to pE_\alpha(t, v) - q\int_u^0 (e^{itw} - 1)\, d|w|^{-\alpha} + q|u|^{-\alpha}.$$

(ii) $1 < \alpha < 2$. Using the same abbreviations as in (i) we can write

$$I_n = -\int_0^v (e^{itw} - 1 - itw)\, d\{1 - F(a_n w)\} - \{1 - F(a_n v)\} + \{1 - F(0)\} - it\int_0^v w\, d\{1 - F(a_n w)\}.$$

However by an integration by parts

$$\int_0^v w\, d\{1 - F(a_n w)\} = v\{1 - F(a_n v)\} - \frac{1}{a_n}\int_0^\infty [1 - F(v)]\, dv + \int_v^\infty [1 - F(a_n x)]\, dx.$$

Similarly

$$II_n = \int_u^0 (e^{itw} - 1 - itw)\, dF(a_n w) + F(0) - F(a_n u) + it \int_u^0 w\, dF(a_n w)$$

where

$$\int_u^0 w\, dF(a_n w) = -uF(a_n u) - \frac{1}{a_n} \int_{-\infty}^0 F(v)\, dv + \int_{-\infty}^u F(a_n x)\, dx. \tag{6}$$

From here the calculations are similar as for case (i) except for the last term in (5) and (6). But

$$\int_v^\infty [1 - F(a_n x)]\, dx = v\{1 - F(a_n v)\} \int_1^\infty \frac{1 - F(a_n vy)}{1 - F(a_n v)}\, dy,$$

which is handled by (iii) of Lemma 1. □

2. Main limit theorems. Let us order the r.v. of the sample according to increasing moduli

$$|\tilde{X}_1| \le |\tilde{X}_2| \le \cdots \le |\tilde{X}_n|.$$

We investigate first the limiting behaviour of the hybrid characteristic-distribution function

$$\chi_n(t, y) = E\{e^{itS_n^{(k)}}; |X_{n-k+1}| \le y\} \tag{7}$$

where $\qquad S_n^{(k)} \equiv \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{n-k} - (n-k)\nu$

LEMMA 3. *Assume* (1) *and* (3) *hold. Then as* $n \to \infty$ *for* $y \ge 0$.

$$\chi_n\left(\frac{t}{a_n}, a_n y\right) \to -\frac{1}{(k-1)!} \int_0^y v^{-\alpha(k-1)} e^{-K_\alpha(-v,v,t)}\, dv^{-\alpha}.$$

PROOF. It follows from (27) in [1] that for $y \ge 0$

$$\begin{aligned} \frac{(k-1)!(n-k)!}{n!} \chi_n(t, y) &= \int_0^y \{1 - F(v) + F(-v)\}^{k-1} \psi(v)^{n-k}\, dF(v) \\ &\quad + \int_{-y}^0 \{1 - F(-v) + F(v)\}^{k-1} \psi(v)^{n-k}\, dF(v) \end{aligned} \tag{8}$$

where ($v \ge 0$)

$$\psi(v) \equiv \psi(v, t) = e^{-it\nu} \int_{-v}^v e^{itx}\, dF(x).$$

Change t into t/a_n and y into $a_n y$ in $\chi_n(t, y)$. Replace v by $a_n v$ in both integrations. As $n \to \infty$, $(n-k)!/n! \sim n^{-k}$. Hence

$$\begin{aligned} \lim_{n\to\infty} \chi_n\left(\frac{t}{a_n}, a_n y\right) = -\lim_{n\to\infty} \int_0^y &\{n[1 - Fa_n v) \\ &+ F(-a_n v)]\}^{k-1} \psi(a_n v)^{n-k}\, d\{n[1 - F(a_n v)]\} \\ &+ \lim_{n\to\infty} \int_{-y}^0 \{n[1 - F(-a_n v) + F(a_n v)]\}^{k-1} \psi(-a_n v)^{n-k}\, d\{nF(a_n v)\}. \end{aligned}$$

But $(v \geq 0)$

$$n\left\{1 - \psi\left(a_n v, \frac{t}{a_n}\right)\right\} = n\left\{1 - e^{[-i(t/a_n)\nu]} \int_{-v}^{v} e^{itw}\, dF(a_n w)\right\} \to K_\alpha(-v, v, t)$$

in view of Lemma 2, and the fact that by (3) $n/a_n^2 \to 0$. For $v \geq 0$ as $n \to \infty$

$$\psi(a_n v)^{n-k} \to \exp\{-K_\alpha(-v, v, t)\}.$$

Collecting terms we obtain the result. □

For $0 < \alpha < 1$ Lemma 3 can be found in [1, page 384] with slightly more technical proof. For $1 < \alpha < 2$ our result is more useful than the corresponding statement in [1].

