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SPLITTING AT BACKWARD TIMES IN REGENERATIVE SETS

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By a backward time is meant a random time which only depends on the future, in the same sense as a stopping time only depends on the past. We show that backward times taking values in a regenerative set M split M into conditionally independent subsets. The conditional distributions of the past may further be identified with the Palm distributions P_t with respect to the local time random measure ξ of M both a.e. $E\xi$ and wherever $\{P_t\}$ has a continuous version. Continuity of $\{P_t\}$ occurs essentially where $E\xi$ has a continuous density, and the latter continuity set may be described rather precisely in terms of the growth rate and regularity properties of the Lévy measure of M.

1. Introduction. Reversal of a Markov process X at a fixed time clearly yields a new Markov process Y, though the time homogeneity is usually lost. Still it makes sense to ask to what extent the (non-homogeneous) strong Markov property carries over to Y. More precisely, we consider random times τ such that the corresponding variables on the reversed time scale are stopping times for Y (we shall call such variables τ *backward times*, since they only "depend regressively on the future" of X), and the question is under what conditions τ splits X into conditionally independent paths, and in case of splitting, what can be said about the conditional distributions of the pre-τ process. Reviewing the splitting literature with this general problem in mind, one recognizes coterminal and cooptional times as special backward times for which splitting has been established under various conditions ([14], [4]). Note also that the desired splitting is known in full generality in the case of discrete time and state space ([7], Lemma 3.12).

In the present paper we shall mainly deal with the case when τ takes values in a perfect and closed regenerative set M with empty interior. (Recall that a random set $M \subset R_+$ is regenerative, if every stopping time τ in M splits M into conditionally independent subsets, such that the post-τ set has the same distribution as M, apart from a shift by τ. For a formal definition and basic properties, see e.g. [3], [5], [12]. The discrete case is similar but simpler. The assumption that M be closed is not very restrictive, since the closure of a regenerative set with empty interior is automatically regenerative.) The above case plays a key role, owing to the celebrated Itô [6] representation of X as a Poisson process of excursions on the inverse local time scale for M. On the original time scale, an equivalent description of X is in terms of the local time random measure ξ of M and the associated set of excursions, which may be regarded as marks attached to the gaps of M. To avoid obscuring details, we shall treat the unmarked case first, and then indicate briefly to what extent the arguments carry over to the general case (which requires some caution with the choice of σ-fields).

To describe our main results, we shall need some notation. Let the basic probability space be $(\Omega, \mathscr{F}, P)$, and denote P-integration by E. Write $f(P)$ for the distribution of the random element f. Lebesgue measure on real intervals will be denoted by λ. By $\mu_1 \ll \mu_2$ we mean that μ_1 is absolutely continuous with respect to μ_2. For arbitrary μ_1 and μ_2 we further define

$$\mu_1 \wedge \mu_2 = \sup\{\mu : \mu \le \mu_1 \quad \text{and} \quad \mu \le \mu_2\},$$

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which clearly corresponds to taking the minimum of the $(\mu_1 + \mu_2)$-densities.

Since the *renewal measure* $\rho = E\xi$ is locally finite, the *Palm distributions* P_t on Ω and with respect to ξ exist and are given by

$$P_t A = \frac{E[\xi(dt); A]}{E\xi(dt)}, \quad A \in \mathscr{F}, \ t > 0 \quad \text{a.e. } \rho,$$

(cf. [9]). Intuitively, P_t is the conditional distribution, given that $t \in M$. Though this conditioning makes sense only when $\lambda M > 0$ a.s., (see e.g. [3]; this case will be referred to below as *Kingman's case* [10]), a formal justification may be given in general in the form of a limit theorem (Theorem 4.2).

In Theorem 2.1 we prove that any backward time $\tau \in M$ splits M into conditionally independent subsets, the conditional distributions on the left being given by $\{P_t\}$, both a.e. $\tau(P) \wedge \rho$ and wherever $\{P_t\}$ has a continuous version. Thus, in particular, the conditional distributions on the past are given by the Palm distributions if either $\tau(P) \ll \rho$, or $\{P_t\}$ is continuous a.e. $\tau(P)$. The main object of Sections 3–5 is to examine when these conditions are fulfilled. According to Theorem 3.1, the first condition holds (for arbitrary random times) whenever τ is located at a fixed local time distance from a gap in M, including the cases of left and right endpoints, hence in most interesting cases.

For the purpose of dealing with the second condition, note that the inverse (cumulative) local time is a subordinator T. Let ν be its associated Lévy measure, and write $\{\mu^t\}$ and $\{\hat{\mu}^t\}$ for the corresponding semigroups of infinitely divisible distributions and their characteristic functions. Utterly weak restrictions on ν (see Section 5) ensure the validity of the condition

(C): *$\hat{\mu}^t$ is integrable for every $t > 0$.*

Note that (C) holds iff $\mu^t \ll \lambda$ for every $t > 0$ with a bounded density $p(t, \cdot)$, and that $p(t, x)$ is then strictly positive within the supporting interval and jointly uniformly continuous for t bounded away from zero. In this case also $\rho \ll \lambda$ with a lower semicontinuous density $p = \int_0^\infty p(t, \cdot)\, dt$. (In Kingman's case, p is proportional to the p-function of M.)

In Theorem 4.3 we prove that, under (C), the continuity set of p is inherited by the family $\{P_t\}$. In the positively recurrent case, i.e. when $m = ET_1 < \infty$, this statement applies to the point at infinity also, with P acting as limiting distribution. In fact, a much stronger statement at infinity, involving uniform mixing under P_t, will be proved in Theorem 4.4. Finally, Theorem 4.6 shows, again under (C), that almost every P_t resembles P at the origin.

The regularity properties of $\{P_t\}$ are hence linked to those of p, so we are led to study the renewal density in further detail, which is done in Section 5. In Theorem 5.2 we give a rather precise description of the continuity set of p in terms of the growth rate at zero and the regularity properties on $(0, \infty)$ of the Lévy measure ν.

For a convenient probabilistic description of M, we choose ξ rather than M as our basic random object, which is possible since ξ and M determine each other uniquely (a.s. and up to a normalizing factor for ξ). Accordingly (unless otherwise stated), we take Ω to be the space of Radon measures μ on R_+, and $\mathscr{F}$ as the σ-field generated by the mappings $\mu \to \mu B$ for all $B \in \mathscr{B}(R_+)$, the class of bounded Borel sets in R_+. Endowing Ω with the vague topology makes it Polish with $\mathscr{F}$ as its Borel σ-field. Hence there is no problem to define conditional distributions on Ω. Weak convergence of probability distributions on $(\Omega, \mathscr{F})$ is in the sense of the vague topology. In Section 4 we shall also consider convergence in the supremum norm $\|\cdot\|$ for signed measures on Ω and related spaces. For general background on random measures, we refer to [9].

The shift, killing and reflection operators θ_t, k_t and Q_t on Ω are defined for arbitrary $t > 0$ by

$$(\theta_t\mu)B = \mu(B + t), \ (k_t\mu)B = \mu(B \cap [0, t]), \ (Q_t\mu)B = \mu(t - B \cap [0, t]) + \mu(B \cap (t, \infty)),$$

where $B \in \mathscr{B}(R_+)$ is arbitrary. Note that they are all $\mathscr{F}$-measurable. We further define $\mathscr{F}_t$ and $\mathscr{G}_t$ to be the σ-fields in Ω generated by k_t and θ_t respectively. A measurable mapping $\tau:\Omega \to R_+$ will be called a *forward* (=stopping) or *backward time* respectively, if $\{\tau \le t\} \in \mathscr{F}_t$ or $\{\tau \ge t\} \in \mathscr{G}_t$ a.s. (P) for every $t \ge 0$. We further define the (cadlag Markov [12]) processes G_t, D_t, A_t and R_t on R_+ by

$$G_t = \sup\{s \le t : s \in M\}, \quad D_t = \inf\{s > t : s \in M\}, \quad A_t = t - G_t, \quad R_t = D_t - t.$$

Recall that, if $\lambda M = 0$, a fixed $t > 0$ lies a.s. in a gap (contiguous interval) I of M. In this case, G_t and D_t are the left (*g*auche) and right (*d*roite) endpoints of I, whereas A_t is the *a*ge and R_t the *r*emainder of I at t.

We conclude by listing some further notational conventions. The restriction of a measure μ to a set B is denoted by $B\mu$. On R_+, a measure μ will be identified with its cumulative function $\mu(t) = \mu_t = \mu[0, t]$. The Fourier transform of μ will be denoted by $\hat{\mu}$ and the Laplace transform by $\tilde{\mu}$. Powers μ^n or μ^t are in the sense of convolution $(*)$. The symbol $\perp$ denotes mutual singularity. Finally, indicator functions are denoted by 1_A o' $1\{\cdot\}$, and Dirac measures by δ_x.

