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# A GLOBAL INTRINSIC CHARACTERIZATION OF BROWNIAN LOCAL TIME

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Let $B(t)$ be a Brownian motion with local time $s(t, x)$. Paul Lévy showed that for each $x$, $s(t, x)$ is a.s. equal to the limit as $\delta$ approaches zero of $\delta^{1/2}$ times the number of excursions from $x$, exceeding $\delta$ in length, that are completed by $B$ up to time $t$. The aim of the present paper is to show that the exceptional null sets, which may depend on $x$, can be combined into a single null set off which the above convergence is uniform in $x$. The proof uses nonstandard analysis to construct a simple combinatorial representation for the local time of a Brownian motion constructed by R. M. Anderson.

**1. Introduction and statement of results.** The local time $s(t, x)$ of a Brownian motion $B(t)$ was first introduced by P. Lévy who showed that if $n(t, x, \delta)$ is the number of excursions of $B$ away from $x$ that are greater than $\delta$ in length and are completed by time $t$, then for each $x$ in $R$

$$\lim_{\delta\to 0^+} \delta^{1/2} n(t, x, \delta) = 2(2/\pi)^{1/2} s(t, x) \qquad \text{for all } t \text{ in } [0, \infty) \quad \text{a.s.} \tag{1.1}$$

(see Itô and McKean (1965, page 43)). In fact it is easy to see that the convergence is uniform for $t$ in compact subsets of $[0, \infty)$. This characterization of local time is intrinsic in that $s(t, x)$ is recovered from the random set $Z(t, x) = \{s \le t \mid B(s) = x\}$. In Trotter (1958) the local time of $B$ is obtained as a continuous sojourn density of the Brownian path. That is, there exists a jointly continuous local time $s(t, x)$ such that

$$s(t, x) = \frac{1}{2}\frac{d}{dx}\int_0^t I_{(-\infty,x]}(B(s))\, ds \qquad \text{for all } (t, x) \text{ in } [0, \infty) \times R \quad \text{a.s.} \tag{1.2}$$

(Our definition of local time is the sojourn density with respect to the speed measure of a diffusion, whence the factor ½.) Hence $s(t, x)$ is defined for all $(t, x)$ simultaneously (although the definition is clearly not intrinsic), yet (1.1), like the other intrinsic descriptions given in Itô and McKean (1965), recovers the local time for only a single value of $x$, for $\omega$ outside an exceptional null set. It is the aim of this paper to show that the uncountably many null sets that arise for different values of $x$ in (1.1) may be combined into a single null set and hence establish a global intrinsic characterization of local time. In fact our main result states that the convergence in (1.1) is uniform in $x$ a.s.

THEOREM 1.1. *The following holds with probability one: For every $t' > 0$,*

$$\lim_{\delta\to 0^+} \sup_{(t,x)\in[0,t']\times R} | \delta^{1/2} n(t, x, \delta) - 2(2/\pi)^{1/2} s(t, x) | = 0.$$

The above result is established by means of an intuitive nonstandard representation of local time. In Anderson (1976) a Brownian motion $B(t)$ is constructed by taking the standard part of an infinitesimal random walk $X$ that takes a step of size $\pm(\Delta t)^{1/2}$, each

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with probability ½, for every time interval of length $\Delta t$, where $\Delta t$ is a positive infinitesimal. How should we define the local time of $X$? To answer this question, consider the heuristic formula

$$s(t, x) \doteq \frac{1}{2} \int_0^t I_{[x,x+\Delta x)}(B(s))\, ds(\Delta x)^{-1}.$$

Replace $B$ by $X$, $\int$ by $\sum$, $ds$ by $\Delta t$, $t$ by $j\Delta t$, $x$ by $k(\Delta t)^{1/2}$, and $\Delta x$ by $(\Delta t)^{1/2}$ to obtain the following definition for the local time of $X$:

$$L(j\Delta t, k(\Delta t)^{1/2}) = \frac{1}{2} \textstyle\sum_{i=0}^{j-1} I_{\{k(\Delta t)^{1/2}\}}(X(i\Delta t))(\Delta t)^{1/2}.$$

In Section 3 it is shown that the local time of $B$ may be obtained from $L$ via the standard part map, just as $B$ was constructed from $X$ in Anderson (1976). In Section 4, Theorem 1.1 is proved after two other global intrinsic characterizations of local time are established. More specifically, the characterizations of local time in terms of the limiting behavior (as $\delta$ approaches zero) of the Lebesgue measure of the set of excursions that are less than $\delta$ in length (see Itô and McKean (1965, page 43)), and of the Lebesgue measure of the set of points within $\delta/2$ of $Z(t, x)$ (see Kingman (1973)) are also shown to hold uniformly in $x$ with probability one. The proof of Theorem 1.1 then follows easily from these results. The key idea in our approach is that the zero set of $B$ may be analyzed by means of the zero set of $X$, which is a discrete set of points in $\{i\Delta t \mid i \in {}^*\mathbb{N}_0\}$ and lends itself to combinatorial methods. In particular, it is possible to list the zeros of $X$ in increasing order, which clearly cannot be done for a Brownian motion.

In Section 2 the probability space on which we will work is introduced and a brief description of the nonstandard constructions of Brownian motion and Lebesgue measure in Anderson (1976) is given.

The presentation assumes a basic knowledge of nonstandard analysis (see, for example, Stroyan and Luxemburg (1976)) and a familiarity with the results in Loeb (1975) and Anderson (1976). All the essential prerequisites may be found in Loeb (1979). It is nonetheless hoped that much of this work be accessible, at least on an intuitive level, to a reader with little or no knowledge of nonstandard analysis. In fact, after Theorem 4.7 is established, the proof of the main results described above are totally standard.

**2. Preliminaries.** The nonstandard real numbers, ${}^*R$, contains the real numbers $R$ and also contains infinitesimally small non-zero numbers which are less than every positive real in absolute value. Since ${}^*R$ contains the reciprocals of these infinitesimals, it also includes infinitely large numbers which are greater in absolute value than every positive real. The latter statement is also true of ${}^*\mathbb{Z}$ and ${}^*\mathbb{N}_0$, the sets of nonstandard integers and non-negative integers. Each element $x$ of ${}^*R$ that is not infinitely great is infinitesimally close to a unique real number called the standard part of $x$. For technical reasons we need to assume that ${}^*R$ is contained in an $\omega_1$-saturated enlargement. Although this assumption is necessary for proofs, the foregoing description is sufficient for an intuitive understanding of this paper.

The standard part map from ${}^*R^n$ to $R^n \cup \{\infty\}$ is denoted by $^\circ$, or occasionally by $st$, and we write $x_1 \approx x_2$ if $^\circ(x_1 - x_2) = 0$. For any subset $A$ of ${}^*R^n$, the set of near-standard points in $A$ (i.e. $\{x \in A \mid {}^\circ \| x \| < \infty\}$) is denoted by $ns(A)$. If $U$ is an internal subset of ${}^*R^n$ and $F$ is an internal function from $U$ to ${}^*R$, then $F$ is $S$-continuous on $U$ if whenever $u_1$ and $u_2$ are in $ns(U)$ and satisfy ${}^\circ u_1 = {}^\circ u_2$, then ${}^\circ F(u_1) = {}^\circ F(u_2) \in R$. It is easy to see that whenever $F$ is $S$-continuous on $U$, we may define a continuous function $f$ from ${}^\circ(ns(U))$ to $R$ by $f({}^\circ u) = {}^\circ F(u)$.

Our probability space is almost identical to that used in Anderson (1976). More specifically, consider the following setting:

(1) Let $\Delta t$ be a positive infinitesimal and define $T = \{i\Delta t \mid i \in {}^*\mathbb{N}_0\}$ and $S = \{i(\Delta t)^{1/2} \mid i$

$\in {}^*\mathbb{Z}\}$. Elements of $T$ are denoted by $\underset{\sim}{s}$, $\underset{\sim}{t}$, and $\underset{\sim}{u}$, while elements of $S$ are denoted by $\underset{\sim}{x}$ and $\underset{\sim}{y}$. We let $\mathscr{C}$ (respectively $\mathscr{D}$) denote the internal algebra of all internal subsets of $T$ (respectively $S$), and let $\lambda$ (respectively $\mu$) denote the internal measure defined on $(T, \mathscr{C})$ (respectively $(S, \mathscr{D})$) by $\lambda(A) = |A| \Delta t$ (respectively $\mu(A) = |A| (\Delta t)^{1/2}$), where $|A|$ is the internal cardinality of $A$.

(2) Let $\Omega$ be the set of internal mappings from $T$ to $\{-1, 1\}$ and $\mathscr{A}$ be the internal product $\sigma$-algebra. Sample points in $\Omega$ are written as $\omega = (\omega_{\underset{\sim}{t}})_{\underset{\sim}{t} \in T}$, and for each $\underset{\sim}{t}$ in $T$, the internal sub-$\sigma$-algebra of $\mathscr{A}$ generated by the collection of coordinate mappings $\{\omega_{\underset{\sim}{s}} \mid \underset{\sim}{s} \le \underset{\sim}{t}\}$ is denoted by $\mathscr{A}_{\underset{\sim}{t}}$. An internal probability measure $\bar{P}$ is then defined on $(\Omega, \mathscr{A})$ by

$$\bar{P}(\{\omega \mid (\omega_0, \cdots, \omega_{\underset{\sim}{t}}) \in A_0 \times \cdots \times A_{\underset{\sim}{t}}\}) = \textstyle\prod_{\underset{\sim}{s}=0}^{\underset{\sim}{t}} Q(A_{\underset{\sim}{s}}),$$

where $\{A_{\underset{\sim}{s}} \mid 0 \le \underset{\sim}{s} \le \underset{\sim}{t}\}$ is an internal sequence of subsets of $\{-1, 1\}$ and $Q$ assigns probability ½ to each element of $\{-1, 1\}$. Internal expectation with respect to $\bar{P}$ is denoted by $\bar{E}$.

(3) For each $\underset{\sim}{t}$ in $T$, a shift mapping $\theta_{\underset{\sim}{t}}$ from $\Omega$ to $\Omega$ is defined by $(\theta_{\underset{\sim}{t}}\omega)_{\underset{\sim}{s}} = \omega_{\underset{\sim}{t}+\underset{\sim}{s}}$. It is easy to see that for all $A$ in $\mathscr{A}$ and $\underset{\sim}{t}$ in $T$,

$$\bar{P}(\theta_{\underset{\sim}{t}}^{-1}(A) \mid \mathscr{A}_{\underset{\sim}{t}}) = \bar{P}(A). \tag{2.1}$$

(4) We define an infinitesimal random walk $X : T \times \Omega \to S$ by

$$X(\underset{\sim}{t}, \omega) = \textstyle\sum_{0 < \underset{\sim}{s} \le \underset{\sim}{t}} \omega_{\underset{\sim}{s}} (\Delta t)^{1/2}.$$

(5) Lebesgue measure on the line is denoted by $m$ and $\mathscr{B}$ represents the Borel sets of $R$, or of some measurable subset of $R$.

DEFINITION 2.1. If $U$ is an internal subset of ${}^*R^n$, an *internal stochastic process* on $U$ is an internal function $Y$ from $U \times \Omega$ to ${}^*R$ such that for each $u$ in $U$ and $B$ in ${}^*\mathscr{B}$, $\{\omega \mid X(u, \omega) \in B\} \in \mathscr{A}$.

The fundamental construction in Loeb (1975) allows us to extend ${}^\circ\bar{P}$ to a unique probability measure $L(\bar{P})$ on $\sigma(\mathscr{A})$, the standard $\sigma$-algebra generated by $\mathscr{A}$. We denote the completion of $(\Omega, \sigma(\mathscr{A}), L(\bar{P}))$ by $(\Omega, \mathscr{F}, P)$. Similarly we may construct standard measure spaces $(T, \sigma(\mathscr{C}), L(\lambda))$ and $(S, \sigma(\mathscr{D}), L(\mu))$ with completions $(T, L(\mathscr{C}), L(\lambda))$ and $(S, L(\mathscr{D}), L(\mu))$, respectively. (We have used Corollary 1 in Henson (1979) to obtain the uniqueness of $L(\lambda)$ and $L(\mu)$.)