The main r.v. under consideration will be

$$T_{n,k} \equiv \frac{1}{|\tilde{X}_{n-k+1}|} \{\tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{n-k} - (n-k)\nu\}.$$

THEOREM 1. *Assume* (1) *holds where* $\alpha \in (0, 1) \cup (1, 2)$. *Then as* $n \to \infty$

$$E\{\exp it\, T_{n,k}\} \to \{pE_\alpha(t, 1) + qE_\alpha(-t, 1) + \theta \frac{\alpha}{\alpha - 1}(p - q)it\}^{-k}$$

where $\theta = \min(1, [\alpha])$.

PROOF. Clearly

$$E\left\{\exp it \frac{S_n^{(k)}}{|\tilde{X}_{n-k+1}|}\right\} = \int_0^\infty E\{e^{itS_n^{(k)}/y}, |\tilde{X}_{n-k+1}| \in dy\} = \int_0^\infty \chi_n\left(\frac{t}{a_n y}, a_n\, dy\right).$$

So as $n \to \infty$

$$E\{\exp it\, T_{n,k}\} \to -\frac{1}{(k-1)!} \int_0^\infty y^{-\alpha(k-1)} \exp\left\{-K_\alpha\left(-y, y, \frac{t}{y}\right)\right\} dy^{-\alpha}.$$

However by an easy calculation one can show that for $y > 0$

$$K_\alpha\left(-y, y, \frac{t}{y}\right) = y^{-\alpha} K_\alpha(-1, 1, t)$$

from which the result is immediate. □

The most remarkable fact which follows from Theorem 1 is that $T_{n,k}$ converges in distribution to a random variable T_k (say) which is the sum of k independent r.v. all distributed as T_1. The r.v. T_1 has characteristic function

$$E[e^{itT_1}] = \left\{pE_\alpha(t, 1) + qE_\alpha(-t, 1) + \theta \frac{\alpha}{\alpha - 1}(p - q)it\right\}^{-1},$$

where the expression inside the brackets is an entire function of the possibly complex valued variable t [2]. Also as $t = 0$, $E_\alpha(0, 1) = 1$. Hence T_1 has an analytic characteristic function and in particular has moments of all order. For example

$$ET_1 = \frac{\alpha}{1 - \alpha}(p - q) \equiv \gamma$$

$$\operatorname{Var} T_1 = \frac{\alpha}{2 - \alpha} + \left(\frac{\alpha(p - q)}{\alpha - 1}\right)^2 \equiv \beta.$$

By a standard diagonalisation argument we obtain:

COROLLARY 1. *Assume* (1) *holds. Then there exist sequences k_n and k'_n tending to infinity such that*

(i) $$k_n^{-1} T_{n,k_n} \to_P \gamma;$$

(ii) $$(\beta k'_n)^{-1/2}[T_{n,k'_n} - \gamma k'_n] \to_{\mathscr{D}} \mathscr{N}(0, 1),$$

Motivated by the above corollary we try to find "best-possible" sequences k_n for which we obtain convergence to a degenerate or normal law.

3. Conditions on k for degenerated or normal limit law. In general we try to determine constants $c_n = c_{n,k_n}$ and $d_n = d_{n,k_n}$ such that $d_n^{-1}\{T_{n,k_n} - c_n\}$ converges in distribution either to a degenerated or to a normal limit law.

Clearly, under

(L) $$\begin{cases} c_n = k\gamma \\ d_n = k \end{cases}$$

a degenerate limit will appear; under

(N) $$\begin{cases} c_n = k\gamma \\ d_n = \beta\sqrt{k} \end{cases}$$

a $\mathscr{N}(0, 1)$ will be found.