2. Decomposition. The main purpose of this section is to prove, in Theorem 2.1, a decomposition of our regenerative set M at backward times in M. Applications and extensions of this result are discussed at the end of the section. Though splitting at more general backward times is formally outside the scope of the present paper, we cannot resist proving in Theorem 2.2 that arbitrary backward times split a Lévy process.

By saying that a random time τ *splits* ξ (or M), we shall simply mean that the conditional distributions of ξ, given $\tau = t$, are a.s. such that the *past* k_t and the *future* θ_t of t are independent. Splitting of cadlag random processes is defined analogously. (It is evident from our proofs that all splitting results below can be extended to splitting in the sense of conditional independence of k_τ and a suitably defined "σ-field of post-τ events".) In the same spirit, we may regard the conditional distributions of the past of τ as a measurable family of distributions on Ω indexed by R_+ and unique a.s. $\tau(P)$. The Palm distribution at t and with respect to ξ was defined a.e. ρ in Section 1. For the sake of brevity, its k_t-image will be called the *left Palm distribution at t.*

THEOREM 2.1. *Backward times in M are splitting. The conditional distributions of the past for two backward times τ and τ' agree* a.e. $\tau(P) \wedge \tau'(P)$, *and those for τ coincide with the left Palm distributions P_t, both* a.e. $\rho \wedge \tau(P)$ *and wherever $\{P_t\}$ has a continuous version.*

PROOF. We may assume that $\lambda M = 0$ a.s., the modifications needed in Kingman's case being obvious. We shall proceed in five steps.

1. *Here we prove the splitting for a special type of backward times.* More precisely, we assume that either $\tau = 0$ or $A_{\tau+1} = 1$, and that $\{\tau + 1 \ge t\} \in \mathscr{A}_t$ a.s. for every $t > 1$, where $\mathscr{A}_t = \sigma\{A_s;\, s \ge t\}$. Since $\{A_t\}$ is Markov [12], the random time $\tau' = [\tau + 1 -]$ splits $\{A_t\}$, say with conditional distributions $P_{n,x}$ of k_n, given that $\tau' = n > 0$ and $A_{\tau'} = x$, i.e. given that $\tau = n - x$. This is equivalent to saying that τ splits ξ with conditional distributions P_t of k_t, given that $\tau = t$, where $P_t = P_{n,x}$ for $t = n - x$. The Markov property further implies that the P_t can be chosen independently of τ. The same proof applies to the set of random times τ such that, for a fixed $\varepsilon > 0$, either $\tau = 0$ or $A_{\tau+\varepsilon} = \varepsilon$, and moreover $\{\tau + \varepsilon \ge t\} \in \mathscr{A}_t$ a.s. for all $t > \varepsilon$. Since the latter τ-set is non-increasing in ε, the P_t can be chosen independently of ε.

2. *Here we show that τ* a.s. *avoids right endpoints.* To see this, fix a $t > 0$, and let β_s be the number of gaps up to time D_t of size $>s$. From the nature of T it is clear that the first n gaps of length $>s$ are exchangeable. Since β is invariant under permutations of these

intervals whenever $\beta_s = n$ and $R_t > s$, it follows that the gaps of size $>s$ prior to D_t are conditionally exchangeable, given β, on the set $\{s > R_t\}$. (Note that R_t is β-measurable.) Letting $s \downarrow R_t$, we may conclude that the intervals $>R_t$ are conditionally exchangeable, given β. Hence

$$(1) \qquad P[D_t - G_t > \varepsilon \mid \beta] = \beta_\varepsilon/\beta_{R_t} \le \beta_\varepsilon/\beta_h \qquad \text{a.s. on } \{R_t < h\}, \quad 0 < \text{h} < \varepsilon.$$

Now fix an $\varepsilon > 0$. Let γ_s be the number of gaps up to time τ of size $>s$, and note that $\gamma_\varepsilon/\gamma_h \to 0$ a.s. on $\{\tau > 0\}$ as $h \to 0$, since γ is a.s. unbounded on this event. Hence we may choose $h > 0$ so small that $P\{\gamma_\varepsilon/\gamma_h > \varepsilon, \tau > 0\} \le \varepsilon$. Putting $t_n = nh$ and $I_n = (t_n, t_{n+1})$, we then obtain (with $A_{0-} = 0$)

$$(2) \quad P\{A_{\tau-} > \varepsilon\} \le \varepsilon + P\{A_{\tau-} > \varepsilon, \gamma_\varepsilon/\gamma_h \le \varepsilon\} = \varepsilon + \textstyle\sum_{n=1} P\{\tau \in I_n, A_{\tau-} > \varepsilon, \gamma_\varepsilon/\gamma_h \le \varepsilon\}.$$

Here the nth term equals, with $t = t_n$ and with β based on this particular t,

$$\begin{aligned} &P\{\tau = D_t \in I_n, D_t - G_t > \varepsilon, \beta_\varepsilon/\beta_h \le \varepsilon\} \\ &= E[P[D_t - G_t > \varepsilon \mid \mathscr{G}_t, \beta]; \tau = D_t \in I_n, \beta_\varepsilon/\beta_h \le \varepsilon] \\ &= E[P[D_t - G_t > \varepsilon/\beta]; \tau = D_t \in I_n, \beta_\varepsilon/\beta_h \le \varepsilon] \le \varepsilon\, P\{\tau \in I_n\}, \end{aligned}$$

where we have used in turn the definition of backward times, splitting at D_t, and (1). Inserting this into (2) yields $P\{A_{\tau-} > \varepsilon\} \le 2\varepsilon$, and since ε was arbitrary, it follows that $A_{\tau-} = 0$ a.s., as asserted.

3. *Here we prove that ξ obeys a $0-1$ law on the immediate left of τ.* For this purpose, fix $s, t > 0$, and note that the processes $\{T_x, x \in [0, s]\}$ and $\{T_s - T_{s-x}, x \in [0, s]\}$ have the same distribution, apart from the interchange of right and left continuity. This remains conditionally true, given ξ_t and θ_t with $\xi_t > s$, since on the latter event, these two quantities are invariant under the reflection which relates the two processes. Letting $s \to \xi_t$ along a fixed countable set, it follows easily, e.g. by considering the associated random measures, that the above conditional symmetry extends to the (open) interval $(0, \xi_t)$. The latter symmetry is clearly equivalent to a similar symmetry of ξ on the interval $[0, G_t]$. By the definition of backward times, a further conditioning on τ makes no difference provided that $\tau > t$, and on that event the symmetry remains true under conditioning on τ only. Letting $t \to \tau$ along a fixed countable set, and noting that then $G_t \to \tau$ a.s. by step 2 above, we may conclude that ξ is symmetric (even unconditionally) on $[0, \tau]$. The asserted $0-1$ property now follows from the Blumenthal $0-1$ law for ξ at 0.

4. *Here we extend the splitting to arbitrary τ.* Write $\mathscr{G}_\tau = \sigma(\tau, \theta_\tau)$, and similarly for other random times. A bar over $\mathscr{G}$ will denote P-completion. Define

$$\tau_n = \sup\{t \in M \cap [0, \tau) : R_t \ge 1/n\}, \; n \in N,$$

(sup $\varnothing$ being interpreted as 0). Then $\mathscr{G}_{\tau_n} \supset \mathscr{G}_\tau$, and since $\tau_n \to \tau$ a.s. by step 2, it is easily verified that $\bar{\mathscr{G}}_{\tau_n} \downarrow \bar{\mathscr{G}}_{\tau-}$, where $\mathscr{G}_{\tau-} = \bigcap \mathscr{G}_{\tau-1/n}$. But $\bar{\mathscr{G}}_{\tau-} = \bar{\mathscr{G}}_\tau$ by step 3. Hence, by martingale theory, $P(\cdot \mid \mathscr{G}_{\tau_n}) \to P(\cdot \mid \mathscr{G}_\tau)$ a.s. Now fix a $t > 0$, and let $A \in \mathscr{F}_t$. Then $P(A \mid \mathscr{G}_{\tau_n}) = P_{\tau_n}A$ a.s. on $\{\tau_n > t\}$ by step 1, so we get $P_{\tau_n}A \to P(A \mid \mathscr{G}_\tau)$ a.s. on $\{\tau > t\}$ since $\{\tau_n > t\} \uparrow \{\tau > t\}$ a.s. Hence $P(A \mid \mathscr{G}_\tau)$ is a.s. measurable on $\{\tau > t\}$ with respect to τ and the tail of the sequence $\{\tau - \tau_n\}$. But the tail σ-field of $\{\tau - \tau_n\}$ is trivial by step 3, and so the measurability is on τ alone, i.e. $P(A \mid \mathscr{G}_\tau) = P(A \mid \tau)$ a.s. on $\{\tau > t\}$. This is clearly equivalent to the asserted splitting. The stated independence of the choice of τ follows immediately from the splitting by an independent randomization of τ, the randomization parameter being thought of as $\mathscr{G}_\infty$-measurable.