DEFINITION 2.2. An internal stochastic process $Y$ on $U$ is $S$-continuous if $Y(\cdot, \omega)$ is $S$-continuous on $U(P$—a.s.).

It follows immediately from our earlier remarks and Theorem 2 in Loeb (1975) that if $Y$ is an $S$-continuous stochastic process on $U$ then we may define a (standard) stochastic process on $(\Omega, \mathscr{F}, P)$ with continuous paths on its index set ${}^\circ(ns(U))$ by

$$y({}^\circ u, \omega) = \begin{cases} {}^\circ Y(u, \omega) & \text{if } Y(\cdot, \omega) \text{ is } S\text{-continuous on } U \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

NOTATION 2.3. We write $y = st(Y)$ for the process $y$ defined by (2.2). This of course includes the deterministic case.

In Anderson (1976) it is shown that $X$ is an $S$-continuous process on $T$ and that the continuous stochastic process $B = st(X)$ is a Brownian motion on $(\Omega, \mathscr{F}, P)$. The same article has a nonstandard construction of Lebesgue measure which we now present in a slightly modified form. Since $ns(T) = \cup_{n=1}^{\infty} {}^*[0, n] \cap T \in L(\mathscr{C})$, we may consider $L(\lambda)$ as a measure on $L_{ns}(\mathscr{C})$, the trace of $L(\mathscr{C})$ on $ns(T)$. If a similar convention is made for $L(\mu)$, then Theorem 14 in Anderson (1976) implies the following:

THEOREM 2.4. *The mappings* ${}^\circ : (ns(T), L_{ns}(\mathscr{C}), L(\lambda)) \to ([0, \infty), \mathscr{B}, m)$ *and* ${}^\circ : (ns(S), L_{ns}(\mathscr{D}), L(\mu)) \to (\mathscr{R}, \mathscr{B}, m)$ *are measurable and measure-preserving.* □

We will also use the nonstandard representation of the Itô integral with respect to $B$ that is developed in [1].

## 3. The Nonstandard Construction of Local Time.

DEFINITION 3.1. The **-local time of X* is the internal stochastic process $L$ on $T \times S$ defined by

$$L(\underset{\sim}{t}, \underset{\sim}{x}, \omega) = \tfrac{1}{2} \sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2}.$$

A related internal stochastic process on $T \times S$ is defined by

$$J(\underset{\sim}{t}, \underset{\sim}{x}, \omega) = \sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{(\underset{\sim}{x},\infty)}(X(\underset{\sim}{s}))\omega_{\underset{\sim}{s}+\Delta t}(\Delta t)^{1/2}.$$

Hence $L(\underset{\sim}{t}, \underset{\sim}{x}, \omega)$ is just the number of times $X$ visits $\underset{\sim}{x}$ before $\underset{\sim}{t}$. Intuitively local time measures the length of time $X$ spends at $\underset{\sim}{x}$. The significance of $L(\underset{\sim}{t}, \underset{\sim}{x}, \omega)$ is that it captures the intuitive idea exactly, whereas in the usual models one has to resort to less direct methods (e.g. the Radon-Nikodym theorem).

Note that $J(\cdot, \underset{\sim}{x}, \omega)$ behaves exactly like $X$ when $X > \underset{\sim}{x}$ and is constant when $X \leq \underset{\sim}{x}$.

Our immediate objectives are first to show that $L$ is an $S$-continuous process on $T \times S$ and then prove that $st(L)$ is the local time of $B$. The first result is proved by means of the following nonstandard version of a classical result of Kolmogorov.

LEMMA 3.2. *Suppose $b \in (0, \infty)$, $c \in (1, \infty)$, and $Y$ is an internal stochastic process on $T \times S$ satisfying the folowing conditions:*

*(1) For each $\underset{\sim}{x}$ in $ns(S)$, $Y(\cdot, \underset{\sim}{x})$ is an $S$-continuous process on $T$.*

*(2) For each $\underset{\sim}{t}$ in $ns(T)$, there exists a positive real constant $c(\underset{\sim}{t})$ such that whenever $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are infinitesimally close elements of $ns(S)$, we have*

$$\bar{E}(\max_{\underset{\sim}{s}\leq \underset{\sim}{t}} | Y(\underset{\sim}{s}, \underset{\sim}{x}) - Y(\underset{\sim}{s}, \underset{\sim}{x}') |^b) \leq c(\underset{\sim}{t}) | \underset{\sim}{x} - \underset{\sim}{x}' |^c.$$

*Then $Y$ is an $S$-continuous process on $T \times S$.* □

The proof of the above result is virtually identical to the classical proof (see for example [2, Th.4.1.8, page 164]) and may be found in [12, page 196]. Note that there is no need to select a separable version of the process $Y$ since the parameter set $T \times S$ is a *-countable set of points.

LEMMA 3.3. *There exist real constants $c_1$ and $c_2$ such that for all $(\underset{\sim}{t}, \underset{\sim}{x})$ in $T \times S$:*

*(a) $\bar{P}(X(\underset{\sim}{t}) = \underset{\sim}{x}) \leq c_1(\Delta t)^{1/2}(\underset{\sim}{t} + \Delta t)^{-1/2}$*

*(b) $\bar{E}(L(\underset{\sim}{t}, \underset{\sim}{x})^2) \leq c_2 \underset{\sim}{t}$.*

PROOF. Part (a) is a trivial application of the Transfer Principle and Stirling's Formula. For (b) we note that

$$\begin{aligned}
\bar{E}((L(\underset{\sim}{t}, \underset{\sim}{x}))^2) &= \tfrac{1}{4}\, \bar{E}((\textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2})^2) \\
&\leq \tfrac{1}{2} \textstyle\sum_{\underset{\sim}{s}_1=0}^{\underset{\sim}{t}-\Delta t} \sum_{\underset{\sim}{s}_2=\underset{\sim}{s}_1}^{\underset{\sim}{t}-\Delta t} \bar{E}(I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}_1))I_{\{0\}}(X(\underset{\sim}{s}_2) - X(\underset{\sim}{s}_1))\Delta t) \\
&\leq c_1^2/2 \textstyle\sum_{\underset{\sim}{s}_1=0}^{\underset{\sim}{t}-\Delta t} \sum_{\underset{\sim}{s}_2=\underset{\sim}{s}_1}^{\underset{\sim}{t}-\Delta t} (\underset{\sim}{s}_1 + \Delta t)^{-1/2}(\underset{\sim}{s}_2 - \underset{\sim}{s}_1 + \Delta t)^{-1/2}(\Delta t)^2 \\
&\leq c_1^2/2 \int_0^{\underset{\sim}{t}} \int_{s_1}^{\underset{\sim}{t}} s_1^{-1/2}(s_2 - s_1)^{-1/2}\, {}^*ds_2\, {}^*ds_1 \\
&\leq 2c_1^2 \underset{\sim}{t}.
\end{aligned}$$

(Here $^*ds$ denotes internal Lebesgue integration.) □

We are now ready to prove the $S$-continuity of $L$. Our approach is analogous to that in McKean (1969, pages 68–71) in that Lemma 3.2 is used to establish the $S$-continuity of $J$ and then the nonstandard version of a well-known formula of Tanaka gives us the $S$-continuity of $L$.

THEOREM 3.4. (a) *The internal stochastic process $J$ is S-continuous on $T \times S$ and for each $\underset{\sim}{x}$ in $ns(S)$ satisfies*

$$(3.1)\qquad {}^\circ J(\underset{\sim}{t}, \underset{\sim}{x}) = \int_0^{{}^\circ \underset{\sim}{t}} I_{({}^\circ \underset{\sim}{x},\infty)}(B(s))\, dB(s) \qquad \textit{for all } \underset{\sim}{t} \textit{ in } ns(T) \text{ a.s.}$$

(*b*) *For almost all $\omega$ we have*

$$(3.2)\qquad (X(\underset{\sim}{t}) - \underset{\sim}{x})^+ - (-\underset{\sim}{x})^+ \approx J(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x})$$

*for all $(\underset{\sim}{t}, \underset{\sim}{x})$ in $ns(T \times S)$.*

(*c*) *The process $L$ is S-continuous on $T \times S$.*

REMARK 3.5. Equation (3.2) may be rewritten as

$$(3.3)\qquad (X(\underset{\sim}{t}) - \underset{\sim}{x})^+ - (-\underset{\sim}{x})^+ \approx \int_0^{\underset{\sim}{t}} I_{(\underset{\sim}{x},\infty)}(X(\underset{\sim}{s}))\, dX(\underset{\sim}{s}) + L(\underset{\sim}{t}, \underset{\sim}{x}) \qquad \text{for all } (\underset{\sim}{t}, \underset{\sim}{x}) \text{ in } ns(T \times S) \text{ a.s.,}$$

which is somewhat more transparent than the standard form of Tanaka's formula,

$$(3.4)\qquad (B(t) - x)^+ - (-x)^+ = \int_0^t I_{(x,\infty)}(B(s))\, dB(s) + s(t, x) \text{ for all } t \text{ in } [0, \infty) \text{ a.s., for all } x.$$

Indeed, when $X(\underset{\sim}{t})$ exceeds $\underset{\sim}{x}$, $(X(\underset{\sim}{t}) - \underset{\sim}{x})^+$ behaves exactly like $X$, when $X(\underset{\sim}{t})$ is less than $\underset{\sim}{x}$, $(X(\underset{\sim}{t}) - \underset{\sim}{x})^+$ remains constant at zero, and when $X(\underset{\sim}{t})$ equals $\underset{\sim}{x}$, $(X(\underset{\sim}{t}) - \underset{\sim}{x})^+$ will increase by $(\Delta t)^{1/2}$ if $X(\underset{\sim}{t})$ does, and will remain constant otherwise. This last contribution leads us to add on the sum of the upcrossings of $X$ from $\underset{\sim}{x}$ to $\underset{\sim}{x} + (\Delta t)^{1/2}$ up to time $\underset{\sim}{t}$, which is infinitesimally close to $L(\underset{\sim}{t}, \underset{\sim}{x})$ since, starting from $\underset{\sim}{x}$, $X$ is equally likely to go up or down.

PROOF. (a) It suffices to show that the conditions of Lemma 3.2 are satisfied by $J$ with $b = 4$ and $c = 2$. Since $P({}^\circ X(\underset{\sim}{s}) = {}^\circ \underset{\sim}{x}) = 0$ for each $(\underset{\sim}{s}, \underset{\sim}{x})$ in $ns(T \times S)$ with ${}^\circ \underset{\sim}{s} > 0$, it follows that $I_{(\underset{\sim}{x},\infty)}(X(\underset{\sim}{s}))$ is a 2-lifting of $I_{({}^\circ \underset{\sim}{x},\infty)}(B(s))$ in the sense of Anderson (1976, Definition 30). Therefore by Anderson (1976, Theorems 33 and 35), for each $\underset{\sim}{x}$ in $ns(S)$, $J(\cdot, \underset{\sim}{x})$ is an $S$-continuous process on $T$ and satisfies (3.1). To verify condition (2) of Lemma 3.2, we fix $\underset{\sim}{t}$ in $ns(T)$ and $\underset{\sim}{x} < \underset{\sim}{x}'$ in $ns(S)$ such that $\underset{\sim}{x} \approx \underset{\sim}{x}'$. Since $\{(J(\underset{\sim}{t}, \underset{\sim}{x}) - J(\underset{\sim}{t}, \underset{\sim}{x}'), \mathscr{A}_{\underset{\sim}{t}}) \mid \underset{\sim}{t} \in T\}$ is an internal martingale, a well-known square function inequality for martingales (Burkholder (1973, Theorem 15.1)) and the Transfer Principle imply that for some real constant $c$