To facilitate the arithmetic we are forced to introduce some notation

(9) $$\varphi_{n,k}(t) = E\left\{\exp \frac{it}{d_n}[T_{n,k} - c_n]\right\} \qquad (t \in \mathbb{R})$$

(10) $$J \equiv J_n(t, y) = \int_{-y}^{y} \exp\left(\frac{itx}{yd_n}\right) dF(x) \qquad (t \in \mathbb{R}, y \geq 0)$$

(11) $$g \equiv g_n(y) = \frac{1}{d_n}\left\{\frac{c_n}{n-k} + \frac{\nu}{y}\right\} \qquad (y \geq 0)$$

and

(12) $$\phi \equiv \phi_n(t, y) = e^{-itg}J.$$

Using again (8) we see that

$$\varphi_{n,k}(t) = e^{-it(c_n/d_n)} \int_0^\infty \chi_n\left(\frac{t}{y\, d_n}, dy\right)$$

or

(13) $$\varphi_{n,k}(t) = \frac{n!}{(k-1)!\,(n-k)!}\int_0^\infty \{1 - F(y) + F(-y)\}^{k-1}\{\phi_n(t, y)\}^{n-k}\, d\{F(y) - F(-y)\}.$$

We now successively transform the integral on the right. First put

(14) $$s = 1 - F(y) + F(-y) \equiv H(y) \qquad (y \geq 0)$$

and

(15) $$1 - F(y) - F(-y) \equiv K(y); \qquad (y \geq 0)$$

we denote by H^i the inverse function of H so that $H^i(s) = y$. In a similar fashion as in Lemma 1 we obtain

LEMMA 4.

(i) *If* $\beta > \alpha - 1$ *then* $\int_0^1 u^\beta \frac{H(uy)}{H(y)}\,du \to (\beta - \alpha + 1)^{-1}$ *as* $y \to \infty$;

(ii) *if* $\beta < \alpha - 1$ *then* $\int_1^\infty u^\beta \frac{H(uy)}{H(y)}\,du \to (\alpha - \beta - 1)^{-1}$ *as* $y \to \infty$;

(iii) *if* $\beta > \alpha - 1$ *then* $\int_0^1 u^\beta \frac{K(uy)}{H(y)}\,du \to \frac{p-q}{\beta - \alpha + 1}$ *as* $y \to \infty$.

(iv) $s = H(y) \sim y^{-\alpha}L^{-\alpha}(y)$ *as* $y \to \infty$ *is equivalent to*

$$y = H^i(s) \sim s^{-1/\alpha}L^*(s^{-1/\alpha}) \quad as \quad s \to 0.$$

We have from (13)

$$\varphi_{n,k}(t) = \frac{n!}{(k-1)!(n-k)!}\int_0^1 s^{k-1}\{\phi_n(t, H^i(s))\}^{n-k}\,ds.$$

Let $\{p_n\}_1^\infty$ and $\{q_n\}_1^\infty$ be sequences of positive constants to be chosen shortly. Change s into z by the substitution

$$s = q_n + p_n z. \tag{16}$$

Then

$$\varphi_{n,k}(t) = I_1 \int_{-q_n/p_n}^{(1-q_n)/p_n} I_2(z)\{\phi_n(t, y)\}/(1-s)\}^{n-k}\,dz \tag{17}$$

where

$$I_1 = n!p_n(q_n)^{k-1}(1-q_n)^{n-k}/(k-1)!(n-k)! \tag{18}$$

and

$$I_2(z) = \left\{1 + \frac{p_n}{q_n}z\right\}^{k-1}\left\{1 - \frac{p_n}{1-q_n}\right\}^{n-k}.$$

It is easy to show that $I_2(z)$ will have a "useful" limit for $n \to \infty$, $k \to \infty$ and $n - k \to \infty$ only if we put

$$q_n = \frac{k-1}{n-1}, \quad p_n^2 = \frac{(k-1)(n-k)}{(n-1)^3} \tag{19}$$

so that $q_n/p_n \to \infty$, $(1-q_n)/p_n \to \infty$.

LEMMA 5. *If* $n \to \infty$, $k \to \infty$, $n - k \to \infty$ *then with* (19)

(i) $I_1 \to (\sqrt{2\pi})^{-1}$;

(ii) $I_2(z) \to \exp\{-z^2/2\}$ *uniformly in compact z-intervals.*

PROOF. (i) *Follows from* (18) *and Stirling's formula*; (ii) We follow Smirnov [7, page 95]. It is easy to show that

$$I_2'(z) = -zI_2(z)\left\{1 + \frac{p_n}{q_n}z\right\}^{-1}\left\{1 - \frac{p_n}{1-q_n}z\right\}^{-1}.$$