5. *Here we establish the connection with the Palm distributions.* For this purpose, fix $b > a > 0$, and define

$$\tau_x = \sup\{t \in [0, b]: \;\; \xi[t, b] = x\}, \; x > 0.$$

Then each τ_x is clearly a backward time, so by the randomization argument in step 4 and Fubini's theorem, there exists a family of conditional distributions P_t on $\mathscr{F}_t$, serving simultaneously for almost all τ_x (λ). By the same argument, the P_t coincide a.e. $\rho \wedge \tau(P)$ with the conditional distributions on the past of τ for any specific backward time τ.

Write $I = [a, b]$. Note that

$$\xi I = \sup\{x:\ \tau_x \in I\} = \int_0^\infty 1\{\tau_x \in I\}\, dx,$$

and hence that $\xi = \int 1\{\tau_x \in \cdot\}\, dx$ on $[0, b]$. Letting $A \in \mathscr{F}_a$, we get

$$E[\xi I; A] = E\int_0^\infty 1[\tau_x \in I, A]\, dx = \int_0^\infty P[\tau_x \in I, A]\, dx = \int_0^\infty dx \int_I P_t A \quad P\{\tau_x \in dt\}$$

$$= \int_I P_t A \int_0^\infty P\{\tau_x \in dt\}\, dx = \int_I P_t A\, E\int_0^\infty 1\{\tau_x \in dt\}\, dx = \int_I P_t A\, E\xi(dt).$$

Since a, b and A were arbitrary, this shows that $k_s(P_t)$ equals the Palm distribution of k_s at t for fixed $s < t$ and for $t > 0$ a.s. ρ. It remains to let $s \to t$ along a fixed countable set.

If τ is a.s. a left endpoint, then $\tau(P) \ll \rho$ by Theorem 3.1 below, so in that case the conditional distributions on the left can be chosen as the Palm distributions. This applies in particular to the auxiliary random times τ_n in step 4. Now suppose that $\{P_t\}$ has a version which is left continuous on some set $C \subset R_+$. Since $\tau_n \uparrow \tau$ a.s., and moreover $P_{\tau_n} A \to P(A \mid \tau)$ a.s. on $\{\tau > s\}$ for any $A \in \mathscr{F}_s$, $s > 0$, we may then choose the conditional distributions on the left to agree with $\{P_t\}$ even on C. □

The case when the regenerative set M is embedded in a Markov process is more complex, mainly due to the fact that, by definition, backward times may depend not only on "the future" but also on "the immediate past". When only M is considered, this causes no trouble because of the $0 - 1$ property on the left proved in step 3 above, but in the Markov setup, the analogous splitting statement is no longer true in general. In fact, even for the age process $\{A_t\}$ based on M, splitting at backward times requires conditioning on both τ and $A_{\tau-}$, and then the simple connection with the Palm distributions is lost.

One way out of this trouble is to *assume* that the conclusion in step 2 is fulfilled, i.e. that τ a.s. avoids right endpoints. Another way is to think of the excursions as marks associated with the *left* endpoints, and to define $\{\mathscr{G}_t\}$ accordingly. In both cases, the above proof goes through without changes. A third possibility is to impose conditions on X which imply a $0 - 1$ law on the left of backward times. We may e.g. assume X to be a Lévy process based on an infinite Lévy measure. This is a consequence of the following theorem and the fact that X is a.s. continuous on the time set where X hits a fixed point u (hence $X(\tau) = X(\tau-) = u$ in this case). Write $\hat{\mu}^t$ for the characteristic function of $X(t)$.

THEOREM 2.2. *Let X be a Lévy process, and τ a backward time based on X. Then the pre-τ and post-τ segments of X are conditionally independent, given τ and $X(\tau-)$. Under (C), the conditional distributions on the past, given $\tau = t > 0$ and $X(\tau-) = x$, may further be taken from a fixed jointly continuous family $\{P_{t,x}\}$.*

The continuity here is in the sense of weak convergence with respect to the Skorohod J_1 topology in $D[0, 1]$ (cf. [1]) for the rescaled processes $\{X(st),\ s \in [0, 1]\}$. Since, as it turns out, the conditional processes on the past have exchangeable increments, it is equivalent (and in fact simpler) to state the continuity in terms of the associated canonical random elements (cf. Theorem 2.3 in [8]).

PROOF. It τ takes values in a fixed countable set, the splitting is an immediate consequence of the Markov property, and it follows for general τ by approximation from the left as in steps 3–4 of the previous proof.

To prove the second assertion, note that (C) implies $X_t(P) \ll \lambda$ for every $t > 0$ with a density $p(t, \cdot)$ which is bounded and jointly continuous for t bounded away from zero. Letting $0 < s < t$ and $A \in \mathscr{F}_s$, we get a.s.

$$P[A \mid X_t = x] = \frac{E[p(t-s, x-X_s); A]}{Ep(t-s, x-X_s)}, \; x \in R,$$

so by dominated convergence, the left-hand side is jointly continuous in total variation on $\mathscr{F}_s$. Since the increments of X are conditionally exchangeable on $[0, t]$, the stated continuity now follows easily by Theorem 2.3 in [8]. □

One interesting consequence of Theorems 2.1 and 2.2 (and of Theorem 4.2 below) is the fact that, for Lévy processes X which hit points and satisfy (C), horizontal and vertical window conditioning are asymptotically equivalent as the window size tends to zero. More precisely, writing $H_\varepsilon(t, x)$ and $V_\varepsilon(t, x)$ for the events that the graph of X intersects the sets $[t-\varepsilon, t+\varepsilon] \times \{x\}$ and $\{t\} \times [x-\varepsilon, x+\varepsilon]$ respectively, it is seen that $P[\cdot \mid H_\varepsilon(t, x)]$ and $P[\cdot \mid V_\varepsilon(t, x)]$ both approach $P_{t,x}$ on the left and $\theta_t(P)$ on the right of t (in an obvious sense).

3. Distributions of random times in M. The local time ξ of M may be defined as a constant times the dual previsible projection of the point process M_ε of left endpoints belonging to gaps $>\varepsilon$ (provided that $\nu(\varepsilon, \infty) > 0$). Taking expectations, it follows that EM_ε is proportional to ρ. Hence $\tau(P) \ll \rho$ for any random time τ in M_ε, and this result extends by an obvious truncation argument to arbitrary random times in the set of left endpoints. (This simple fact was used in the proof of Theorem 2.1.) We shall extend the above result to a much larger class of random times, which also includes the right endpoints. Though the present study was motivated by Theorem 2.1, the results may have some independent interest.

THEOREM 3.1. *Let τ be an M-valued random time at a fixed local time distance from a gap. Then $\tau(P) \ll \rho$.*

PROOF. For definiteness we may assume that $\lambda M = 0$ a.s., the opposite case being similar. Let $\{B_n\}$ be a countable partition of the interval $(0, \infty]$ into relatively compact Borel sets. Then every gap of a fixed sample path of M can be characterized uniquely as being the kth gap with size in B_n for some k and n. This yields a measurable partition $\{A_{nk}\}$ of Ω, where A_{nk} is the event where the (unique) gap related to τ is the one indexed by n and k. Let $\tau_{nk} \in M$ be the a.s. unique point at the given right or left distance c from the gap with index (n, k), (if such a point exists; otherwise we put $\tau_{nk} = 0$). Since $\tau = \sum_{n,k} 1_{A_{nk}}\tau_{nk}$, we get $\tau(P) = \sum_{n,k} P[\tau_{nk} \in \cdot; A_{nk}]$, so it is enough to prove that $\tau_{nk}(P) \ll \rho$ on $(0, \infty)$ for fixed (n, k) with $\nu B_n > 0$. To simplify the notation we may henceforth omit the subscripts n and k.

It is easily seen from the Itô (1972) representation of M that

$$\tau = T'_\vartheta + \textstyle\sum_{r=1}^{\kappa} \beta_r, \tag{1}$$

where T' is a subordinator based on $B^c\nu$; ϑ is obtained from the position of the kth atom of an independent Poisson process with rate νB by adding or subtracting c and reducing to zero when the resulting quantity is negative; $\beta_1, \beta_2, \cdots$ are i.i.d. random variables which are independent of T' and ϑ with distribution $B\nu/\nu B$; and κ is some Z_+-valued random variable which is zero on the set $\{\vartheta = 0\}$. The crucial property of ϑ is that $\vartheta(P) \ll \lambda$ on $(0, \infty)$, and since $\vartheta = 0$ implies $\tau = 0$, we may assume without loss that $\vartheta > 0$ a.s. We may further reduce as above to the case when $\kappa = m$ is fixed.