$$\begin{aligned}
&\bar{E}(\max_{\underset{\sim}{s} \le \underset{\sim}{t}} (J(\underset{\sim}{s}, \underset{\sim}{x}) - J(\underset{\sim}{s}, \underset{\sim}{x}'))^4)\\
&\quad\le c\bar{E}((\textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{(\underset{\sim}{x},\underset{\sim}{x}']}(X(\underset{\sim}{s}))(\omega_{\underset{\sim}{s}+\Delta t})^2 \Delta t)^2)\\
&\quad\le 2c\bar{E}(\textstyle\sum_{\underset{\sim}{s}_1=0}^{\underset{\sim}{t}-\Delta t} \sum_{\underset{\sim}{s}_2=\underset{\sim}{s}_1}^{\underset{\sim}{t}-\Delta t} I_{(\underset{\sim}{x},\underset{\sim}{x}']}(X(\underset{\sim}{s}_1)) I_{(\underset{\sim}{x},\underset{\sim}{x}']}(X(\underset{\sim}{s}_2))(\Delta t)^2)\\
&\quad\le 2c\bar{E}(\textstyle\sum_{\underset{\sim}{s}_1=0}^{\underset{\sim}{t}-\Delta t} \sum_{\underset{\sim}{s}_2=\underset{\sim}{s}_1}^{\underset{\sim}{t}-\Delta t} I_{(\underset{\sim}{x},\underset{\sim}{x}']}(X(\underset{\sim}{s}_1)) I_{(\underset{\sim}{x}-\underset{\sim}{x}',\underset{\sim}{x}'-\underset{\sim}{x})}(X(\underset{\sim}{s}_2) - X(\underset{\sim}{s}_1))(\Delta t)^2)\\
&\quad\le 4cc_1^2(\underset{\sim}{x}' - \underset{\sim}{x})^2(\textstyle\sum_{\underset{\sim}{s}_1=0}^{\underset{\sim}{t}-\Delta t} \sum_{\underset{\sim}{s}_2=\underset{\sim}{s}_1}^{\underset{\sim}{t}-\Delta t} (\underset{\sim}{s}_1 + \Delta t)^{-1/2}(\underset{\sim}{s}_2 - \underset{\sim}{s}_1 + \Delta t)^{-1/2}(\Delta t)^2),
\end{aligned}$$

where we have used Lemma 3.3 (a) to obtain the last line. The $*$-finite sum in the above expression is bounded above (and is infinitesimally close to) the integral $\int_0^{\underset{\sim}{t}} \int_{s_1}^{\underset{\sim}{t}} (s_1(s_2 - s_1))^{-1/2}\, {}^*ds_2\, {}^*ds_1$, which is less than $4\underset{\sim}{t}$. Condition (2) of Lemma 3.2 now follows from the above bounds and the proof of (a) is complete.

(b) If $Y(\underset{\sim}{t}, \underset{\sim}{x}) = (X(\underset{\sim}{t}) - \underset{\sim}{x})^+ - (-\underset{\sim}{x})^+$, then a brief examination of the different possible cases shows that

$$Y(\underset{\sim}{t} + \Delta t, \underset{\sim}{x}) = Y(\underset{\sim}{t}, \underset{\sim}{x}) + I_{(\underset{\sim}{x},\infty)}(X(\underset{\sim}{t}))\omega_{\underset{\sim}{t}+\Delta t}(\Delta t)^{1/2} + I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{t}))I_{\{1\}}(\omega_{\underset{\sim}{t}+\Delta t})(\Delta t)^{1/2}.$$

By induction one obtains

$$Y(\underset{\sim}{t}, \underset{\sim}{x}) = \textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{(\underset{\sim}{x},\infty)}(X(\underset{\sim}{s}))\omega_{\underset{\sim}{s}+\Delta t}(\Delta t)^{1/2} + \sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))I_{\{1\}}(\omega_{\underset{\sim}{s}+\Delta t})(\Delta t)^{1/2}$$

$$= J(\underline{t}, \underline{x}) + L(\underline{t}, \underline{x}) + \textstyle\sum_{\underline{s}=0}^{\underline{t}-\Delta t} I_{\{\underline{x}\}}(X(\underline{s}))(I_{\{1\}}(\omega_{\underline{s}+\Delta t}) - ½)(\Delta t)^{1/2}$$

$$= J(\underline{t}, \underline{x}) + L(\underline{t}, \underline{x}) + ½ \textstyle\sum_{\underline{s}=0}^{\underline{t}-\Delta t} I_{\{\underline{x}\}}(X(\underline{s}))\omega_{\underline{s}+\Delta t}(\Delta t)^{1/2}$$

$$= J(\underline{t}, \underline{x}) + L(\underline{t}, \underline{x}) + ½(J(\underline{t}, \underline{x} - (\Delta t)^{1/2}) - J(\underline{t}, \underline{x})).$$

The $S$-continuity of $J$ established in (a) now gives us (3.2).

Part (c) follows immediately from (a), (b) and the $S$-continuity of $X$. □

At this point we could use Tanaka's formula (3.4), (3.1) and (3.2) to immediately conclude that $st(L)$ is the local time of $B$. With only a little more effort, however, we can prove directly that $st(L)$ is a continuous sojourn density for $B$ and obtain Tanaka's formula as a corollary.

THEOREM 3.6. *The process $st(L)(t, x)$ is the jointly continuous local time $s(t, x)$ of $B$, that is, it satisfies $st(L)(t, x) = ½\,(d/dx) \int_0^t I_{(-\infty,x]}(B(s))\,ds$ for all $(t, x)$ in $[0, \infty) \times R$* a.s.

PROOF. This is an immediate consequence of Theorem 3.4(c) and the following lemma.

LEMMA 3.7. *Let $G : T \to S$ be internal and $S$-continuous. Assume that $F(\underline{t}, \underline{x}) = ½ \sum_{\underline{s}<\underline{t}} I_{\{\underline{x}\}}(G(\underline{s}))(\Delta t)^{1/2}$ is $S$-continuous on $T \times S$. Then*

$$st(F)(t, x) = \frac{1}{2}\frac{d}{dx}\int_0^t I_{(-\infty,x]}(st(G)(s))\,ds.$$

PROOF. Let $f = st(F)$, $g = st(G)$ and $D$ be a countable dense subset of $R$ such that $\int_0^\infty I_{\{x\}}(g(s))\,ds = 0$ for all $x$ in $D$. (The existence of $D$ is clear since a $\sigma$-finite measure can assign positive mass to at most countably many singletons.) Fix $(t, x)$ in $[0, \infty) \times D$ and $(\underline{t}, \underline{x})$ in $T \times S$ such that $°(\underline{t}, \underline{x}) = (t, x)$. Theorem 2.4 implies that

$$\begin{aligned}\int_0^t I_{(-\infty,x)}(g(s))\,ds &= \int_T I_{[0,\underline{t})}(\underline{s}) I_{(-\infty,x)}(°G(\underline{s}))\,dL(\lambda)\\ &\le \int_T I_{[0,\underline{t})}(\underline{s}) I_{(-\infty,\underline{x}]}(G(\underline{s}))\,dL(\lambda)\\ &\le \int_T I_{[0,\underline{t})}(\underline{s}) I_{(-\infty,x]}(°G(\underline{s}))\,dL(\lambda)\\ &= \int_0^t I_{(-\infty,x]}(g(s))\,ds.\end{aligned}$$

Since $x$ is in $D$, the left and right-hand sides are equal, and therefore

$$\int_0^t I_{(-\infty,x]}(g(s))\,ds = \int_T I_{[0,\underline{t})}(\underline{s}) I_{(-\infty,\underline{x}]}(G(\underline{s}))\,dL(\lambda)$$

$$= °\textstyle\sum_{\underline{s}<\underline{t}} I_{(-\infty,\underline{x}]}(G(\underline{s}))\Delta t \qquad \text{(by Loeb (1975, Theorem 3))}$$

$$= °\textstyle\sum_{\underline{s}<\underline{t}} \sum_{\underline{y}\le\underline{x}} I_{\{\underline{y}\}}(G(\underline{s}))(\Delta t)^{1/2}(\Delta t)^{1/2}$$

$$= 2\,°\textstyle\sum_{\underline{y}\le\underline{x}} F(\underline{t}, \underline{y})(\Delta t)^{1/2}$$

$$= 2\int_{ns(S)} I_{(-\infty,x)}(°\underline{y})\,°F(\underline{t}, \underline{y})\,dL(\mu) \qquad \text{(by Loeb (1975, Theorem 3))}$$

$$= 2\int_{ns(S)} I_{(-\infty,x)}(°\underline{y}) f(t, °\underline{y})\,dL(\mu).$$

By applying Theorem 2.4 on the right-hand side, we obtain

$$\int_0^t I_{(-\infty,x]}(g(s))\,ds = 2\int_{-\infty}^x f(t,y)\,dy \tag{3.5}$$

for all $(t, x)$ in $[0, \infty) \times D$. Since both sides of (3.5) are right-continuous in $x$, (3.5) holds for all $(t, x)$ in $[0, \infty) \times R$ and the result follows by differentiating (3.5) with respect to $x$. □

REMARK 3.8. Tanaka's formula (3.4) now follows immediately from the above result and (3.1) by applying the standard part map to (3.2).

Although we have only proven Tanaka's formula and the existence of a jointly continuous local time for the particular Brownian motion $B = st(X)$, these results now follow easily for every continuous Brownian motion. Indeed, it is easy to see that Tanaka's formula and the existence and joint continuity of $s(t, x)$ is a property of Wiener measure on the space of continuous function on $[0, \infty)$, and not the particular Brownian motion under consideration.

**4. The global intrinsic characterization.** In order to study the relationship between the local time $s(t, x)$ and the random set $Z(t, x) = \{s \le t \mid B(s) = x\}$, we will compare the *-local time $L(\underset{\sim}{t}, \underset{\sim}{x})$ with certain functions of the random set $\{\underset{\sim}{s} < \underset{\sim}{t} \mid X(\underset{\sim}{s}) = \underset{\sim}{x}\}$ for $(\underset{\sim}{t}, \underset{\sim}{x}) \approx (t, x)$. In studying $Z(t, x)$ the following notation is used.

NOTATION 4.1.
(1) $I(t, x) = \{I \subset [0, t] \mid I \text{ is a connected component of } Z(\infty, x)^c\}$
(2) $a(t, x, \delta) = \{s \in R \,\|\, s - u| < \delta/2 \text{ for some } u \text{ in } Z(t, x)\}$
(3) $a'(t, x, \delta) = \cup \{I \mid I \in I(t, x), m(I) \le \delta\}$
(4) $n(t, x, \delta) = \operatorname{card}\{I \in I(t, x) \mid m(I) > \delta\}$.