Since $I_2(0) = 1$, we find

$$I_2(z) = \exp - \left\{ \int_0^z v \left\{ 1 + \frac{p_n}{q_n} v \right\}^{-1} \left\{ 1 - \frac{p_n}{1 - q_n} v \right\}^{-1} dv \right\}.$$

But

$$p_n/q_n \leq (k-1)^{-1/2} \text{ and } p_n/(1-q_n) \leq (n-k)^{-1/2}.$$

Hence the result. □

If we can find the limiting form of the remaining expression

$$I_3 = \{\phi_n(t, y)/(1-s)\}^{n-k} \tag{20}$$

the limit of $\varphi_{n,k}(t)$ is readily obtained. However we have to restrict ourselves to compact intervals. To see that this is allowed we first prove:

LEMMA 6. *There exists a constant C, independent of k and n so that for $n \to \infty$ and $k \to \infty$ and $T > 0$*

$$J_+ \equiv I_1 \int_T^{(1-q_n)/p_n} I_2(z) \{\phi_n(t, y)/(1-s)\}^{n-k} \, dz < CT^{-2};$$

$$J_- \equiv I_1 \int_{-q_n/p_n}^{-T} I_2(z) \{\phi_n(t, y)/(1-s)\}^{n-k} \, dz < CT^{-2}.$$

PROOF. We only prove the result for J_+. Returning to integration with respect to s we find

$$J_+ = \frac{n!}{(k-1)!\,(n-k)!} \int_{q_n + p_n T}^1 s^{k-1} \{\phi_n(t, H^i(s))\}^{n-k} \, ds.$$

However by (10) and (12)

$$|\phi_n(t, y)| \leq F(y) - F(-y) = 1 - H(y) = 1 - s.$$

Moreover since $s \geq q_n + p_n T$, $(s - q_n)^2/(p_n T)^2 \geq 1$. Hence

$$|J_+| \leq \frac{1}{p_n^2 T^2} \frac{n!}{(k-1)!(n-k)!} \int_0^1 s^{k-1}(1-s)^{n-k}(s-q_n)^2 \, ds.$$

Now use (19) and some standard properties of the beta function to find that $|J_+| \leq T^{-2}(1 + o(1))$ where $o(1) \to 0$ as $n \to \infty$ and $k \to \infty$. □

We now determine under (L) or (N) for what sequences $\{k_n\} I_3$ converges as $n \to \infty$ where $s = q_n + p_n z$ and where $|z| < Z$ where Z is an arbitrary but fixed constant.

We first remark that the estimations of Lemma 4 are applicable if $s \to 0$ or if $q_n \to 0$ since $p_n/q_n \to 0$. This means that $k = o(n)$.

LEMMA 7. *Under conditions (L) or (N), $g_n(y) \to 0$ as $n \to \infty$, $k \to \infty$, $k = o(n)$, uniformly in z, $|z| < Z$.*

PROOF. If (L) holds, then $g = \dfrac{\gamma}{n-k} + \dfrac{\nu}{ky}$ where $n - k \to \infty$ and $ky \to \infty$. If (N) holds, then

$$\beta g = \frac{\sqrt{k}}{n-k} + \beta \frac{v}{ky} \to 0$$

as well. □

We turn to the term J. As in the proof of Lemma 2, we have to rewrite J. If we want to deal with the two cases $0 < \alpha < 1$ and $1 < \alpha < 2$ simultaneously, we need to introduce the following function which will be fundamental in the (N) case

$$(21) \qquad \rho(y) = \begin{cases} \displaystyle\int_0^1 \left\{ (p-q)u^{-\alpha} - \frac{K(uy)}{H(y)} \right\} du & \text{if } 0 < \alpha < 1, \\ \displaystyle -\int_1^\infty \left\{ (p-q)u^{-\alpha} - \frac{K(uy)}{H(y)} \right\} du & \text{if } 1 < \alpha < 2. \end{cases}$$

We obtain after straightforward algebra that

$$J = 1 - \{e^{it/d_n}[1-F(y)] + e^{-it/d_n}F(-y)\} + \frac{itv}{yd_n} + \frac{its}{d_n}\frac{p-q}{1-\alpha} - \frac{its}{d_n}\rho(y)$$
$$+ \frac{it}{yd_n}\int_0^y \{[e^{itx/yd_n} - 1][1-F(x)] - [e^{-itx/yd_n} - 1]F(-x)\}\, dx.$$