Writing μ' and μ'' for the infinitely divisible distributions based on $B^c\nu$ and $B\nu$ respectively, and letting ξ' be the local time random measure associated with μ', we get

$$T'_\vartheta(P) = E\mu'^{\vartheta} = \int \mu'^t P\{\vartheta \in dt\} \ll \int \mu'^t \, dt = E\xi'. \tag{2}$$

Next define ξ_n as ξ, except that the gaps in B which occur before local time n are omitted. It is clear from the Itô representation of ξ that $\xi_n(P)$ agrees with the conditional distribution of ξ, given that no B-gaps occur before local time n. Since this is an elementary conditioning, it follows that $E\xi_n \ll E\xi = \rho$. For fixed $t > 0$ we now make a measurable partition $\Omega = \cup A_n$, such that on A_n, $\xi' = \xi_n$ on $[0, t]$. Then $\xi' = \sum 1_{A_n} \xi_n$ on $[0, t]$, and we may conclude as before that $E\xi' \ll \rho$ on $[0, t]$. Since t was arbitrary, the same relation holds on R_+. Hence $T'_\vartheta(P) \ll \rho$ by (2).

By (1) it remains to prove that $\rho*(B\nu)^m \ll \rho$. To see this, note that $\mu''^t*(B\nu)^m \ll \mu''^t$ for all $t > 0$, since μ''^t is a compound Poisson distribution based on $B\nu/\nu B$. Convolving with μ'^t we get $\mu^t*(B\nu)^m \ll \mu^t$, and the desired relation follows from this by integration with respect to t. □

The above technique is readily adapted to prove absolute continuity on $(0, \infty)$ for virtually all random times of practical interest (except in Kingman's case, when the absolute continuity fails even for the basic quantities G_t and D_t). Thus it is natural to conjecture that, when $\lambda M = 0$ a.s., *every* random time in M would have the stated property. A counterexample is given below for a certain rather special class of regenerative sets. The random times under consideration have the further interesting property of being simultaneously forward and backward.

THEOREM 3.2. *There exist regenerative sets M with $\lambda M = 0$ a.s. such that the random times T_x are both forward and backward and have ρ-singular distributions.*

PROOF. Define $\nu = \sum k^{-1}\delta_{c_k}$, where the constants $c_k < (k!)^{-1}$ will be chosen later, and note that $\nu R_+ = \infty$ while $\int x\nu\,(dx) < \infty$. Clearly $T_t = \sum \alpha_k(t) c_k$, where α_k are independent homogeneous Poisson processes on R_+ with rates k^{-1}. Since $P\{\alpha_k(t) \geq 2\} = 0(k^{-2})$ as $k \to \infty$ for every $t > 0$, it follows from the Borel-Cantelli lemma that a.s.

$$\limsup_{k\to\infty} \alpha_k = 1, \quad t \geq 0, \tag{3}$$

and we may assume without loss that (3) holds identically.

Define for fixed $\{c_k\}$

$$A_n = \{\textstyle\sum_1^\infty a_k c_k;\ a_1 = n,\ a_2, a_3, \cdots \in Z_+, \sup a_k < \infty\}, \quad n \in Z_+.$$

Then $\lambda A_n = 0$ for all n, since for fixed $b > 0$ and for every $n \geq 2$,

$$\lambda\{\textstyle\sum_1^\infty a_k c_k;\ a_1, a_2, \cdots \in Z_+, \sup a_k < b\} \leq b^{n-1} \sup\{\sum_n^\infty a_k c_k; \sup a_k < b\} \leq 2b^n/n!,$$

which tends to zero as $n \to \infty$. Together with (3) this implies that, for fixed $\xi \in \Omega$, $t > 0$, $n \in Z_+$ and $c_2, c_3, \cdots$ as above, and for $c_1 \in (0, 1)$ a.e. λ,

$$\alpha_1(t) = n \quad \text{iff} \quad T_t \in A_n. \tag{4}$$

Now suppose that ξ and $\{c_k\}$ are such that (4) holds simultaneously for all $n \in Z_+$ and $t \in N$. If (4) fails at some t, it must be because $T_t \in A_m$ for some $m \neq \alpha_1(t)$. But then (3) shows that (4) fails for all subsequent t, which is a contradiction. Thus the exceptional null-set for c_1 can be chosen independently of n and t. By Fubini's theorem, almost all c_1, $c_2, \cdots$ (λ^∞) are such that (4) holds a.s., simultaneously for all n and t.

The sets A_n are measurable, since the corresponding sets with a fixed bound for $\{a_k\}$ are closed. The function $f_1 = \sum n 1_{A_n}$ is therefore measurable, and (4) implies that a.s. $\alpha_1(t) = f_1(T_t)$ for all t. In the same way, it is possible for almost all $\{c_k\}$ to construct measurable functions $f_2, f_3, \cdots$ such that

$$\alpha_k(t) = f_k(T_t), \quad t > 0,\ k \in N, \quad \text{a.s.} \tag{5}$$

Next note that, by the law of large numbers,

$$\lim_{n\to\infty} \textstyle\sum_1^n \alpha_k(t)/\sum_1^n k^{-1} = t, \quad t > 0, \quad \text{a.s.,} \tag{6}$$

the exceptional null-set being independent of t because of the monotonicity of both sides. Putting $f = g \circ (f_1, f_2, \cdots)$ where

$$g(a_1, a_2, \cdots) = \liminf_{n\to\infty} \textstyle\sum_1^n a_k / \sum_1^n k^{-1},$$

it follows from (5) and (6) that

$$f(T_t) = t, \quad t > 0, \quad \text{a.s.} \tag{7}$$

In particular,

$$\xi_x = \xi_x = f(D_x), \quad x > 0, \quad \text{a.s.},$$

so the process $\{\xi_x\}$ is adapted not only to $\{\mathscr{F}_x\}$ but also to $\{\mathscr{G}_x\}$. Since $\{\xi_x\}$ is further a.s. continuous, its passage times T_t must be both forward and backward.

Finally put $F_t = f^{-1}\{t\}$, $t > 0$, and note that, by (7),

$$\mu_s F_t = P\{T_s \in f^{-1}\{t\}\} = P\{f(T_s) = t\} = \delta_{s,t}, \quad s, t > 0,$$

and therefore

$$\rho F_t = \int_0^\infty \mu_s F_t \, ds = \int_0^\infty \delta_{s,t} \, ds = 0, \quad t > 0. \square$$

4. The Palm distributions. The main object of this section is to examine the continuity, asymptotic, and other properties of the family $\{P_t\}$ of Palm distributions with respect to the local time random measure ξ of M. For the sake of simplicity we consider the unmarked case only. Most results carry over without changes to the case when excursion marks are attached to the intervals.

First we describe the basic structure of the P_t.

LEMMA 4.1. *For $t > 0$ a.e. ρ,*

$$(k_t, \theta_t)(P_t) = k_t(P_t) \times P, \quad Q_t(P_t) = P_t,$$

and

$$k_t(P_t) = \int_0^\infty dx \, k_t(P[\cdot \mid T_x = t])\mu^x(dt)/\rho(dt). \tag{1}$$

PROOF. Let $I = [a, b]$ and $A \in \mathscr{F}_a$, $B \in \mathscr{G}_b$ be arbitrary. Since $\xi I = \int 1\{T_x \in I\} \, dx$, we get by Fubini's theorem and splitting at T_x

$$\begin{aligned} E[\xi I; A \cap B] &= E\int_0^\infty 1\{T_x \in I, A \cap B\} \, dx = \int_0^\infty P\{T_x \in I, A \cap B\} \, dx \\ &= \int_0^\infty E[P(A \cap B \mid T_x); T_x \in I] \, dx \\ &= \int_0^\infty E[P(A \mid T_x)P(\theta_{T_x} B); T_x \in I] \, dx \\ &= \int_0^\infty dx \int_I P(A \mid T_x = t)P(\theta_t B) \, \mu^x(dt) \\ &= \int_I P(\theta_t B) \int_0^\infty dx \, P(A \mid T_x = t) \, \mu^x(dt). \end{aligned}$$

On the other hand, we get by the definition of Palm distributions

$$E[\xi I; A \cap B] = \int_I P_t(A \cap B)\, \rho(dt).$$

Since I was arbitrary, it follows that, for $A \in \mathscr{F}_a$ and $B \in \mathscr{G}_b$ with a and b fixed and for ρ-a.e. $t \in (a, b)$,

$$P_t(A \cap B) = P(\theta_t B) \int_0^\infty dx\, P[A \mid T_x = t]\, \mu^x(dt)/\rho(dt).$$

For a.e. t, we may now use a montone class argument to extend the last relation to arbitrary $A \in \mathscr{F}_t$ and $B \in \mathscr{G}_t$. This proves the asserted independence, as well as the identities of $k_t(P_t)$ and $\theta_t(P_t)$.