It follows easily from the characterization of $s(t, x)$ as the "mesure du voisinage" of $B$ (i.e. (1.1)) that for each $x$ in $R$

$$\lim_{\delta\to 0^+} m(a'(t, x, \delta))\delta^{-1/2} = 2(2/\pi)^{1/2} s(t, x) \quad \text{for all } t \ge 0 \text{ a.s.} \tag{4.1}$$

(see Lévy (1948, page 224)). Since the existence of a sojourn density for $B$ implies that for almost all $\omega$, $Z(t, x)$ has Lebesgue measure zero for all $(t, x)$, Theorem 1 of Kingman (1973) implies that with probability one,

$$m(a(t, x, \delta)) = \delta I_{\{Z(t,x)\neq\emptyset\}} + \delta n(t, x, \delta) + m(a'(t, x, \delta)), \quad \text{for all } (t, x) \text{ in } [0, \infty) \times R. \tag{4.2}$$

Combining (1.1), (4.1) and (4.2), we obtain the intrinsic description of local time in Kingman (1973) (in the special case of a Brownian motion), that is, for each $x$

$$\lim_{\delta\to 0^+} m(a(t, x, \delta))\delta^{-1/2} = 4(2/\pi)^{1/2} s(t, x) \quad \text{for all } t \ge 0 \text{ a.s.} \tag{4.3}$$

As was the case for (1.1), both (4.2) and (4.3) are valid for $\omega$ outside an exceptional null set that may depend on $x$. In order to prove Theorem 1.1 we will first show that the convergence in (4.3) holds uniformly in $x$ a.s., then establish the same result for (4.1), and finally obtain Theorem 1.1 from these results by means of (4.2). The first step is to define internal stochastic processes that represent

$$m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2} \qquad \text{and} \qquad m(a'(t, x, \delta) \cup a'(t, x', \delta))\delta^{-1/2}.$$

NOTATION 4.2. *If* $\underset{\sim}{x}, \underset{\sim}{x}' \in S$, $\underset{\sim}{t}, \underset{\sim}{\delta} \in T$, *and* $\underset{\sim}{\delta} > 0$, then
(a) $T(0) = \min\{\underset{\sim}{t} > 0 \mid X(\underset{\sim}{t}) = 0\}$,
$T(\underset{\sim}{x}, \underset{\sim}{x}') = \min\{\underset{\sim}{t} > 0 \mid X(\underset{\sim}{t}) = \underset{\sim}{x} \text{ or } \underset{\sim}{x}'\}$,
(b) $U(\underset{\sim}{x}, \underset{\sim}{\delta}) = (T(0, \underset{\sim}{x}) \wedge \underset{\sim}{\delta})\underset{\sim}{\delta}^{-1/2}$, $U'(\underset{\sim}{x}, \underset{\sim}{\delta}) = T(0, \underset{\sim}{x})\underset{\sim}{\delta}^{-1/2} I_{\{T(0,\underset{\sim}{x}) \le \underset{\sim}{\delta}\}}$,

(c) $M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) = \sum_{\underline{s}=0}^{\underline{t}-\Delta t} (I_{\{\underline{x}\}}(X(\underline{s}))(U(\underline{x}' - \underline{x}, \underline{\delta})\circ\theta_{\underline{s}}) + I_{\{\underline{x}'\}}(X(\underline{s}))(U(\underline{x} - \underline{x}', \underline{\delta})\circ\theta_{\underline{s}})),$
(d) $M'(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) = \sum_{\underline{s}=0}^{\underline{t}-\Delta t} (I_{\{\underline{x}\}}(X(\underline{s}))(U'(\underline{x}' - \underline{x}, \underline{\delta})\circ\theta_{\underline{s}}) + I_{\{\underline{x}'\}}(X(\underline{s}))(U'(\underline{x} - \underline{x}', \underline{\delta})\circ\theta_{\underline{s}})),$
(e) $M''(\underline{t}, \underline{x}, \underline{\delta}) = \sum_{\underline{s}=0}^{\underline{t}-\Delta t} I_{\{\underline{x}\}}(X(\underline{s}))(U'(0, \underline{\delta})\circ\theta_{\underline{s}}),$
where $\theta_{\underline{s}}$ is the shift mapping defined at the beginning of Section 2.

LEMMA 4.3. *For each $(t, x, x', \delta)$ in $[0, \infty) \times R^2 \times (0, \infty)$, there exists a null set $\Lambda$ such that if $\omega \notin \Lambda$ and $(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) \in T \times S^2 \times T$ and satisfies ${}^\circ(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) = (t, x, x', \delta)$ and $\underline{x} \neq \underline{x}'$, then*

(a) $|{}^\circ M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) - m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2}| \le \delta^{1/2}$,
(b) $|{}^\circ M''(\underline{t}, \underline{x}, \underline{\delta}) - m(a'(t, x, \delta))\delta^{-1/2}| \le \delta^{1/2}$
(c) $m(a'(t, x, \delta) \cup a'(t, x', \delta))\delta^{-1/2} \le {}^\circ M'(\underline{t}, \underline{x}, \underline{x}', \underline{\delta})$.

PROOF. If $(t, x, x', \delta)$ is fixed, let $\Lambda^c$ be the set of probability one for which the following conditions are satisfied:
($C_1$) There are no local extrema of $B$ in $Z(\infty, x) \cup Z(\infty, x')$.
($C_2$) There is no $I$ in $I(\infty, x) \cup I(\infty, x')$ such that $m(I) = \delta$.
($C_3$) Both $x$ and $x'$ are distinct from $B(t)$.
($C_4$) $\limsup_{s\to\infty} B(s) = +\infty$, and $\liminf_{s\to\infty} B(s) = -\infty$.
($C_5$) The local time (in the sense of (1.2)) exists and is jointly continuous, and $X$ is $S$-continuous on $T$.
That ($C_1$) and ($C_2$) are satisfied by almost all Brownian paths are well-known properties of Brownian motion. If ($C_2$) is false, then there exists a rational $r$ such that the length of the excursion away from $x$ containing $r$ is $\delta$. Since the length of the excursion containing $r$ has a density (Lévy (1948, Theorem 44.4)), ($C_2$) holds a.s. That ($C_1$) holds a.s., is left for the reader to check.

Choose $\omega$ in $\Lambda^c$ and $(\underline{t}, \underline{x}, \underline{x}', \underline{\delta})$ as in the statement of the lemma. We may assume without loss of generality that $\underline{\delta} = 2\gamma\Delta t$ for some $\gamma$ in ${}^*N$ since ${}^\circ\gamma = \infty$ and

$$(2\gamma/2\gamma + 1)^{1/2} M(\underline{t}, \underline{x}, \underline{x}', 2\gamma\Delta t) \le M(\underline{t}, \underline{x}, \underline{x}', (2\gamma + 1)\Delta t)$$
$$\le (2\gamma + 2/2\gamma + 1)^{1/2} M(\underline{t}, \underline{x}, \underline{x}', (2\gamma + 2)\Delta t),$$

would then give (a) for every $\underline{\delta} \approx \delta$, and similarly for (b) and (c).

(a) Let $\{\underline{s} < \underline{t} \mid X(\underline{s}) = \underline{x} \text{ or } \underline{x}'\} = \{\underline{t}_0, \ldots, \underline{t}_{N-1}\}$, where $\underline{t}_i < \underline{t}_{i+1}$ and define $\underline{t}_N = \min\{\underline{s} \ge \underline{t} \mid X(\underline{s}) = \underline{x} \text{ or } \underline{x}'\}$ (by ($C_4$)${}^\circ\underline{t}_N < \infty$). If

$$A = \cup_{i=0}^{N-1}((\underline{t}_i, \underline{t}_i + \underline{\delta}/2] \cup (\underline{t}_{i+1} - \underline{\delta}/2, \underline{t}_{i+1}]) \cap (\underline{t}_i, \underline{t}_{i+1}] \cap T,$$

then

$$\lambda(A)\underline{\delta}^{-1/2} = \sum_{i=0}^{N-1} ((\underline{t}_{i+1} - \underline{t}_i) \wedge \underline{\delta})\underline{\delta}^{-1/2} = M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}). \tag{4.4}$$

If $\underline{u} \in A - (\underline{t}_N - \underline{\delta}/2, \underline{t}_N]$, then $|\underline{u} - \underline{t}_i| \le \underline{\delta}/2$ for some $i \le N - 1$. Therefore ${}^\circ\underline{t}_i \le t$, $B({}^\circ\underline{t}_i) = {}^\circ X(\underline{t}_i) \in \{x, x'\}$ and $|{}^\circ\underline{u} - {}^\circ\underline{t}_i| \le \delta/2$, which together imply that ${}^\circ\underline{u} \in a(t, x, \delta') \cup a(t, x', \delta')$ for all $\delta' > \delta$. Hence, for all $\delta' > \delta$

$$\begin{aligned} {}^\circ M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) &= {}^\circ(\lambda(A)\underline{\delta}^{-1/2}) \le \delta^{1/2}/2 + L(\lambda)(st^{-1}(a(t, x, \delta') \cup a(t, x', \delta')))\delta^{-1/2} \\ &= \delta^{1/2}/2 + m(a(t, x, \delta') \cup a(t, x', \delta'))\delta^{-1/2} \end{aligned} \tag{4.5}$$

(the last by Theorem 2.4). The existence of a sojourn density for $B$ implies that

$$m(\{s \mid |s - u| = \delta/2 \text{ for some } u \text{ in } Z(t, x) \cup Z(t, x')\}) = 0$$

and therefore we may let $\delta'$ approach $\delta$ in (4.5) to obtain

$${}^\circ M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta}) \le \delta^{1/2}/2 + m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2}. \tag{4.6}$$

To obtain an inequality in the opposite direction, suppose ${}^\circ\underline{s} \in a(t, x, \delta) \cup a(t, x', \delta)$ and $t > {}^\circ\underline{s} > {}^\circ\underline{t}_0 = \inf\{s \mid B(s) \in \{x, x'\}\}$. The last equality holds by ($C_1$) and implies that we are excluding a subset of $a(t, x, \delta) \cup a(t, x', \delta)$ of Lebesgue measure no more than $\delta$. There exists $u$ in $Z(t, x) \cup Z(t, x')$ such that $|u - {}^\circ\underset{\sim}{s}| < \delta/2$ and $(C_3)$ implies that $u < t$. By $(C_1)$ there exists $\underset{\sim}{u} \approx u$ such that $X(\underset{\sim}{u}) = \underset{\sim}{x}$ or $\underset{\sim}{x}'$, since otherwise $X(\underset{\sim}{u}) > \underset{\sim}{x}$ (or $\underset{\sim}{x}'$) or $X(\underset{\sim}{u}) < \underset{\sim}{x}$ (or $\underset{\sim}{x}'$) for all $\underset{\sim}{u} \approx u$, hence for all $u$ within $n^{-1}$ of $u$ (for some $n$ in $N$), and therefore $B$ would have a local extremum in $Z(\infty, x) \cup Z(\infty, x')$. Since $\underset{\sim}{u} < \underset{\sim}{t}$, $|\underset{\sim}{u} - \underset{\sim}{s}| < \underset{\sim}{\delta}/2$ and $\underset{\sim}{t}_0 < \underset{\sim}{s} < \underset{\sim}{t}$, it follows that $\underset{\sim}{s} \in A$ and therefore

$$
\begin{aligned}
m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2} &= L(\lambda)(st^{-1}(a(t, x, \delta) \cup a(t, x', \delta)))\delta^{-1/2} \\
&\qquad \text{(by Theorem 2.4)} \\
&\le \delta^{1/2} + L(\lambda)(A)\delta^{-1/2} \\
&= \delta^{1/2} + {}^\circ M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) \text{ (by (4.4))}.
\end{aligned} \tag{4.7}
$$

Combining (4.6) and (4.7), we obtain (a).

(b) Let

$$
A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) = \{\underset{\sim}{s} \in T \mid \text{there exist } \underset{\sim}{u}_1 < \underset{\sim}{t} \text{ and } \underset{\sim}{u}_2 \in T \text{ such that } \underset{\sim}{u}_1 \le \underset{\sim}{s} < \underset{\sim}{u}_2, X(\underset{\sim}{u}_i) = \underset{\sim}{x} \text{ for } i = 1, 2, \text{ and } \underset{\sim}{u}_2 - \underset{\sim}{u}_1 \le \underset{\sim}{\delta}\}.
$$

It follows from the definition of $M''$ that

$$
M''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) = \lambda(A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}))\underset{\sim}{\delta}^{-1/2}.
$$

Suppose that ${}^\circ\underset{\sim}{s} \in a'(t, x, \delta)$, so that there exist $u_1 < u_2 \le t$ such that $u_1 < {}^\circ\underset{\sim}{s} < u_2$, $B(u_i) = x (i = 1, 2)$, and $u_2 - u_1 \le \delta$. By $(C_2)$, we see that $u_2 - u_1 < \delta$, and since $u_i$ cannot be a local extrema of $B(\cdot)$, there exist $\underset{\sim}{u}_i \approx u_i$ such that $X(\underset{\sim}{u}_i) = \underset{\sim}{x} (i = 1, 2)$. Since $\underset{\sim}{u}_2 - \underset{\sim}{u}_1 \le \underset{\sim}{\delta}$, $\underset{\sim}{u}_1 \le \underset{\sim}{s} < \underset{\sim}{u}_2$, and $\underset{\sim}{u}_1 < \underset{\sim}{t}$, it follows that $\underset{\sim}{s} \in A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta})$ and therefore, we have shown

$$
st^{-1}(a'(t, x, \delta)) \subset A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}). \tag{4.8}
$$

Let $\underset{\sim}{s} \in A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) - st^{-1}(Z(t, x) \cup (t - \delta, t])$, and let $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ be as in the definition of $A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta})$. Clearly ${}^\circ\underset{\sim}{u}_1 \le {}^\circ\underset{\sim}{s} \le {}^\circ\underset{\sim}{u}_2$, $B({}^\circ\underset{\sim}{u}_i) = x (i = 1, 2)$, and ${}^\circ\underset{\sim}{u}_2 - {}^\circ\underset{\sim}{u}_1 \le \delta$. Since $B({}^\circ\underset{\sim}{s}) \ne x$, and ${}^\circ\underset{\sim}{u}_2 \le {}^\circ\underset{\sim}{s} + \delta \le t$, it follows that ${}^\circ\underset{\sim}{s} \in a'(t, x, \delta)$. Therefore we have shown

$$
A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) - st^{-1}(Z(t, x) \cup (t - \delta, t]) \subset st^{-1}(a'(t, x, \delta)). \tag{4.9}
$$

By combining (4.8) and (4.9), and using Theorem 2.4, one may conclude that

$$
\begin{aligned}
{}^\circ M''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) &= {}^\circ(\lambda(A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}))\underset{\sim}{\delta}^{-1/2}) \\
&\ge L(\lambda)(st^{-1}(a'(t, x, \delta)))\delta^{-1/2} \quad \text{(by (4.8))} \\
&= m(a'(t, x, \delta))\delta^{-1/2} \quad \text{(by theorem 2.4)} \\
&\ge {}^\circ\lambda(A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}))\delta^{-1/2} - \delta^{1/2} \quad \text{(by (4.9))} \\
&= {}^\circ M''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) - \delta^{1/2},
\end{aligned}
$$

where we have used the fact that $L(\lambda)(st^{-1}(Z(t, x))) = m(Z(t, x)) = 0$ (by $(C_5)$). Hence (b) follows immediately.