The two functions

$$(22) \qquad \pi_n \equiv \pi_n(t, y) = \frac{1}{s}\{e^{it/d_n}[1-F(y)] + e^{-it/d_n}F(-y)\}$$

and

$$(23) \qquad \mu_n \equiv \mu_n(t, y) = \frac{d_n}{itys}\int_0^y \{[e^{itx/yd_n} - 1][1-F(x)] - [e^{-itx/yd_n} - 1]F(-x)\}\, dx$$

are helpful in rewriting J in the form

$$(24) \qquad J = (1-s)\left\{1 + \frac{s}{1-s}\left[1 - \pi_n + \frac{it}{sd_n}\left(\frac{v}{y} + \frac{p-q}{1-\alpha}s - s\rho(y)\right) - \left(\frac{t}{d_n}\right)^2 \mu_n\right]\right\}.$$

We estimate in turn π_n, μ_n and $\frac{s}{1-s}$. Recall that under (L) or (N), $d_n \to \infty$ as $k \to \infty$.

a. $\pi_n(t, y)$.
From (22) we have

$$s\pi_n(t, y) = \{1 - F(y) + F(-y)\}\left\{1 - \frac{1}{2}\left(\frac{t}{d_n}\right)^2 + o(d_n^{-2})\right\} + \frac{it}{d_n}\{1 - F(y) - F(-y)\}$$

or

$$(25) \qquad \begin{aligned} \pi_n(t, y) &= 1 + \frac{it}{d_n}\frac{K(y)}{H(y)} - \frac{1}{2}\left(\frac{t}{d_n}\right)^2 + o(d_n^{-2}) \\ &= 1 + \frac{it}{d_n}(p-q) - \frac{1}{2}\left(\frac{t}{d_n}\right)^2 + o(d_n^{-1}). \end{aligned}$$

b. $\mu_n(t, y)$.

In an entirely similar fashion by Lemma 4

$$\mu_n(t, y) = \int_0^1 v \frac{H(yv)}{H(y)} dv + \frac{it}{2d_n} \int_0^1 v^2 \frac{K(yv)}{H(y)} dv + o\left(\frac{k}{d_n^2}\right) \tag{26}$$
$$= \frac{1}{2-\alpha} + \frac{it}{2d_n} \frac{p-q}{3-\alpha} + o(1).$$

c. $s/(1-s)$.
From the definition

$$\frac{s}{1-s} = \frac{q_n}{1-q_n} \left\{1 + \frac{p_n}{q_n} z\right\} \left\{1 - \frac{p_n}{1-q_n} z\right\}^{-1}.$$

But $q_n/(1-q_n) = (k-1)/(n-k)$; hence

$$(n-k)\frac{s}{1-s} - (k-1) = z\left\{\frac{(n-1)(k-1)}{n-k}\right\}^{1/2} \left\{1 - \frac{p_n}{1-q_n} z\right\}^{-1}.$$

Hence for $|z| < Z$ uniformly as $n \to \infty$, $k \to \infty$, $k = o(n)$

$$(n-k)\frac{s}{1-s} - (k-1) = z\left\{\frac{nk}{n-k}\right\}^{1/2} (1 + o(1)). \tag{27}$$

We now evaluate the limit of I_3. Since for $n \to \infty$, $k \to \infty$, $k = o(n)$ both $g \to 0$ and $J/(1-s) \to 1$ we can write that I_3 converges iff

$$R_n(t, y) \equiv (n-k)\left\{e^{-itg} \frac{J}{1-s} - 1\right\}$$

converges.

Expand e^{itg} and use (24) together with the estimation obtained in a and b above. Collecting terms according to powers of t, we obtain

$$\begin{aligned} R_n(t, y) = {} & it\left\{\frac{s(n-k)}{1-s}\frac{1}{d_n} X_n(y) - (n-k)g\right\} \\ & - \frac{t^2}{2}\left\{(n-k)g^2 - 2g\frac{s(n-k)}{1-s}\frac{X_n(y)}{d_n} + \frac{s(n-k)}{(1-s)}\frac{1}{d_n^2}\frac{\alpha}{2-\alpha}\right\} \\ & + o(g^2(n-k)) + O\left(g^2 \frac{s(n-k)}{(1-s)}\frac{X_n(y)}{d_n}\right) \\ & + O\left(g\frac{s(n-k)}{(1-s)}\frac{1}{d_n^2}\right) + O\left(\frac{s(n-k)}{(1-s)}\frac{1}{d_n}\right), \end{aligned} \tag{28}$$

where

$$X_n(y) = \frac{p-q}{1-\alpha} - \rho(y) + \frac{\nu}{sy} - \frac{K(y)}{H(y)}.$$