To prove the symmetry assertion, note that, for fixed $x > 0$, the processes $\{T_s, s \in [0, x]\}$ and $\{T_x - T_{x-s}, s \in [0, x]\}$ have the same distribution except for an interchange of right and left continuity. This symmetry of T is clearly preserved under conditioning on the invariant variable T_x, and under this conditioning, the above symmetry is equivalent to symmetry of the inverse process $\{\xi_t, t \in [0, T_x]\}$ under the reflexion $t \to T_x - t$. Hence $Q_t(P[\cdot \mid T_x = t]) = P[\cdot \mid T_x = t]$ a.s., and inserting this into (1) yields the desired conclusion. □

Lemma 4.1 and the methods proving it give rise to some nice formulae. Note in particular that (with E_t denoting P_t-integration)

$$E_t \xi_t^n = n! \frac{\rho^{n+1}(dt)}{\rho\,(dt)}, \quad t > 0 \text{ a.e. } \rho, \quad n \in \mathrm{N},$$

and further that, whenever ρ has a Lebesgue density p,

$$\frac{E_t\xi\,(ds)}{ds} = \frac{p(s)p(t-s)}{p(t)}, \quad \text{a.e. } s, t, \quad 0 < s < t. \tag{2}$$

Intuitively (and in Kingman's case even formally), P_t is the conditional distribution, given that $t \in M$. For a precise statement, let $P_{t,\varepsilon}$ be defined analogously by conditioning on $\{t \in M_\varepsilon\}$, where M_ε is the ε-neighbourhood of M. Let us further define the measures ρ_ε by

$$\rho_\varepsilon I = \int_I P\{t \in M_\varepsilon\}\, dt = E\lambda(M_\varepsilon \cap I).$$

Theorem 4.2. *For any finite interval I,*

$$\lim_{\varepsilon \to 0} \frac{1}{\rho\varepsilon I} \int_I P_{t,\varepsilon}\, \rho_\varepsilon(dt) - \frac{1}{\rho I} \int_I P_t\, \rho(dt) = 0. \tag{3}$$

Moreover, $P_{t,\varepsilon} \to_w P_t$ *wherever* $\{P_t\}$ *is weakly continuous.*

This approximation of P_t corresponds to Kingman's [11] intrinsic description of ξ. Other elementary local time constructions give rise to alternative but less natural approximations.

Proof. Define for each $\varepsilon > 0$ the random measure $\zeta_\varepsilon = \lambda(M_\varepsilon \cap \cdot)$ on R_+, and note that $\rho_\varepsilon = E\zeta_\varepsilon$. Let $m_\varepsilon = E\zeta_\varepsilon(T_1)$, and put $\xi_\varepsilon = \zeta_\varepsilon/m_\varepsilon$. By [11] we have $\xi_\varepsilon(t) \to \xi(t)$ a.s. as $\varepsilon \to 0$, simultaneously for all t, and we shall need the fact that this convergence is true in the L_1 sense also, i.e. that $\xi_\varepsilon(t)$ is uniformly integrable. This is obvious for $t = T_n$, since in this case both sides have expectation n, and it extends immediately to the random times $t \wedge T_n$, $t > 0$, $n \in N$. It remains to observe that, by Wald's identity,

$$E(\xi_\varepsilon(t) - \xi_\varepsilon(t \wedge T_n)) \le E([\xi(t)] + 1 - n)_+ \to 0.$$

The L_1 convergence implies that $E[\xi_\varepsilon I; A] \to E[\xi I; A]$ holds uniformly in A for every finite interval I, and (3) follows if we divide by the corresponding expressions with $A = \Omega$, since $E\xi I > 0$.

Suppose next that $P\cdot$ is continuous at t. Clearly

$$\{t \in M_\varepsilon\} = \{D_{t-\varepsilon} \le t + \varepsilon\} = \{G_{t+\varepsilon} \ge t - \varepsilon\}, \quad 0 < \varepsilon < t.$$

Let $s < t - \varepsilon$ and $u > t + \varepsilon$ be fixed, and conclude by splitting at $D_{t-\varepsilon}$ and $G_{t+\varepsilon}$ respectively that $k_s(P_{t,\varepsilon}[\cdot \mid \theta_u])$ is a mixture of $k_s(P_{t'})$ while $\theta_u(P_{t,\varepsilon}[\cdot \mid k_s])$ is a mixture of $\theta_u \circ \theta_{t'}^{-1}(P)$ for $t' \in (t - \varepsilon, t + \varepsilon)$. Hence the former measures converge uniformly (an any metrization of the weak topology) to $k_s(P_t)$ and $\theta_{u-t}(P) = \theta_u(P_t)$ respectively. By the definition of weak convergence, the uniformity of this convergence implies that even

$$(k_s, \theta_u)(P_{t,\varepsilon}) \to_w k_s(P_t) \times \theta_u(P_t) = (k_s, \theta_u)(P_t), \quad s < t < u.$$

Our argument further shows that, for any $c > 0$,

$$\limsup_{\varepsilon \to 0} P_{t,\varepsilon}\{\xi[s, u] > c\} \le P_t\{\xi(t - s) \ge c/2\} + P\{\xi(u - t) \ge c/2\} \to 0, \quad s, u \to t,$$

so the above convergence of $P_{t,\varepsilon}$ extends to the whole line by Theorem 4.9 in [9]. □

Motivated by Theorem 2.1, we next examine the continuity of the family $\{P_t\}$.

THEOREM 4.3. *If (C) holds and p is continuous at some $u > 0$, then P_t is weakly continuous at u. Indeed,*

$$\lim_{t \to u} \|(k_s, \xi_t)(P_t) - (k_s, \xi_u)(P_u)\| = 0, \quad s < u. \tag{4}$$

PROOF. By Lemma 4.1,

$$\frac{E[\xi\,(dt); \xi_t \in B]}{dt} = p(t)P_t\{\xi \in B\} = \int_B p(t, x)\,dx.$$

Splitting at D_s and using Fubini's theorem, we thus obtain for any $C \in \mathscr{F}_s$ and $B \in \mathscr{B}(R_+)$

$$\frac{E[\xi\,(dt); C, \xi_t - \xi_s \in B]}{dt} = \frac{EP(C \mid D_s)E[\xi\,(dt - D_s); \xi_{t-D_s} \in B/D_s]}{dt}$$
$$= EP(C \mid D_s) \int_B p(t - D_s, x)\,dx,$$

and hence by extension, for any $A \in \mathscr{F}_s \times \mathscr{B}(R_+)$,

$$q_A(t) \equiv \frac{E[\xi\,(dt); (\xi, \xi_t - \xi_s) \in A]}{dt} = E \iint_A P(dy \mid D_s)p(t - D_s, x)\,dx.$$

Now the functions $p(\cdot, x)$ are uniformly equicontinuous for x bounded away from zero, so q_A is continuous whenever the R_+-projection of A is contained in some interval $[r, r']$ with $0 < r < r' < \infty$. But the integrability at infinity is uniform in t, since q_A is continuous for $A = \Omega_r \equiv \Omega \times [r, \infty)$ (cf. the proof of Theorem 5.2), so q_A is in fact continuous whenever $A \subset \Omega_r$, and this continuity is clearly uniform in A for fixed r. Writing $A_r = A \cap \Omega_r$, it follows in particular that q_{A_r} is equicontinuous for fixed r.

Abbreviate $q_A = q$, $q_{A_r} = q_r$, $q_{\Omega_r} = p_r$. Suppose that p is continuous at $u > s$, and let $\varepsilon > 0$ be fixed. By monotone convergence we may choose an $r > 0$ such that, for all A,

$$0 \le q(u) - q_r(u) \le p(u) - p_r(u) < \varepsilon.$$

New we may choose $\delta > 0$ so small that for all A

$$|p(t) - p(u)| < \varepsilon, \quad |q_r(t) - q_r(u)| < \varepsilon, \quad |t - u| < \delta.$$

For such t and all A,

$$0 \leq q(t) - q_r(t) \leq p(t) - p_r(t) \leq |p(t) - p(u)| + |p(u) - p_r(u)| + |p_r(u) - p_r(t)| < 3\varepsilon,$$

so

$$q(t) - q(u) \leq |q(t) - q_r(t)| + |q_r(t) - q_r(u)| + |q_r(u) - q(u)| < 5\varepsilon.$$

Thus $q = q_A$ is equicontinuous at u, and hence so is $P_t\{(\xi, \xi_t - \xi_s) \in A\} = q_A/p$. This yields (4) since ζ_s is $\mathscr{F}_s$-measurable.

By Lemma 4.1, $\|(\theta_{t-s}(P_t) - (\theta_{u-s}(P_u)\| \to 0$ for fixed $s < u$ as $t \to u$, so

$$([t - s, \infty)\xi)(P_t) \to_w ([u - s, \infty)\xi)(P_u), \quad t \to u.$$

Since moreover $\{P_t\}$ is tight at the origin for t near u, it follows easily by Theorem 4.2 in [1] that $P_t \to_w P_u$. □

The above connection between the continuity sets of $P.$ and p is not merely technical. In fact, it is easily seen from the definition of Palm distributions and from the proof of Theorem 5.3 below that $P.$ is discontinuous on the set S'_d defined in Section 5.