(c) If

$$
A'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) = \{\underset{\sim}{s} \in T \mid \text{there exist } \underset{\sim}{u}_1 < \underset{\sim}{t} \text{ and } \underset{\sim}{u}_2 \in T \text{ such that } \underset{\sim}{u}_1 \le \underset{\sim}{s} < \underset{\sim}{u}_2, X(\underset{\sim}{u}_i) = \underset{\sim}{x} \text{ or } \underset{\sim}{x}' \text{ (for } i = 1, 2), \text{ and } \underset{\sim}{u}_2 - \underset{\sim}{u}_1 \le \underset{\sim}{\delta}\},
$$

then, by the definition of $M'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta})$,

$$
M'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) = \lambda(A'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}))\underset{\sim}{\delta}^{-1/2}.
$$

Since $A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) \cup A''(\underset{\sim}{t}, \underset{\sim}{x}', \underset{\sim}{\delta}) \subset A'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta})$, (4.8) implies that

$$
\begin{aligned}
m(a'(t, x, \delta) \cup a'(t, x', \delta))\delta^{-1/2} &= L(\lambda)(st^{-1}(a'(t, x, \delta) \cup a'(t, x', \delta)))\delta^{-1/2} \\
&\le {}^\circ\lambda(A''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) \cup A''(\underset{\sim}{t}, \underset{\sim}{x}', \underset{\sim}{\delta}))\delta^{-1/2} \\
&\le {}^\circ\lambda(A'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}))\delta^{-1/2} \\
&= {}^\circ M'(\underset{\sim}{t}\, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}),
\end{aligned}
$$

as required. □

The limiting behavior of $m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2}$ and $m(a'(t, x, \delta))\delta^{-1/2}$ as $\delta$ approaches zero is now determined by the corresponding behavior of $°M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta})$ and $°M''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta})$. In order to study the latter two processes we use some elementary combinatorial probability theory to examine the internal distributions of $U(\underset{\sim}{y}, \underset{\sim}{\delta})$ and $U'(\underset{\sim}{y}, \underset{\sim}{\delta})$.

NOTATION 4.4. If $\underset{\sim}{\delta} > 0$ is in $T$ and $n$ is a natural number, let

$$p_n(\underset{\sim}{\delta}) = \bar{E}(((T(0) \wedge \underset{\sim}{\delta})/\underset{\sim}{\delta})^n)(\underset{\sim}{\delta}/\Delta t)^{1/2},$$

and

$$p'_n(\underset{\sim}{\delta}) = \bar{E}((T(0)I_{\{T(0)\leq \underset{\sim}{\delta}\}}/\underset{\sim}{\delta})^n)(\underset{\sim}{\delta}/\Delta t)^{1/2}.$$

If in addition $\underset{\sim}{x} \in S$, then define

$$q(\underset{\sim}{x}, \underset{\sim}{\delta}) = \begin{cases} \bar{E}(U(\underset{\sim}{x}, \underset{\sim}{\delta}))(\Delta t)^{-1/2} & \text{if } \underset{\sim}{x} \neq 0 \\ 1/2\, p_1(\underset{\sim}{\delta}) & \text{if } \underset{\sim}{x} = 0, \end{cases}$$

and

$$q'(\underset{\sim}{x}, \underset{\sim}{\delta}) = \begin{cases} \bar{E}(U'(\underset{\sim}{x}, \underset{\sim}{\delta}))(\Delta t)^{-1/2} & \text{if } \underset{\sim}{x} \neq 0 \\ \tfrac{1}{2}\, p'_1(\underset{\sim}{\delta}) & \text{if } \underset{\sim}{x} = 0. \end{cases}$$

LEMMA 4.5. (a) *If $\underset{\sim}{\delta} = \gamma\Delta t$ for some $\gamma$ in $*N - N$ then for all $n$ in $N$,*

$$°p_n(\underset{\sim}{\delta}) = (2\pi)^{-1/2}(n - \tfrac{1}{2})^{-1} + (2/\pi)^{1/2}$$

*and*

$$°p'_n(\underset{\sim}{\delta}) = (2\pi)^{-1/2}(n - \tfrac{1}{2})^{-1}.$$

(b) *If $\underset{\sim}{x} \in S$ and $\underset{\sim}{\delta} > 0$ is in $T$, then*

$$p_1(\underset{\sim}{\delta})/2 \leq q(\underset{\sim}{x}, \underset{\sim}{\delta}) \leq p_1(\underset{\sim}{\delta})/2 + |\underset{\sim}{x}|/2\underset{\sim}{\delta}^{1/2}$$

*and*

$$p'_1(\underset{\sim}{\delta})/2 \leq q'(\underset{\sim}{x}, \underset{\sim}{\delta}) \leq p'_1(\underset{\sim}{\delta})/2 + |\underset{\sim}{x}|/2\underset{\sim}{\delta}^{1/2}.$$

PROOF. (a) As in the proof of Lemma 4.3 we may assume without loss of generality that $\underset{\sim}{\delta} = 2\gamma\Delta t$ for some $\gamma$ in $*N - N$. If $j \in *N$, then

$$\bar{P}(T(0) = 2j\Delta t) = 2/j\binom{2(j-1)}{j-1}2^{-2j}$$

by an elementary counting argument (see Feller (1968, III.3, Lemma 2)). An easy application of the "reflection principle" implies that

$$\bar{P}(T(0) > 2\gamma\Delta t) = \bar{P}(X(2\gamma\Delta t) = 0) = \binom{2\gamma}{\gamma}2^{-2\gamma}.$$

Therefore if $n \in N$, then

$$\begin{aligned} p_n(\underset{\sim}{\delta}) &= \textstyle\sum_{j=1}^{\gamma} (2j\Delta t/2\gamma\Delta t)^n(2\gamma)^{1/2}(2/j)\binom{2(j-1)}{j-1}2^{-2j} + (2\gamma)^{1/2}\binom{2\gamma}{\gamma}2^{-2\gamma} \\ &= 2\sqrt{2}\left(\textstyle\sum_{j=1}^{\gamma} (j^{n-1}/\gamma^{n-1/2})\binom{2(j-1)}{j-1}2^{-2j}\right) + (2\gamma)^{1/2}\binom{2\gamma}{\gamma}2^{-2\gamma}. \end{aligned} \tag{4.10}$$

By Stirling's Formula and the Transfer Principle if $i \in *N$, then

$$2^{-2i}\binom{2i}{i} = (1 + \delta_i)(\pi i)^{-1/2},$$

where $\delta_i \approx 0$ if $i \in *N - N$. Therefore

$$(2\gamma)^{1/2}\binom{2\gamma}{\gamma}2^{-2\gamma} = (2/\pi)^{1/2}(1 + \delta_\gamma) \approx (2/\pi)^{1/2}. \tag{4.11}$$

Since the first term in (4.10) is $p'_n(\underset{\sim}{\delta})$, we have

$$\begin{aligned} p'_n(\underset{\sim}{\delta}) &= 2\sqrt{2}\textstyle\sum_{j=1}^{\gamma} (j^{n-1}/\gamma^{n-1/2})(j^2/2j(2j-1))(1 + \delta_j)(\pi j)^{-1/2} \\ &= (2\pi)^{-1/2}\textstyle\sum_{j=1}^{\gamma} (j/\gamma)^{n-3/2}(1 + \delta'_j)\gamma^{-1}, \end{aligned} \tag{4.12}$$

where $\delta_j' \approx 0$ if $j \in {}^*N - N$ and $°(\sup_{j\in {}^*N} \delta_j') < \infty$. We note that

$$(4.13) \qquad \textstyle\sum_{j=1}^{\gamma} (j/\gamma)^{n-3/2}\gamma^{-1} \approx \int_0^1 t^{n-3/2}\,dt = (n - ½)^{-1}$$

and therefore

$$\begin{aligned} °\textstyle\sum_{j=1}^{\gamma} (j/\gamma)^{n-3/2}\delta_j'\gamma^{-1} &\le °((\textstyle\sum_{j=1}^{\gamma^{1/4}} \delta_j')\gamma^{-1/2}) + °((\max_{\gamma^{1/4}\le j\le\gamma} \delta_j') \textstyle\sum_{j=1}^{\gamma} (j/\gamma)^{n-3/2}\gamma^{-1}) \\ (4.14) \qquad &\le °((\max_{1\le j\le \gamma^{1/4}} \delta_j')\gamma^{-1/4}) + °(\max_{\gamma^{1/4}\le j\le\gamma} \delta_j') \int_0^1 t^{n-3/2}\,dt \\ &= 0. \end{aligned}$$

By substituting (4.13) and (4.14) into (4.12), we see that

$$°p_n'(\underline{\delta}) = (2\pi)^{-1/2}(n - ½)^{-1},$$

and therefore by (4.10) and (4.11),

$$°p_n(\underline{\delta}) = (2\pi)^{-1/2}(n - ½)^{-1} + (2/\pi)^{1/2}.$$

(b) We prove the result only for $q(\underline{x}, \underline{\delta})$, as the argument for $q'(\underline{x}, \underline{\delta})$ is identical. Since the inequality is obvious for $\underline{x} = 0$ and $q(\underline{x}, \underline{\delta}) = q(-\underline{x}, \underline{\delta})$ by symmetry, we may assume $\underline{x} > 0$. Then

$$\begin{aligned} q(\underline{x}, \underline{\delta}) &= \bar{E}(T(0, \underline{x}) \wedge \underline{\delta})(\underline{\delta}\Delta t)^{-1/2} \\ &= \int_{\{\omega_{\Delta t}=1\}} (\Delta t + T(-(\Delta t)^{1/2}, \underline{x} - (\Delta t)^{1/2})\circ\theta_{\Delta t}) \wedge \underline{\delta}\, d\bar{P}(\underline{\delta}\Delta t)^{-1/2} \\ &\quad + \int_{\{\omega_{\Delta t}=-1\}} T(0) \wedge \underline{\delta}\, d\bar{P}(\underline{\delta}\Delta t)^{-1/2}. \end{aligned}$$

By symmetry the second term is $½p_1(\underline{\delta})$. The first term is bounded above by

$$½\bar{E}(\Delta t + T(-(\Delta t)^{1/2}, \underline{x} - (\Delta t)^{1/2}))(\underline{\delta}\Delta t)^{-1/2} = \underline{x}/2\underline{\delta}^{1/2},$$

and the result follows. □

If $°\underline{x} = °\underline{x}'$, $\underline{x} \ne \underline{x}'$ and $°\underline{\delta} > 0$, then the previous two results show that

$$|m(a(°\underline{t}, °\underline{x}, °\underline{\delta}))(°\underline{\delta})^{-1/2} - °M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta})| \le (°\underline{\delta})^{1/2} \text{ a.s.}$$

and $$°(2q(\underline{x}' - \underline{x}, \underline{\delta})(L(\underline{t}, \underline{x}) + L(\underline{t}, \underline{x}'))) = 4(2/\pi)^{1/2}s(°\underline{t}, °\underline{x}).$$

Hence the original problem of comparing $m(a(t, x, \delta))\delta^{-1/2}$ and $4(2/\pi)^{1/2}s(t, x)$ has been translated into the nonstandard problem of comparing $M(\underline{t}, \underline{x}, \underline{x}', \underline{\delta})$ and $2q(\underline{x}' - \underline{x}, \underline{\delta}) \cdot (L(\underline{t}, \underline{x}) + L(\underline{t}, \underline{x}'))$. Similarly a comparison of $m(a'(t, x, \delta))$ and $2(2/\pi)^{1/2}s(t, x)$ is equivalent to a comparison of $M''(\underline{t}, \underline{x}, \underline{\delta})$ and $2p_1'(\underline{\delta})L(\underline{t}, \underline{x})$.