Denote the coefficient of it in (28) by $A_n(t, y)$; that of $-\frac{t^2}{2}$ by $B_n(t, y)$ and the maximal o or O term by $C_n(t, y)$ so that

$$R_n(t, y) = itA_n(t, y) - \frac{t^2}{2} B_n(t, y) + O(C_n(t, y)).$$

LEMMA 8. (i) *Assume the conditions* (L) *hold.*
Then as $n \to \infty$, $k \to \infty$, $k = o(n)$ *uniformly in* z, $|z| < Z$

$$R_n(t, y) \to 0.$$

(ii) *Assume the conditions* (N) *hold.*
Choose k_n *in such a way that* $\nu \dfrac{\sqrt{k_n}}{y} \to 0$, $\sqrt{k_n}\rho(y) \to 0$ *and*

$$\sqrt{k_n}\left\{\frac{K(y)}{H(y)} - (p-q)\right\} \to 0 \qquad \textit{where} \quad k_n/n = H(y).$$

Then as $n \to \infty$

$$R_n(t, y) \to i\left(\frac{t}{\beta}\right)\gamma z - \frac{1}{2}\left(\frac{t}{\beta}\right)^2 \frac{\alpha}{2-\alpha}.$$

PROOF. (i) First, $X_n(y) = \dfrac{p-q}{1-\alpha} - (p-q) + o(1) - \rho(y) + \dfrac{\nu}{sy}$ while $(n-k)g = \gamma + \nu(n-k)/(ky)$. Note that $\rho(y) \to 0$ as $y \to \infty$. Hence

$$A_n(t, y) = \frac{s(n-k)}{k(1-s)}\{\gamma - \rho(y) + o(1)\} - \gamma + \frac{\nu}{y}\frac{(n-k)}{k(1-s)} - \frac{\nu(n-k)}{ky}.$$

The last two terms combine to $\nu \dfrac{(n-k)s}{k(1-s)} \cdot \dfrac{1}{y}$. Applying (27) we find that $A_n(t, y) \to 0$, since $y \to \infty$.

For $B_n(t, y)$ the last term tends to zero by (27). The first yields

$$(n-k)g^2 = \gamma^2/(n-k) + 2g\nu/(ky) + \nu^2(n-k)/(ky)^2.$$

For the last term we note that $ky^2 = n \cdot \left(\dfrac{k}{n}\right)y^2$. But $\dfrac{k}{n} \sim s$ and $y \sim s^{-1/\alpha}L^*(s^{-1/\alpha})$ by (iv) of Lemma 4. So $sy^2 \sim \dfrac{k}{n}y^2 \sim s^{1-2/\alpha}L^*(s^{-1/\alpha}) \to \infty$ since $\alpha < 2$.

Finally, the second term yields by (27) again

$$\frac{s(n-k)}{k(1-s)}gX_n(y) \sim g\left\{\gamma + o(1) + \frac{\nu}{sy}\right\} \sim \nu\frac{g}{sy}.$$

But $g/(sy) = \gamma/[(n-k)sy] + \nu/(ksy^2) \to 0$ since $(n-k)sy \sim nsy \sim ky \to \infty$ and $ksy^2 \to \infty$ since $sy^2 \to \infty$.

The terms in $C_n(t, y)$ have already been estimated.

(ii) Now $c_n = k\gamma$ while $d_n = \beta\sqrt{k}$. Hence after some algebra

$$\beta A_n(t, y) = \frac{s(n-k)}{(1-s)\sqrt{k}}\left(\frac{p-q}{1-\alpha} - \frac{K(y)}{H(y)}\right) - \gamma\sqrt{k} - \frac{s(n-k)}{(1-s)\sqrt{k}}\rho(y) + \nu\frac{s(n-k)}{(1-s)\sqrt{k}}\frac{1}{y}.$$

As

$$\frac{s(n-k)}{(1-s)\sqrt{k}} = \sqrt{k} + z\left\{\frac{n}{n-k}\right\}^{1/2}(1 + o(1)) + o(1)$$

by (27) we have to choose k in such a way that $\sqrt{k}\rho(y) \to 0$, $\nu\dfrac{\sqrt{k}}{y} \to 0$ and $\sqrt{k}\left(p - q - \dfrac{K(y)}{H(y)}\right) \to 0$. Hence $\beta A_n(t, y) \to \gamma z$.