The above argument is easily modified to apply to the case when $u = \infty$, yielding convergence of P_t towards P whenever $p(t) \to m^{-1} > 0$. A strengthened version of this result involving a uniform mixing property will be proved next.

THEOREM 4.4. *If (C) holds and $m < \infty$, then*

$$\lim_{s\to\infty} \sup_{u-r\geq s} \int_u^{u+1} \|(k_r, \theta_{r+s})(P_t) - k_r(P) \times (\theta_{r+s}\circ Q_t)(P)\| \, dt = 0. \tag{5}$$

If moreover $p(t) \to m^{-1}$, then (5) *can be strengthened to*

$$\lim_{s\to\infty} \sup_{t-r\geq s} \|(k_r, \theta_{r+s})(P_t) - k_r(P) \times (\theta_{r+s}\circ Q_t)(P)\| = 0. \tag{6}$$

The convergence and mixing property in (6) has a counterpart for P itself. Write $\bar{P}$ for the stationary version of P, which exists when $m < \infty$.

LEMMA 4.5. *If (C) holds and $m < \infty$, then*

$$\lim_{s\to\infty} \sup_{t\geq 0} \|(k_t, \theta_{t+s})(P) - k_t(P) \times \bar{P}\| = 0.$$

PROOF. Since $m < \infty$ and moreover $T_1(P) \ll \lambda$ by (C), the embedded renewal process $\sum_{n=1}^{\infty} \delta_{T_n}$ can be coupled to its stationary version, (cf. [13]). Applying the same coupling to ξ, we get

$$\lim_{t\to\infty} \|\theta_t(P) - \bar{P}\| = 0.$$

Since $P(0) = 0$, it follows in particular that $\{R_t\}$ is tight. For fixed $\varepsilon > 0$, we may therefore choose $r > 0$ so large that

$$\sup_{t\geq 0} P\{R_t > r\} < \varepsilon, \qquad \sup_{t\geq r} \|\theta_t(P) - \bar{P}\| < \varepsilon.$$

Taking $s \geq 2r$, it follows by splitting at D_t that

$$\begin{aligned}\|(k_t, \theta_{t+s})(P) - k_t(P) \times \bar{P}\| &\leq E\,\|\theta_{t+s}(P[\cdot \mid D_t]) - \bar{P}\| \\ &\leq P\{D_t > t + r\} + E[\|\theta_{t+s-D_t}(P) - \bar{P}\|; D_t \leq t + r] \leq 2\varepsilon. \ \square\end{aligned}$$

PROOF OF THEOREM 4.4. We prove (6) first. Let $A \in \mathscr{F}_s$, and conclude as in the proof of Theorem 4.3 that

(7) $$q_A(t) \equiv \frac{E[\xi(dt); A]}{dt} = EP(A \mid D_s)p(t - D_s).$$

Put for $r > 0$

$$q_{A,r}(t) = EP(A \mid D_s) \int_r^\infty p(t - D_s, x)\, dx,$$

and note that $q_{A,r}(t) \to PA/m$ uniformly in A by dominated convergence. Abbreviating $q = q_A$, $q_r = q_{A,r}$, $p_r = q_{\Omega,r}$, we hence obtain for fixed r

$$|q(t) - q_r(t)| \le |p(t) - p_r(t)| \to \left| \frac{1}{m} - \frac{1}{m} \right| = 0,$$

so $q(t) \to PA/m$, and therefore

$$P_t A = \frac{q(t)}{p(t)} \to \frac{PA/m}{1/m} = PA,$$

uniformly in A. This proves that

(8) $$\lim_{t\to\infty} \|k_s(P_t - P)\| = 0, \quad s > 0.$$

Let $\varepsilon > 0$ be arbitrary. By Lemma 4.5 and the fact that $P(0) = 0$, we may choose $s > 0$ so large that

$$\|\theta_s(P - \bar{P})\| < \varepsilon, \quad \bar{P}\{D_0 > s\} < \varepsilon.$$

For $t \ge 3s$, the first inequality yields

$$\|k_{2s} \circ Q_t(P - \bar{P})\| \le \|\theta_{t-2s}(P - \bar{P})\| < \varepsilon.$$

By (8), we may next choose $t_0 \ge 3s$ so large that

$$\|k_{2s}(P_t - P)\| < \varepsilon, \quad t \ge t_0.$$

Combining these bounds and noting that $\bar{P} = Q_t(\bar{P})$, we get

$$\|k_s \circ \theta_s(P_t - Q_t(P))\| < 3\varepsilon, \quad P_t\{R_s > s\} < 3\varepsilon, \quad Q_t(P)\{R_s > s\} < 2\varepsilon.$$

Hence it follows by splitting at D_s that

$$\begin{aligned} \|k_{t-s}(P_t - P)\| &= \|\theta_s(P_t - Q_t(P))\| = \|D_s(P_t - Q_t(P))\| \\ &\le P_t\{D_s > 2s\} + Q_t(P)\{D_s > 2s\} + \|k_s \circ \theta_s(P_t - Q_t(P))\| < 8\varepsilon. \end{aligned}$$

Indeed, Theorem 2.1 remains valid for each P_t, and the family of conditional distributions is the same as for P because of (2). Since $\|k_{t-s}(P_t - P)\|$ is nonincreasing in $s \le t$, our argument proves that

(9) $$\lim_{s\to\infty} \sup_{t\ge s} \|k_{t-s}(P_t - P)\| = 0.$$

From (9) and Lemma 4.5 it follows that the family $A_r(P_t)$, $0 \le r \le t < \infty$, is tight. Splitting at G_r for arbitrary $r \in [s, t]$ and using (9), we may conclude as in the proof of Lemma 4.5 that

$$\lim_{s\to\infty} \sup_{s\le r\le t} \|(k_{r-s}, \theta_r)(P_t) - k_{r-s}(P) \times \theta_r(P_t)\| = 0.$$

It remains to note that $\theta_r(P_t)$ can be replaced by $\theta_r \circ Q_t(P)$ because of (9).

Turning to the proof of (5), conclude from Lemma 5.1 below that (C) implies

$$\lim_{u\to\infty} \int_u^{u+1} \left| p(t) - \frac{1}{m} \right| dt = 0.$$

Arguing as before, we thus obtain

$$\lim_{u\to\infty} \int_u^{u+1} \sup_A \left| q_A(t) - \frac{PA}{m} \right| dt = 0.$$

Since moreover $\liminf p(t) = m^{-1}$, we get for large t

$$| P_t A - PA | = \left| \frac{q_A(t)}{p(t)} - PA \right| \le \left(\left| q_A(t) - \frac{PA}{m} \right| + PA \left| p(t) - \frac{1}{m} \right| \right) \Big/ p(t)$$

$$\le 4m \sup_A \left| q_A(t) - \frac{PA}{m} \right|.$$

Hence

$$\lim_{u\to\infty} \int_u^{u+1} \|k_s(P_t - P)\| \, dt = 0, \quad s > 0.$$

Proceeding as in the proof of (9), we may conclude that

$$\text{(10)} \qquad \lim_{s\to\infty} \sup_{u\ge s} \int_u^{u+1} \|k_{t-s}(P_t - P)\| \, dt = 0.$$

Splitting P_t at G_{t-r} for $0 \le r \le t - 2s$ yields

$$\|(k_{t-r-2s}, \theta_{t-r})(P_t) - k_{t-r-2s}(P) \times \theta_{t-r}(P_t)\| \le E_t \| k_{t-r-2s}(P_{G_{t-r}} - P)\|$$

$$\le E_t[\| k_{t-r-2s}(P_{G_{t-r}} - P)\|; A_{t-r} \le s] + P_t\{A_{t-r} > s\}.$$

Since the random elements involved here are defined on $M \cap [s, t-s]$, it follows from (10) and Lemma 4.5 that the right-hand side can be uniformly approximated in the mean sense for large s by the corresponding expression in $\bar{P}$ and its expectation $\bar{E}$. But then A_t becomes stationary, so we may introduce a random variable A with $A(P) \equiv A_t(\bar{P})$. For $r \le u - 2s$ we then obtain the uniform bound

$$\int_u^{u+1} E[\| k_{t-r-2s}(P_{t-r-A} - P)\|; A \le s] \, dt + P\{A > s\}.$$

Here the second term tends to zero, and by Fubini's theorem the first term equals

$$E\left[\int_u^{u+1} \| k_{t-r-2s}(P_{t-r-A} - P)\| \, dt; A \le s\right] \le E\left[\int_{u-r-A}^{u+1-r-A} \| k_{t-s}(P_t - P)\| \, dt; A \le s\right].$$

Since $u - r - A \ge s$, we may conclude from (10) that the inner integral on the right tends uniformly to zero as $s \to \infty$. Hence so does the whole expression. It remains to note that $\theta_{t-r}(P_t)$ can be replaced by $\theta_{t-r} \circ Q_t(P)$ because of (10). □

It is interesting to notice that (6) remains true under the same conditions for the wider class of conditional distributions on the past occurring in Theorem 2.1. In fact, these distributions were obtained as set-wise limits of Palm distributions P_t, so any bound on the values of $k_s(P_t)$ for $s < t$ remains valid for the wider class. This applies in particular to (9). The remainder of the proof didn't depend on the nature of P_t.