The following lemma uses a square function inequality in Burkholder (1973, Theorem 21.1) that implies, among other things, that for each $p > 0$ there exists a real constant $C_p$ such that for every martingale $\{(f_n, \mathscr{F}_n) \mid n \in N\}$, if $\langle f, f\rangle_n = \sum_{i=1}^n E((f_i - f_{i-1})^2 \mid \mathscr{F}_{i-1})$ ($f_0 = 0$ and $\mathscr{F}_0$ is the trivial $\sigma$-field), then

$$(4.15) \qquad E(\max_{i\le n}|f_i|^p) \le C_pE(\langle f, f\rangle_n^{p/2} + \max_{i\le n}|f_i - f_{i-1}|^p).$$

LEMMA 4.6. *There exists a real constant $c_3$ such that whenever $\underline{x}$ and $\underline{x}'$ are distinct elements of $S$, and $\underline{t}$ and $\underline{\delta} = \gamma\Delta t \le 1$ are elements of $T$ with $\gamma \in {}^*N - N$, then*

(a) $\bar{E}((M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) - 2q(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})(L(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x}')))^4) \leq c_3(\underset{\sim}{t} \vee 1)\underset{\sim}{\delta}$,
(b) $\bar{E}((M''(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{\delta}) - 2p_1'(\underset{\sim}{\delta})L(\underset{\sim}{t}, \underset{\sim}{x}))^4) \leq c_3(\underset{\sim}{t} \vee 1)\underset{\sim}{\delta}$,
(c) $\bar{E}((M'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) - 2q'(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})(L(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x}')))^4) \leq c_3(\underset{\sim}{t} \vee 1)\underset{\sim}{\delta}$.

PROOF. (a) Choose $\underset{\sim}{t}$, $\underset{\sim}{x}$, $\underset{\sim}{x}'$ and $\underset{\sim}{\delta}$ as in the statement of the theorem, and define

$$Y(\underset{\sim}{t}) \equiv M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) - 2q(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})(L(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x}'))$$
$$= \textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} (I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2}(V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}}) + I_{\{\underset{\sim}{x}'\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2}(V(\underset{\sim}{x} - \underset{\sim}{x}', \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})),$$

where $$V(\underset{\sim}{y}, \underset{\sim}{\delta}) = U(\underset{\sim}{y}, \underset{\sim}{\delta})(\Delta t)^{-1/2} - q(\underset{\sim}{y}, \underset{\sim}{\delta}).$$

An easy computation using (2.1) shows that $\{(Y(\underset{\sim}{s}), \mathscr{A}_{\underset{\sim}{s}}) \mid \underset{\sim}{s} \in T\}$ is an internal martingale. Therefore the Transfer Principle and (4.15) imply that

$$\bar{E}(Y(\underset{\sim}{t})^4) \leq C_4\bar{E}((\textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))\Delta t\bar{E}((V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})^2 \mid \mathscr{A}_{\underset{\sim}{s}})$$
$$+ I_{\{\underset{\sim}{x}'\}}(X(\underset{\sim}{s}))\Delta t\bar{E}((V(\underset{\sim}{x} - \underset{\sim}{x}', \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})^2 \mid \mathscr{A}_{\underset{\sim}{s}}))^2)$$
$$+ C_4\bar{E}(\textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} (I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2}V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})^4$$
$$+ (I_{\{\underset{\sim}{x}'\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2}V(\underset{\sim}{x} - \underset{\sim}{x}', \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})^4).$$

Denote the first term of the above expression by $E_1$, and the second term by $E_2$. We note that

$$\bar{E}((V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})^\circ\theta_{\underset{\sim}{s}})^2 \mid \mathscr{A}_{\underset{\sim}{s}}) = \mathrm{Var}(U(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta}))(\Delta t)^{-1}$$
$$\leq \bar{E}((T(0) \wedge \underset{\sim}{\delta}/\underset{\sim}{\delta})^2)\underset{\sim}{\delta}(\Delta t)^{-1}$$
$$= p_2(\underset{\sim}{\delta})(\underset{\sim}{\delta}/\Delta t)^{1/2}.$$

Therefore $E_1$ is bounded by

$$C_4 p_2(\underset{\sim}{\delta})^2(\underset{\sim}{\delta}/\Delta t)\Delta t\bar{E}((\textstyle\sum_{\underset{\sim}{s}=0}^{\underset{\sim}{t}-\Delta t} I_{\{\underset{\sim}{x}\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2} + I_{\{\underset{\sim}{x}'\}}(X(\underset{\sim}{s}))(\Delta t)^{1/2})^2)$$
$$\leq C_4 p_2(\underset{\sim}{\delta})^2\underset{\sim}{\delta}8\bar{E}(L(\underset{\sim}{t}, \underset{\sim}{x})^2 + L(\underset{\sim}{t}, \underset{\sim}{x}')^2) \leq C_4 p_2(\underset{\sim}{\delta})^2\underset{\sim}{\delta}16c_2\underset{\sim}{t}$$

(the last by Lemma 3.3). Since ${}^\circ p_2(\underset{\sim}{\delta})$ is finite, $E_1$ is bounded by $\alpha_1\underset{\sim}{t}\underset{\sim}{\delta}$ for some real $\alpha_1$. In order to bound $E_2$, we use (2.1) to see that

$$\bar{E}((V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})\circ\theta_{\underset{\sim}{s}})^4 \mid \mathscr{A}_{\underset{\sim}{s}}) = \bar{E}(V(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})^4)$$
$$\leq \alpha_2\bar{E}((T(0) \wedge \underset{\sim}{\delta}/\underset{\sim}{\delta})^4)\underset{\sim}{\delta}^2(\Delta t)^{-2} = \alpha_2 p_4(\underset{\sim}{\delta})(\underset{\sim}{\delta}/\Delta t)^{3/2},$$

where $\alpha_2$ is real. Hence $E_2$ is bounded by

$$C_4\alpha_2 p_4(\underset{\sim}{\delta})(\underset{\sim}{\delta}/\Delta t)^{3/2}(\Delta t)^{3/2}\bar{E}(2L(\underset{\sim}{t}, \underset{\sim}{x}) + 2L(\underset{\sim}{t}, \underset{\sim}{x}')) \leq C_4\alpha_2 p_4(\underset{\sim}{\delta})\underset{\sim}{\delta}^{3/2}4\sqrt{c_2}\sqrt{\underset{\sim}{t}}$$

(the last by Lemma 3.3). Since $\underset{\sim}{\delta} \leq 1$ and ${}^\circ p_4(\underset{\sim}{\delta})$ is finite, it follows that $E_2 \leq \alpha_3\underset{\sim}{\delta}(\underset{\sim}{t} \vee 1)$ for some real $\alpha_3$, and the result is now immediate from this, and the above bound on $E_1$.

The proofs of (b) and (c) are identical to the above and are therefore omitted. □

The above result is now "standardized" by means of Lemmas 4.3 and 4.5.

THEOREM 4.7. *There exist functions* $c: R \times (0, 1] \to [0, \infty)$ *and* $c': R \times (0, 1] \to [0, \infty)$, *and a real constant* $c_4$ *such that for all* $\delta$ *in* $(0, 1]$, $x$, $x'$ *in* $R$, *and* $t \geq 0$, *the following conditions hold*:

(i) $2(2/\pi)^{1/2} \leq c(x, \delta) \leq 2(2/\pi)^{1/2} + |x|\delta^{-1/2}$, *and*
$(2/\pi)^{1/2} \leq c'(x, \delta) \leq (2/\pi)^{1/2} + |x|\delta^{-1/2}$,
(ii) $E((m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2} - c(x' - x, \delta)(s(t, x) + s(t, x')))^4) \leq c_4(t \vee 1)\delta$,

(iii) $E((m(a'(t, x, \delta))\delta^{-1/2} - 2(2/\pi)^{1/2}s(t, x))^4) \le c_4(t \vee 1)\delta$,
(iv) $E(((m(a'(t, x, \delta) \cup a'(t, x', \delta))\delta^{-1/2} - c'(x' - x, \delta)(s(t, x) + s(t, x')))^+)^4)$
$\le c_4(t \vee 1)\delta$.

PROOF. If $(x, \delta) \in R \times (0, 1]$ and $(\underset{\sim}{x}, \underset{\sim}{\delta}) \in S \times T$ satisfy $x \in [\underset{\sim}{x} - (\Delta t)^{1/2}, \underset{\sim}{x})$ and $\delta \in [\underset{\sim}{\delta} - \Delta t, \underset{\sim}{\delta})$, then define $c(x, \delta) = 2{}^\circ q(\underset{\sim}{x}, \underset{\sim}{\delta})$ and $c'(x, \delta) = 2{}^\circ q'(\underset{\sim}{x}, \underset{\sim}{\delta})$. Condition (i) now follows by taking standard parts in Lemma 4.5(b).

Fix $\delta$, $t$, $x$ and $x'$ as in the statement of the theorem and choose $(\underset{\sim}{\delta}, \underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}')$ in $T^2 \times S^2$ such that ${}^\circ(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}') = (t, x, x')$, $\delta \in [\underset{\sim}{\delta} - \Delta t, \underset{\sim}{\delta})$ and $x' - x \in [\underset{\sim}{x}' - \underset{\sim}{x} - (\Delta t)^{1/2}, \underset{\sim}{x}' - \underset{\sim}{x})$ (in particular $\underset{\sim}{x} \neq \underset{\sim}{x}'$). Since $c(x' - x, \delta) = 2{}^\circ q(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})$ we see that

$$E((m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2} - c(x' - x, \delta)(s(t, x) + s(t, x')))^4)$$
$$\le 8E((m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2} - {}^\circ M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}))^4)$$
$$+ 8E({}^\circ(M(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{\delta}) - 2q(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})(L(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x}')))^4).$$

The first term is bounded by $8\delta^2$ by Lemma 4.3(a), and since $E({}^\circ|W|) \le {}^\circ\bar{E}(|W|)$ for any internal $\mathscr{A}$-measurable random variable $W$, Lemma 4.6(a) bounds the second term by $8c_3(t \vee 1)\delta$. Since $\delta \le 1$, (ii) follows.

The proof of (iii) is similar to the above argument.