The estimations for $B_n(t, y)$ and $C_n(t, y)$ are similar and are omitted. □

It is now easy to finish the proof of the following theorem.

THEOREM 2. (i) *Assume* (1) *holds. Then for any sequence* $k_n \to \infty$ *for which* $k_n = o(n)$ *as* $n \to \infty$.

$$\frac{1}{k_n} T_{n,k_n} \to_P \gamma.$$

(ii) *Assume* (1) *holds. Take any sequence* $k_n \to \infty$ *for which* $k_n = o(n)$ *as* $n \to \infty$ *such that for* y *defined by*

$$k_n/n = \{yL(y)\}^\alpha, \qquad \nu\sqrt{k_n}/y \to 0, \qquad \sqrt{k_n}\rho(y) \to 0, \qquad \sqrt{k_n}\left\{(p-q) - \frac{K(y)}{H(y)}\right\} \to 0.$$

Then

$$(\beta k_n)^{-1/2}\{T_{n,k_n} - k_n\gamma\} \to_{\mathscr{D}} \mathscr{N}(0, 1).$$

PROOF. (i) Let $\varepsilon > 0$ be arbitrary. Then with Lemma 6

$$|\varphi_{n,k_n}(t) - 1| \le |J_+| + |J_-| + \left| I_1 \int_{-T}^{T} I_2(z) I_3\, dz - 1 \right|$$

$$\le \frac{2C}{T^2} + \left| I_1 \int_{-T}^{T} I_2(z) I_3\, dz - \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{-z^2/2}\, dz \right| + 2\frac{1}{\sqrt{2\pi}} \int_{T}^{\infty} e^{-z^2/2}\, dz.$$

Now choose $T > \sqrt{C/\varepsilon}$ and such that $\frac{1}{\sqrt{2\pi}} \int_T^\infty e^{-z^2/2}\, dz < \varepsilon$. Then by Lemma 8

$$|\varphi_{n,k_n}(t) - 1| < 5\varepsilon.$$

(ii) The proof is similar and is omitted. □

COROLLARY 2. *If* F *is continuous and symmetric then as* $k_n \to \infty$ *such that* $k_n = o(n)$ *as* $n \to \infty$

$$\text{(i)} \quad \frac{1}{k_n} T_{n,k_n} \to_P 0;$$

$$\text{(ii)} \quad \left\{\frac{\alpha k_n}{2-\alpha}\right\}^{-1/2} T_{n,k_n} \to_{\mathscr{D}} \mathscr{N}(0, 1).$$

PROOF. This follows from $p = q$, $K(y) = 0$, $\nu = 0$ and $\rho(y) = 0$. □

4. An example. Theorem 2 can be applied to stable laws, to some Pareto and extreme value distributions, etc. In all these cases, a sequence k_n can be more or less specified.

THEOREM 3. *Assume that as* $x \to \infty$

$$\begin{cases} 1 - F(x) = px^{-\alpha} + bx^{-\alpha-\beta} + o(x^{-\alpha-\beta}) \\ F(-x) = qx^{-\alpha} + b'x^{-\alpha-\beta} + o(x^{-\alpha-\beta}). \end{cases}$$

where $\alpha \in (0, 1) \cup (1, 2)$, $0 < p \le 1$, $p + q = 1$, $0 < \beta \le \alpha$ *and* b *and* b' *real constants. Then the conditions of Theorem* 2 (ii) *are satisfied if* $k_n = o(n^\gamma)$ *where*

$$\gamma = \begin{cases} \min\left\{\dfrac{2\beta}{\alpha + 2\beta}, \dfrac{2(1-\alpha)}{2-\alpha}\right\} & \text{if } 0 < \alpha < 1 \\ \min\left\{\dfrac{2\beta}{\alpha + 2\beta}, \dfrac{2}{2+\alpha}\right\} & \text{if } 1 < \alpha < 2, \quad \nu \ne 0 \\ \dfrac{2\beta}{\alpha + 2\beta} & \text{if } 1 < \alpha < 2, \quad \nu = 0. \end{cases}$$