From (7) it is clear that $k_s(P_t) \ll k_s(P)$ holds under (C) for all $s < t$ and for $t > 0$ a.e. ρ. We shall prove that the two measures actually agree asymptotically as $s \to 0$.

THEOREM 4.6. *Under (C),*

$$\lim_{s\to 0} \| k_s(P_t - P)\| = 0, \quad t > 0 \text{ a.e. } \rho.$$

PROOF. Letting $0 < s < t$ and splitting at D_s, we get for $A \in \mathscr{F}_s$

$$P_tA = \frac{EP(A \mid D_s)p(t - D_s)}{p(t)},$$

so

$$p(t) \mid P_tA - PA \mid \leq EP(A \mid D_s)|p(t - D_s) - p(t)| \leq E \mid p(t - D_s) - p(t)|.$$

Now

$$p_r = \int_r^\infty p(\cdot, x)\, dx \uparrow p, \quad r \to 0,$$

1 331 35 so for fixed ε, $t > 0$ and $s' \in (0, t)$ we may choose $r > 0$ so small that

$$p(t) - p_r(t) < \varepsilon, \quad E(p(t - D_{s'}) - p_r(t - D_{s'})) < \varepsilon.$$

Now $Ep(t - D_s) = p(t)$ is independent of s, while

$$Ep_r(t - D_s) = \frac{E[\xi\,(dt); \xi_t - \xi_s > r]}{dt}$$

is non-increasing in s for fixed t and r, so we get in fact

$$E(p(t - D_s) - p_r(t - D_s)) < \varepsilon, \quad s \leq s'.$$

Since p_r is bounded and continuous and since $D_s \to 0$ a.s. as $s \to 0$, we may next choose $s'' \leq s'$ so small that

$$E \mid p_r(t - D_s) - p_r(t)| < \varepsilon, \quad s \leq s''.$$

Combining these estimates, we get for $s \leq s''$

$$\begin{aligned} p(t)\, \| k_s(P_t - P)\| &\leq E \mid p(t - D_s) - p(t)| \\ &\leq E(p(t - D_s) - p_r(t - D_s)) + E \mid p_r(t - D_s) - p_r(t)| + (p(t) - p_r(t)) \\ &< 3\varepsilon. \;\square \end{aligned}$$

Note that the last theorem is false without condition (C). In fact, letting ν be such as in Theorem 3.2, it is easily seen that $k_s(P_t) \perp k_s(P)$ for $t > 0$ a.e. ρ and for all $s \in (0, t]$.

5. The renewal density. Motivated by the results in Section 4, we devote this final section to a study of the continuity and asymptotic properties of the renewal density $p = d\rho/d\lambda$. We may restrict our attention to the case when $\lambda M = 0$ a.s., since the existence and continuity of p are automatic in Kingman's case [10]. Note that p exists and is lower semicontinuous whenever (C) holds. A simple sufficient condition for (C) is that

$$\lim_{u \to 0} u^{-2} \mid \log u \mid^{-1} \nu_2(u) = \infty, \tag{1}$$

where

$$\nu_2(u) = \int_0^u x^2\, \nu(dx), \quad u \geq 0. \tag{2}$$

For general infinitely divisible distributions on R (which occur in Theorem 2.2), (2) should be replaced by

$$\nu_2(u) = \sigma^2 + \int_{-u}^u x^2\, \nu(dx), \quad u \geq 0,$$

where σ^2 is the variance of the Gaussian component. To see that (C) follows from (1), it suffices to note that, for fixed $t > 0$,

$$-\log |\hat{\mu}(u)|^t = t \int (1 - \cos ux)\, \nu(dx) \geq \frac{1}{3} tu^2 \nu_2(u^{-1}) \geq 2 |\log u|$$

when u is large enough, and hence that $|\hat{\mu}(u)|^t = O(u^{-2})$ at infinity.

It is remarkable that so little is needed to ensure absolute continuity of the μ^t. (This seems to have remained unnoticed, despite an extensive literature on related matters; see e.g. [16].) Note that (1) requires only slightly more than ν to be infinite. In fact, ν is finite if $\lim u^{-2} |\log u|^r \nu_2(u) = 0$ for some $r > 1$, and infinite if $\limsup u^{-2}\nu_2(u) = \infty$.

Under (C), p admits a Stone [15] type decomposition:

Lemma 5.1. *If (C) holds, then p exists and may be decomposed into p' and $p'' \geq 0$, where p'' is integrable while p' is >0 on $(0, \infty)$ and continuous on $[0, \infty]$ with limit m^{-1} at infinity.*

Here and below, we shall make use of the formula

$$\rho = \bar{\mu}*(\delta_0 + \mu^1 + \mu^2 + \cdots), \tag{3}$$

where $\bar{\mu} = \int_0^1 \mu^t\, dt$. Note incidentally that the renewal theorem for ρ, i.e. $\theta_t(\rho) \to_v \lambda/m$ (cf. e.g. [3] or [5]), follows immediately from (3) and the classical renewal theorem.

Proof. By [15], the second factor in (3) may be written in the form $\kappa' + \kappa''$, where κ'' is finite while κ' has a continuous density which converges to m^{-1}. Since μ^1 has a uniformly continuous density, we may assume that $\mu^1 \leq \kappa'$, in which case the density is positive also. The above properties of κ' and κ'' are clearly preserved under convolution with $\bar{\mu}$, so we may take p' and p'' to be the densities of $\bar{\mu}*\kappa'$ and $\bar{\mu}*\kappa''$ respectively. □

For the next result, we need to introduce the *index*

$$\alpha = \sup\{r \geq 0: \lim_{u\to 0} u^{r-2}\nu_2(u) = \infty\}, \tag{4}$$

and to define $d = [\alpha^{-1}] - 1$ whenever $\alpha > 0$. Our α should be compared with the classical index β of Blumenthal and Getoor [2], which is given by

$$\beta = \sup\left\{r: \int_0^1 x^r \nu\,(dx) = \infty\right\} = \sup\{r: \limsup_{u\to 0} u^{r-2}\nu_2(u) = \infty\}. \tag{5}$$

The general relation is $0 \leq \alpha \leq \beta \leq 1$, though α and β will often agree in practice. Note that $\alpha > 0$ implies (1), and hence also the existence of p.

We shall further define the *singularity set* S of ν to be the set of all $t \in [0, \infty]$ such that ν has no bounded density in any neighbourhood of t. Note in particular that $0 \in S$, since ν is assumed to be infinite. Put $S_0 = \{0\}$, and define $S_1, S_2, \cdots$ recursively by the relation $S_n = S_{n-1} + S$ (addition being in the pointwise sense).

Theorem 5.2. *If $\alpha > 0$, then p exists and is further continuous on $[0, \infty]\backslash S_d$.*

Note in particular that p is continuous on $(0, \infty]$ if $\alpha > ½$, or if $\alpha > 0$ while ν has a bounded density on $[\varepsilon, \infty)$ for every $\varepsilon > 0$. Furthermore, $p(t) \to m^{-1}$ as $t \to \infty$ whenever $\alpha > 0$ while ν has a bounded density at infinity.

Proof. Defining κ', κ'' and $\bar{\mu}$ as before, write

$$\rho = \bar{\mu} + \bar{\mu}*(\mu^1*\kappa' + \kappa''*\mu^1). \tag{6}$$

Here κ' and μ^1 have uniformly continuous densities, and since this property is clearly preserved under convolution with finite measures, it is shared by the second term in (6). Moreover, the density of the second term tends to m^{-1} by the classical renewal theorem. It is thus enough to prove the continuity assertion for $\bar{\mu}$ in place of ρ. Note that, since the densities of μ^t are uniformly equicontinuous for t bounded away from zero, the continuity of p on a closed interval is equivalent to uniform integrability at $t = 0$ of the corresponding densities. This equivalence will be used repeatedly without further comments.