Since $c'(x' - x, \delta) = 2{}^\circ q'(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})$, Lemma 4.3(c) implies that

$$E(((m(a'(t, x, \delta) \cup a'(t, x', \delta))\delta^{-1/2} - c'(x' - x, \delta)(s(t, x) + s(t, x')))^+)^4)$$
$$\le E({}^\circ((M'(\underset{\sim}{t}, \underset{\sim}{x}, \underset{\sim}{x}' \underset{\sim}{\delta}) - 2q'(\underset{\sim}{x}' - \underset{\sim}{x}, \underset{\sim}{\delta})(L(\underset{\sim}{t}, \underset{\sim}{x}) + L(\underset{\sim}{t}, \underset{\sim}{x}')))^+)^4).$$

As in the proof of (ii) we may use Lemma 4.6(c) to bound the above by $c_3(t \vee 1)\delta$. □

REMARK 4.8. If $\delta$ and $|x - x'|\delta^{-1/2}$ are both small, then (i) implies that $c(x' - x, \delta)$ is close to $2(2/\pi)^{1/2}$ and therefore $c(x' - x, \delta)(s(t, x) + s(t, x'))$ is close to $4(2/\pi)^{1/2}s(t, x)$. Hence from (ii) we see that both $m(a(t, x, \delta))\delta^{-1/2}$ and $m(a(t, x, \delta) \cup a(t, x', \delta))\delta^{-1/2}$ are close to $4(2/\pi)^{1/2}s(t, x)$, and in particular, there is almost a "complete overlap" between $a(t, x, \delta)$ and $a(t, x', \delta)$.

We will need the following rather crude result on the escape rate of a Brownian motion. It follows immediately from Taylor (1974, Theorem 8), but since no proof is given there, we give an elementary justification of the particular result that is required.

LEMMA 4.9. *Assume that* $\{\alpha_n \mid n \in N\}$ *and* $\{\beta_n \mid n \in N\}$ *are sequences of real numbers converging to zero such that* $\sum_{n=1}^\infty \beta_n^{2m}\alpha_n^{-(m+1)} < \infty$ *for some* $m$ *in* $N$. *If* $t \ge 0$ *is fixed and*

$$A_n = \{\omega \mid \inf_{u \le t} \sup_{s \in [0,\alpha_n)} |B(u + s) - B(u)| \le \beta_n\},$$

*then* $P(A_n$ occurs infinitely often$) = 0$.

PROOF. Fix $t \ge 0$ and define $W_n = \{j\alpha_n/2 \mid j \in N, j \le 2t/\alpha_n + 1\}$ and $V(\beta) = \inf\{t \mid |B(t)| = \beta\}$. Then

$$P(A_n) \le P(\min_{u \in W_n} \sup_{s \in [0,\alpha_n/2)} |B(u + s) - B(u)| \le 2\beta_n)$$
$$\le (2t/\alpha_n + 1)P(V(3\beta_n) \ge \alpha_n/2)$$
$$\le (2t/\alpha_n + 1)(2/\alpha_n)^m E(V(3\beta_n)^m) \tag{4.16}$$
$$\le C(m)(3\beta_n)^{2m}\alpha_n^{-(m+1)}E(V(1)^m),$$

where $C(m)$ is a real constant and we have used the fact that $V(\beta)$ and $\beta^2V(1)$ have the same distributions. Since (4.16) is summable over $n$ for an appropriate choice of $m$, the result follows from the Borel-Cantelli Lemma. □

In order to prove that (4.1) and (4.3) hold uniformly in $x$, we first show that they hold uniformly for $x$ in a sequence of lattices $\{S_n\}$ that increase in size as $\delta = \delta_n$ decreases to zero.

**Lemma** 4.10. *Let $S_n = \{kn^{-5} \mid k \in \mathbb{Z}, |k| \le n^6\}$ and if $t \ge 0$ is fixed, define*

$$B_n = \{\omega \mid \sup_{x\in S_n} |m(a(t, x, n^{-8}))n^{-4} - 4(2/\pi)^{1/2}s(t, x)| \ge n^{-1/5}\}$$
$$\cup \{\omega \mid \sup_{x\in S_n} |m(a(t, x, n^{-8}) \cup a(t, x + n^{-5}, n^{-8}))/n^{-4}$$
$$-c(n^{-5}, n^{-8})(s(t, x) + s(t, x + n^{-5}))| \ge n^{-1/5}\},$$

*and*

$$C_n = \{\omega \mid \sup_{x\in S_n} |m(a'(t, x, n^{-8}))n^{-4} - 2(2/\pi)^{1/2}s(t, x)| \ge n^{-1/5}\}$$
$$\cup \{\omega \mid \sup_{x\in S_n} m(a'(t, x, n^{-8}) \cup a'(t, x + n^{-5}, n^{-8}))/n^{-4}$$
$$-c'(n^{-5}, n^{-8})(s(t, x) + s(t, x + n^{-5})) \ge n^{-1/5}\},$$

*Then $P(B_n \cup C_n$ occurs infinitely often$) = 0$.*

**Proof.** Theorem 4.7 (ii) implies that

$$P(B_n) \le \text{card}(S_n)n^{4/5}(\max_{x\in S_n} E((m(a(t, x, n^{-8}))/n^{-4} - 4(2/\pi)^{1/2}s(t, x))^4)$$
$$+ \max_{x\in S_n} E((m(a(t, x, n^{-8}) \cup a(t, x + n^{-5}, n^{-8}))/n^{-4}$$
$$- c(n^{-5}, n^{-8})(s(t, x) + s(t, x + n^{-5})))^4))$$
$$\le 3n^6 n^{4/5}(c_4(t \vee 1)n^{-8} + c_4(t \vee 1)n^{-8})$$
$$= 6\, c_4(t \vee 1)n^{-6/5}.$$

Theorem 4.7 (iii) and (iv) lead to the same upper bound for $P(C_n)$, and hence the result follows by the Borel-Cantelli Lemma. □

We are finally ready to show that (4.3) holds uniformly in $x$.

**Theorem** 4.11. *The following holds with probability one*:
*For every $t' > 0$, $\lim_{\delta\to 0^+} \sup_{(t,x)\in[0,t']\times R} |m(a(t, x, \delta))\delta^{-1/2} - 4(2/\pi)^{1/2}s(t, x)| = 0$.*

**Proof.** Since $s(\cdot, x)|_{[0,t']}$ is a.s. uniformly continuous in $t$ uniformly in $x$ (since $s|_{[0,t']\times R}$ is a.s. uniformly continuous), and $m(a(\cdot, x, \delta))\delta^{-1/2}$ is non-decreasing, it suffices to show that for each $t > 0$

$$\lim_{\delta\to 0^+} \sup_{x\in R} |m(a(t, x, \delta))\delta^{-1/2} - 4(2/\pi)^{1/2}s(t, x)| = 0 \quad \text{a.s.}$$

In fact it suffices to show that

$$\lim_{n\to\infty} \sup_{x\in R} |m(a(t, x, \delta_n))\delta_n^{-1/2} - 4(2/\pi)^{1/2}s(t, x)| = 0 \quad \text{a.s.} \tag{4.17}$$

for some sequence $\{\delta_n\}$ decreasing to zero such that $\lim_{n\to\infty} \delta_{n+1}\delta_n^{-1} = 1$. Indeed, if $\delta_{n+1} \le \delta < \delta_n$, then

$$\begin{aligned}(\delta_{n+1}/\delta_n)^{1/2}m(a(t, x, \delta_{n+1}))\delta_{n+1}^{-1/2} &\le m(a(t, x, \delta))\delta^{-1/2} \\ &\le (\delta_n/\delta_{n+1})^{1/2}m(a(t, x, \delta_n))\delta_n^{-1/2}.\end{aligned} \tag{4.18}$$

Therefore, if (4.17) holds, both the extreme left-hand and right-hand sides of (4.18) converge to $4(2/\pi)^{1/2}s(t, x)$ uniformly in $x$ a.s., and hence so does $m(a(t, x, \delta))\delta^{-1/2}$.

Let $t \ge 0$ be fixed, define $S_n$ and $B_n$ as in Lemma 4.10 and define $A_n$ as in Lemma 4.9 with $\alpha_n = n^{-9}$ and $\beta_n = n^{-5}$. Lemmas 4.9 and 4.10 imply that

$$N = \{\omega \mid A_n \cup B_n \text{ occurs infinitely often, or } B(\cdot, \omega) \text{ or } s(\cdot, \cdot)(\omega) \text{ is not continuous}\}$$

is a null set. Now fix $\omega$ in $N^c$ and choose $M(\omega)$ in $\mathbb{N}$ such that for all $n \geq M$ the following conditions hold:

(a) $\omega \notin A_n \cup B_n$,

(b) $(n-1)^{-9} + n^{-8}/2 \leq (n-1)^{-8}/2$,

(c) $\sup_{s \leq t} |B(s)| < M$.

It suffices to prove the existence of a sequence $\{\epsilon_n \mid n > M\}$ converging to zero such that

$$\sup_{x \in R} |m(a(t, x, n^{-8}))/n^{-4} - 4(2/\pi)^{1/2} s(t, x)| \leq \epsilon_n. \tag{4.19}$$

If $|x| \geq M$ then (c) implies $s(t, x) = 0 = m(a(t, x, n^{-8}))$.

Suppose now that $|x| < M$, and for $n > M$ let $x_n = \sup\{y \in S_n \mid y \leq x\}$. (Note that $x_n$ exists and $x_n + n^{-5} \in S_n$ since $|x| < M \leq n - 1$.) Fix $n > M$. If $v \in a(t, x, n^{-8})$, there exists $u$ in $[0, t]$ such that $B(u) = x$ and $|u - v| < n^{-8}/2$. If $s' = \inf\{s \geq u \mid B(s) = x_{n-1}$ or $x_{n-1} + (n-1)^{-5}\}$, then $s' - u < (n-1)^{-9}$ since $\omega \notin A_{n-1}$ and $x_{n-1} \leq B(s) \leq x_{n-1} + (n-1)^{-5}$ for all $s$ in $[u, s']$. In particular,

$$|v - s'| \leq |v - u| + |u - s'| < n^{-8}/2 + (n-1)^{-9} < (n-1)^{-8}/2,$$

and $s' < t + (n-1)^{-9}$, Therefore,

$$v \in a(t + (n-1)^{-9}, x_{n-1}, (n-1)^{-8}) \cup a(t + (n-1)^{-9}, x_{n-1} + (n-1)^{-5}, (n-1)^{-8}),$$

and we have proven

$$\begin{aligned} a(t, x, n^{-8}) \subset\ & a(t + (n-1)^{-9}, x_{n-1}, (n-1)^{-8}) \\ & \cup a(t + (n-1)^{-9}, x_{n-1} + (n-1)^{-5}, (n-1)^{-8}). \end{aligned} \tag{4.20}$$

If $v \in a(t, x_n, n^{-8}) \cap a(t, x_n + n^{-5}, n^{-8})$, then there exists $u_1, u_2 \leq t$ such that $B(u_1) = x_n$, $B(u_2) = x_n + n^{-5}$ and $|u_i - v| < n^{-8}/2$ for $i = 1, 2$. By the Intermediate Value Theorem, there is a $u_0$ in the closed interval with end points $u_1$ and $u_2$ such that $B(u_0) = x$ and $|u_0 - v| < n^{-8}/2$. Hence, $v \in a(t, x, n^{-8})$ and we have proven

$$a(t, x_n, n^{-8}) \cap a(t, x_n + n^{-5}, n^{-8}) \subset a(t, x, n^{-8}). \tag{4.21}$$

From (4.20) we obtain

$$\begin{aligned} & m(a(t, x, n^{-8}))/n^{-4} \\ & \quad \leq m(a(t + (n-1)^{-9}, x_{n-1}, (n-1)^{-8}) \cup a(t + (n-1)^{-9}, x_{n-1} + (n-1)^{-5}, \\ & \qquad (n-1)^{-8}))/(n-1)^{-4} \times n^4/(n-1)^4 \\ & \quad \leq (m(a(t, x_{n-1}, (n-1)^{-8})) \cup a(t, x_{n-1} + (n-1)^{-5}, (n-1)^{-8}))/(n-1)^{-4} \\ & \qquad + ((n-1)^{-9} + (n-1)^{-8})/(n-1)^{-4})n^4/(n-1)^4. \end{aligned}$$