PROOF. (i) $0 < \alpha < 1$. Since $H(x) = x^{-\alpha} + (b + b')x^{-\alpha-\beta} + o(x^{-\alpha-\beta})$ as $x \to \infty$ the solution of $k/n = H(y)$ with respect to y is

$$y = (k/n)^{-1/\alpha}\left\{1 + o\left(\frac{k}{n}\right)\right\}.$$

Now $\nu = 0$; we only have to verify the other two conditions. $K(y) = (p - q)y^{-\alpha} + (b - b')y^{-\alpha-\beta} + o(y^{-\alpha-\beta})$ yields

$$\sqrt{k}\left\{(p-q) - \frac{(p-q) + (b-b')y^{-\beta} + o(y^{-\beta})}{1 + O(y^{-\beta})}\right\} = \sqrt{k}\, O(y^{-\beta}).$$

This tends to 0 if

$$\sqrt{k}\left(\frac{k}{n}\right)^{\beta/\alpha} \to 0 \qquad \text{or if} \quad k = o\{n^{2\beta/(\alpha+2\beta)}\}.$$

Now $\sqrt{k}\rho(y) \to 0$. Let A be fixed and positive. Then

$$\rho(y) = \frac{p-q}{1-\alpha} - \frac{1}{yH(y)}\int_0^A K(v)\,dv - \frac{(p-q)}{yH(y)}\int_A^y v^{-\alpha}(1 + O(v^{-\beta}))\,dv$$

or

$$|\rho(y)| \le C_1 y^{-\beta} + C_2 y^{\alpha-1}.$$

Hence $\sqrt{k}\rho(y) \to 0$ if

$$k = o\{n^{2\beta/(\alpha+2\beta)}\} \quad \text{and} \quad k = o(n^{2(1-\alpha)/(2-\alpha)}).$$

(ii) $1 < \alpha < 2$.

Again $y = \left(\frac{k}{n}\right)^{-1/\alpha}\left\{1 + o\left(\frac{k}{n}\right)\right\}$ so that $\sqrt{k}/y \sim \sqrt{k}\left(\frac{k}{n}\right)^{1/\alpha} \to 0$ if $k = o(n^{2/(\alpha+2)})$. Remark that this condition can be dropped if $\nu = 0$. The estimation of

$$\sqrt{k}\{(p-q) - K(y)/H(y)\}$$

yields again $\sqrt{k}\,y^{-\beta} \to 0$ as in case (i). Finally

$$\rho(y) = \frac{-1}{1 + O(y^{-\beta})}\int_1^\infty O(y^{-\beta})u^{-\alpha}\,du$$

yielding again $\sqrt{k}\,y^{-\beta} \to 0$. □

5. Some remarks.

(i) The statistical interpretation of Theorem 1 is surprising. For if we use $T_{n,k}$ as an estimator of γ then $T_{n,k}/k$ is asymptotically unbiased and has asymptotic variance β^2/k. Looking back at the definition of $T_{n,k}$ we realise that throwing away more outliers diminishes the asymptotic variance.

(ii) A possible explanation for the independence occuring in Theorem 1 is as follows: all order statistics not smaller than $|\tilde{X}_{n-k+1}|$ subdivide the original sequence $\{X_i\}_1^n$ into k disjoint pieces that are exchangeable and identically distributed. Formally let $T_0 = 1$, and for $m = 1, 2, \cdots, k$ let $T_m = \inf\{j > T_{m-1}: |X_j| \ge |\tilde{X}_{n-k+1}|\}$.

Then let $\vec{Y}_m = \{X_j \cdot |\tilde{X}_{n-k+1}|^{-1}, T_m < j < T_{m+1}\}$ for $m = 1, 2, \cdots, k-1$ and $\vec{Y}_k = \{X_j \cdot |\tilde{X}_{n-k+1}|^{-1}, T_k < j \le n \text{ and } 1 \le j < T_1\}$.

The sequence $\{\vec{Y}_j\}_1^k$ is then exchangeable. In the limit for $n \uparrow \infty$ the distribution of $\vec{Y}_m$ will not depend on k.

(iii) It is quite clear that the theorems in this article give some properties of estimators that can be used for the two parameters α and p in a stable law or more generally for a distribution in the domain of attraction of a stable law.

(iv) Theorems 2 and 3 seem to us the first examples of results dealing with the asymptotic normality of ratios of order statistics. Perhaps a more refined analysis of the Berry-Esseen or of the Edgeworth type might illuminate the role played by the slowly varying function L and as such by the sequence $\{k_n\}$.

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