Let us first assume that ν is restricted to some finite interval $(0, b]$. Starting from

$$\tilde{\mu}^t(u) \equiv \int_0^\infty e^{-ux}\mu^t(dx) = \exp\left\{-t\int_0^\infty (1-e^{-ux})\nu(dx)\right\},$$

we get by formal differentiation under the integral signs

$$\int_0^\infty xe^{-ux}\,\mu^t(dx) = t\tilde{\mu}^t(u)\int_0^\infty xe^{-ux}\,\nu(dx), \tag{7}$$

which may be justified for $u > 0$ by Fubini's theorem, since both sides of (7) are continuous. By analytic continuation and dominated convergence, (7) extends to purely imaginary u, and in particular we get the estimate

$$\left|\int_0^\infty xe^{iux}\,\mu^t(dx)\right| \le tme^{-t\psi(u)},$$

where $\psi(u) = u^2\nu_2(u^{-1})/3$. Since

$$\int x^k\nu(dx) \le b^{k-1}\int x\,\nu(dx) = b^{k-1}m, \quad k \ge 1,$$

we get in the same way by repeated differentiation

$$\left|\int_0^\infty x^n e^{iux}\mu^t(dx)\right| \le e^{-t\psi(u)}\textstyle\sum_{k=1}^n c_{nk}(tm)^k b^{n-k} \tag{8}$$

for some constants $c_{nk} \in N$. To see this, note that the factors t and the integrals $\int x^k e^{-ux}\cdot\nu(dx)$ always come together, and that the sum of the exponents of x in the latter integrals equals n in each term. As for the coefficients, it is easily checked that $c_{n,1} \equiv c_{n,n} \equiv 1$, and that

$$c_{n+1,k} = kc_{n,k} + c_{n,k-1}, \quad n \in N, \quad 1 < k < n.$$

It follows that

$$c_{n,k} \le e^{(n-1)(k-1)}, \quad n \in N, \quad 1 \le k \le n.$$

Indeed, this is obvious for $k = 1$ and $k = n$, and it follows easily by induction for $n \ge 2$ and $1 < k \le n$. Thus the sum in (8) is bounded by a geometric series with quotient tme^{n-1}/b, and we get the estimate

$$\left|\int_0^\infty x^n e^{iux}\mu^t(dx)\right| \le 2tmb^{n-1}e^{-t\psi(u)}, \quad t \le be^{-n}/m. \tag{9}$$

We now define

$$n = n(t) = \left[\log\frac{b}{mt}\right], \quad t \le t_0 \equiv \frac{b}{me},$$

and let ρ' be the measure on R_+ given by

$$\rho'(dx) = \int_0^{t_0} \left(\frac{x}{c}\right)^{n(t)} \mu^t(dx)\, dt,$$

where $c = be^d$. By (9) we get

$$|\hat{\rho}'(u)| \le \frac{2m}{b} \int_0^{t_0} t\left(\frac{b}{c}\right)^{n(t)} e^{-t\psi(u)}\, dt \le \frac{2m}{b}\left(\frac{em}{b}\right)^d \int_0^{t_0} t^{d+1} e^{-t\psi(u)}\, dt$$

$$\le \frac{2m}{b}\left(\frac{em}{b}\right)^d \left\{\frac{t_0^{d+2}}{(d+2)} \wedge \frac{(d+1)!}{(\psi(u))^{d+2}}\right\}.$$

Since $d + 2 = [\alpha^{-1}] + 1 > \alpha^{-1}$, it follows that $\hat{\rho}'$ is integrable, and hence that $\hat{\rho}'$ has a uniformly continuous density. The densities of the measures $(x/c)^{n(t)}\mu^t\,(dx)$ must then be uniformly integrable at $t = 0$, and so the same thing is true for the densities of μ^t on $[c, \infty)$. Thus $\bar{\mu}$ has a continuous density on $[c, \infty)$ which tends to zero at infinity.

Next assume that $\nu = \nu' + \nu''$ where $a = \nu'' R_+ < \infty$, and write μ'^t, ψ' etc. for quantities based on ν' in place of ν. Note that ν and ν' have the same index α. The decomposition of ν induces a division of the gaps into two types, and on the local time scale, the gaps of type 2 form a Poisson process with intensity a. Thus gap number n occurs at a local time with probability density $a^n t^{n-1} e^{-at}/(n-1)!$, $t \ge 0$. Using the regenerative property and the fact that the type 2 gap sizes are independent of the remainder of the process and mutually independent with distribution ν''/a, we get for any $\varepsilon > 0$

$$\text{(10)}\quad \int_0^\varepsilon \mu^t\, dt \le \int_0^\varepsilon \mu'^t\, dt + \sum_{n=1}^\infty \frac{a^n}{(n-1)!} \int_0^\varepsilon t^{n-1} e^{-at}\left(\mu'^t * \int_0^{\varepsilon - t} \mu'^s\, ds\right) dt * (\nu''/a)^n$$

$$\le \sum_{n=0}^\infty \frac{1}{n!} \int_0^\varepsilon t^n \mu'^t\, dt * \nu''^n.$$

By the uniform integrability argument, the asserted (uniform) continuity on the left for $\varepsilon = 1$ will follow, if we can prove the corresponding (uniform) continuity on the right for $\varepsilon = 1$. But for $\varepsilon = 1$, the characteristic function of the sum over $n > d$ is bounded by

$$\sum_{n>d} \frac{a^n}{n!} \int_0^1 t^n\, |\hat{\mu}'|^t\, dt \le \sum_{n>d} \frac{a^n}{n!} \int_0^1 t^{d+1} e^{-t\psi'}\, dt \le e^a \left\{\frac{1}{d+2} \wedge \frac{(d+1)!}{\psi'^{d+2}}\right\},$$

and this being integrable, it follows that the corresponding density is uniformly continuous.

Now let $\nu''' \le \nu''$ be such that $\nu'' - \nu''' \le r\lambda$ for some $r < \infty$, and conclude by induction that

$$\nu''^n \le \nu'''^n + na^{n-1} r\lambda, \quad n \in N.$$

The sum up to d in (10) is then bounded by

$$\text{(11)}\quad \sum_{n=0}^d \frac{1}{n!} \int_0^\varepsilon t^n \mu'^t\, dt * \nu'''^n + \sum_{n=0}^d \frac{na^{n-1} r\varepsilon^{n+1}\lambda}{(n+1)!}.$$

Here the density of the second sum is bounded by $r\varepsilon e^{a\varepsilon}/a$, which tends to zero as $\varepsilon \to 0$, so it is enough to consider the first term in (11). Suppose that ν' is restricted to $(0, b]$, and conclude as above that all the integrals in (11) have uniformly continuous densities on $[c, \infty)$ where $c = be^d$. This settles the behavior at infinity, and it also shows that the density must be continuous outside the set $[0, c] + (A_r)_d$, where A_r is the support of $\nu - \nu \wedge r\lambda$, and $(A_r)_d$ is defined as S_d by iterated addition. Since $(A_r)_d$ is closed and c can be made arbitrarily small, the discontinuity set must in fact be contained in $(A_r)_d$ for any $r > 0$.

It remains to prove that $\cap_r(A_r)_d = S_d$. For $d = 1$, this is obvious from definitions, and so we get in general $S_d = (\cap_r A_r)_d \subset \cap_r (A_r)_d$. Suppose conversely that $x \in \cap_r(A_r)_d$, and choose $x_{nj} \in A_n$ such that $x = x_{n,1} + \cdots + x_{n,d}$ for each n. Next choose a subsequence such that $x_{nj} \to$ some x_j for each j, and note that $x_j \in \cap A_n = S$, where $x = x_1 + \cdots + x_d \in S_d$, as desired. □

Theorem 5.2 is nearly sharp in the following sense. Put

$$\alpha' = \sup\left\{r \geq 0: \lim_{u\to 0} u^r \int_0^\infty (1 - e^{-x/u})\nu(dx) = \infty\right\},$$

and note that $\alpha \leq \alpha' \leq \beta$. (All three indices will agree when ν is sufficiently well-behaved.) Define $d' = [\alpha'^{-1} -] - 1$, and note that $d' = d$ when $\alpha' = \alpha$ and $\alpha^{-1} \notin N$. Further define $S'_0 = \{0\}$, $S'_1 = \{0\} \cup \{x > 0: \nu\{x\} > 0\}$, and recursively $S'_n = S'_{n-1} + S'_1$.

THEOREM 5.3. *If p exists at all, it is discontinuous on $S'_{d'}$.*

PROOF. By the definition of α',

$$\liminf_{u\to\infty} u^{-1}\left\{\int_0^\infty (1 - e^{-ux})\nu(dx)\right\}^{n+1} = 0, \quad n \leq d',$$

and since

$$\int_0^\infty t^n \tilde{\mu}^t(u)\, dt = n!\left\{\int_0^\infty (1 - e^{-ux})\nu(dx)\right\}^{-n-1},$$

the measure $\int_0^\infty t^n \mu^t\, dt$ can have no bounded density near the origin when $n \leq d'$. Arguing as in the preceding proof, we may conclude that p, if it exists, must be unbounded on the right of every atom of $([\varepsilon, \infty)\nu)^n$ for $\varepsilon > 0$ and $0 \leq n \leq d'$, and hence discontinuous on $S'_{d'}$.

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