Since $\omega \notin B_{n-1}$ the above bound implies that

$$\begin{aligned} m(a(t, x, n^{-8}))/n^{-4} & \leq (c((n-1)^{-5}, (n-1)^{-8})(s(t, x_{n-1}) + s(t, x_{n-1} + (n-1)^{-5})) \\ & \qquad + (n-1)^{-1/5} + 2(n-1)^{-4})n^4/(n-1)^4 \\ & \leq ((2(2/\pi)^{1/2} + (n-1)^{-1})(s(t, x_{n-1}) + s(t, x_{n-1} + (n-1)^{-5}) \\ & \qquad + (n-1)^{-1/5} + 2(n-1)^{-4})n^4/(n-1)^4 \end{aligned}$$

(the last by Theorem 4.7(i)). Since $s(t, \cdot)$ is uniformly continuous and bounded, the above estimate allows us to define a sequence $\{\epsilon'_n \mid n > M\}$ decreasing to zero and independent of $x$ such that

$$m(a(t, x, n^{-8}))/n^{-4} \leq 4(2/\pi)^{1/2} s(t, x) + \epsilon'_n \tag{4.22}$$

for all real $x$ and $n > M$. To obtain a bound in the opposite direction we use (4.21) to see that

$$m(a(t, x, n^{-8}))/n^{-4} \geq m(a(t, x_n, n^{-8}) \cap a(t, x_n + n^{-5}, n^{-8}))/n^{-4}$$

$$= (m(a(t, x_n, n^{-8})) + m(a(t, x_n + n^{-5}, n^{-8})) - m(a(t, x_n, n^{-8}) \cup a(t, x_n + n^{-5}, n^{-8})))/n^{-4}$$

Since $\omega \notin B_n$, it follows that

$$m(a(t, x, n^{-8}))/n^{-4} \geq 4(2/\pi)^{1/2}(s(t, x_n) + s(t, x_n + n^{-5})) - c(n^{-5}, n^{-8})(s(t, x_n) + s(t, x_n + n^{-5})) - 3n^{-1/5}$$
$$\geq (2(2/\pi)^{1/2} - n^{-1})(s(t, x_n) + s(t, x_n + n^{-5})) - 3n^{-1/5}$$

(by Theorem 4.7(i)).

Again the uniform continuity and boundedness of $s(t, \cdot)$ leads to the existence of a sequence $\{\epsilon_n'' \mid n > M\}$ decreasing to zero and independent of $x$ such that

$$m(a(t, x, n^{-8}))/n^{-4} \geq 4(2/\pi)^{1/2}s(t, x) - \epsilon_n'' \tag{4.23}$$

for all real $x$ and $n > M$. Clearly (4.22) and (4.23) imply (4.19) and the theorem is proved. □

The proof that the convergence in (4.1) is uniform in $x$ is similar to the above argument in that we must show that the limiting behavior of $m(a'(t, x, \delta_n))\delta_n^{-1/2}$ can be controlled by the limiting behavior of the same expression evaluated at nearby points in $S_n$. The justification of this fact is slightly more delicate than the above proof and in fact uses the previous result.

THEOREM 4.12. *The following holds with probability one*:

*For every* $t' > 0$, $\lim_{\delta\to 0^+} \sup_{(t,x)\in[0,t']\times R} |m(a'(t, x, \delta))\delta^{-1/2} - 2(2/\pi)^{1/2} s(t, x)| = 0$.

PROOF. As in the previous argument it suffices to fix $t \geq 0$ and show that

$$\lim_{n\to\infty} \sup_{x\in R} |m(a'(t, x, \delta_n))\delta_n^{-1/2} - 2(2/\pi)^{1/2}s(t, x)| = 0 \quad \text{a.s.,} \tag{4.24}$$

where $\{\delta_n\}$ decreases to zero and satisfies $\lim_{n\to\infty}\delta_{n+1}\delta_n^{-1} = 1$. Define $S_n$, $A_n$ and $C_n$ as in Lemmas 4.9 and 4.10, where $\alpha_n = n^{-9}/2$ and $\beta_n = n^{-5}$ in the definition of $A_n$. If

$$\epsilon_n(\omega) = \sup_{x\in R} |m(a(t, x, n^{-9}))/n^{-9/2} - 4(2/\pi)^{1/2}s(t, x)|,$$

then Lemmas 4.9 and 4.10 and Theorem 4.11 imply that

$$N = \{\omega \mid A_n \cup C_n \text{ occurs infinitely often}\} \cup \{\omega \mid B(\cdot, \omega) \text{ or } s(\cdot, \cdot)(\omega) \textit{ is not continuous}\} \cup \{\omega \mid \limsup_{n\to\infty} \epsilon_n(\omega) > 0\}$$

is a null set. Choose $\omega$ in $N^c$ and $M(\omega)$ in $N$ such that $\sup_{s\leq t}|B(s)| < M$ and $\omega \notin A_n \cup C_n$ for all $n \geq M$.

If $|x| \geq M$, then clearly $s(t, x) = 0 = m(a'(t, x, n^{-8}))/n^{-4}$.

Assume $|x| < M$, and for $n > M$ let $x_n = \max\{y \in S_n \mid y \leq x\}$. Fix $n > M$ and suppose $v \in a'(t, x, n^{-8}) - a(t, x, n^{-9})$. Therefore there exist $u_1 < u_2 \leq t$ such that $B(u_1) = B(u_2) = x$, $u_2 - u_1 \leq n^{-8}$, $B(s) \neq x$ for all $s$ in $(u_1, u_2)$ and $v \in (u_1, u_2)$. Since $\omega \notin A_n$, there exists $s_1$ in $[u_1, u_1 + n^{-9}/2)$ such that $B(s_1) = x_n$ or $x_n + n^{-5}$. If $s_2 = \sup\{u \leq u_2 \mid B(u) = B(s_1)\}$, then $B(u)$ is between $B(s_2)(= x_n \text{ or } x_n + n^{-5})$ and $B(u_2) = x$ for all $u$ in $[s_2, u_2]$. Since $\omega \notin A_n$ and $\sup_{u\in[s_2,u_2]}|B(u) - B(s_2)| \leq n^{-5}$, it follows that $u_2 - s_2 < n^{-9}/2$. Moreover, since $v \notin a(t, x, n^{-9})$ we see that $s_1 < v < s_2$ where $B(s_1) = B(s_2) \in \{x_n, x_n + n^{-5}\}$ and $s_2 - s_1 \leq n^{-8}$. Therefore either $B(v) = x_n$ or $x_n + n^{-5}$ or $v \in a'(t, x_n, n^{-8}) \cup a'(t, x_n + n^{-5}, n^{-8})$ and we have shown that

$$a'(t, x, n^{-8}) - a(t, x, n^{-9}) \subset a'(t, x_n, n^{-8}) \cup a'(t, x_n + n^{-5}, n^{-8}) \cup Z(t, x_n) \cup Z(t, x_n + n^{-5}).$$

The above inclusion and the fact that $\omega \notin C_n$ imply that

$$\begin{aligned} m(a'(t, x, n^{-8}))/n^{-4} &\leq m(a(t, x, n^{-9}))/n^{-4} \\ &\quad + m(a'(t, x_n, n^{-8}) \cup a'(t, x_n + n^{-5}, n^{-8}))/ n^{-4} \\ &\leq n^{-1/2}(4(2/\pi)^{1/2} s(t, x) + \epsilon_n) \\ &\quad + c'(n^{-5}, n^{-8})(s(t, x_n) + s(t, x_n + n^{-5})) + n^{-1/5} \\ &\leq n^{-1/2}(4(2/\pi)^{1/2} \sup_{x' \in R} s(t, x') + \epsilon_n) \\ &\quad + ((2/\pi)^{1/2} + n^{-1})(s(t, x_n) + s(t, x_n + n^{-5})) + n^{-1/5} \end{aligned}$$

(the last by theorem 4.7(i)). Hence there exists a sequence $\{\epsilon'_n \mid n > M\}$ decreasing to zero and independent of $x$ such that for all $x$

$$m(a'(t, x, n^{-8}))/n^{-4} \leq 2(2/\pi)^{1/2} s(t, x) + \epsilon'_n. \tag{4.25}$$

To obtain a converse inequality, suppose

$$v \in a'(t, x_n, n^{-8}) \cap a'(t, x_n + n^{-5}, n^{-8}).$$

Therefore, there exist intervals $(u_1, u_2)$ and $(u'_1, u'_2)$, contained in $[0, t]$, such that:

(i) $B(u_1) = B(u_2) = x_n$ and $B(u'_1) = B(u'_2) = x_n + n^{-5}$,
(ii) $u_2 - u_1 \leq n^{-8}$ and $u'_2 - u'_1 \leq n^{-8}$,
(iii) $v \in (u_1, u_2) \cap (u'_1, u'_2)$.

If $u_1 < u'_1 < u_2 < u'_2$, then there exist $s_1 \in [u_1, u'_1]$, $s_2 \in [u'_1, u_2]$, and $s_3 \in [u_2, u'_2]$ such that $B(s_i) = x$ for $i = 1, 2, 3$. Note that $v \in (u'_1, u_2) \subset [s_1, s_3]$. If $v \in [s_1, s_2] \subset [u_1, u_2]$, then either $B(v) = x$ or $v \in a'(t, x, n^{-8})$, since $s_2 - s_1 \leq u_2 - u_1 < n^{-8}$. If $v \in [s_2, s_3] \subset [u'_1, u'_2]$ then again, either $B(v) = x$ or $v \in a'(t, x, n^{-8})$. Hence, we have shown

$$a'(t, x_n, n^{-8}) \cap a'(t, x_n + n^{-5}, n^{-8}) \subset a'(t, x, n^{-8}) \cup Z(t, x),$$

and similar arguments lead to the same result for the other possible orderings of $\{u_1, u'_1, u_2, u'_2\}$. The above inclusion implies that

$$\begin{aligned} m(a'(t, x, n^{-8}))/n^{-4} &\geq m(a'(t, x_n, n^{-8}) \cap a'(t, x_n + n^{-5}, n^{-8}))/n^{-4} \\ &= (m(a'(t, x_n, n^{-8})) + m(a'(t, x_n + n^{-5}, n^{-8})) \\ &\quad - m(a'(t, x_n, n^{-8}) \cup a'(t, x_n + n^{-5}, n^{-8})))/n^{-4} \\ &\geq ((2/\pi)^{1/2} - n^{-1})(s(t, x_n) + s(t, x_n + n^{-5})) - 3n^{-1/5}, \end{aligned}$$

where we have used Theorem 4.7(i) and the fact that $\omega \notin C_n$ in the last line. Hence, there exists a sequence $\{\epsilon''_n\}$ decreasing to zero and independent of $x$ such that

$$m(a'(t, x, n^{-8}))/n^{-4} \geq 2(2/\pi)^{1/2} s(t, x) - \epsilon''_n. \tag{4.26}$$

Clearly (4.25) and (4.26) imply (4.24) with $\delta_n = n^{-8}$, and hence the result. □

Finally Theorem 1.1 now follows as an immediate corollary to Theorems 4.11 and 4.12, and (4.2).

REMARK 4.13. Although Theorems 1.1, 4.11 and 4.12 were proven for the particular Brownian motion $B = st(X)$, these results now follow for an arbitrary continuous Brownian motion. Indeed, the nonstandard setting was only used to obtain Theorem 4.7, and from that point on the arguments were totally standard. Since $m(a(t, x, \delta) \cup a(t, x', \delta))$, $m(a'(t, x, \delta) \cup a'(t, x', \delta))$ and $s(t, x)$ are all measurable functions of the Brownian path (considered as an element of $C([0, \infty))$ with the Borel sets for the compact-open topology), Theorem 4.7 is a statement about Wiener measure and not the particular Brownian motion $B$. Hence Theorem 4.7 holds for any given Brownian motion, and therefore the same is true for the global intrinsic characterizations of local time established in Theorems 1.1, 4.11 and 4.12.

The only use of an "advanced" probabilistic method in any of the above arguments was in Theorem 3.4 and Lemma 4.6 where some square function inequalities for martingales were used. It is possible to avoid the use of these martingale inequalities by means of a direct "internal computation" and some easy applications of Hölder's inequality. The proofs would of course become longer, although more in the combinatorial spirit of the nonstandard approach.

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