The Annals of Probability
1981, Vol. 9, No. 5. 769–780

REGULAR BIRTH TIMES FOR MARKOV PROCESSES

BY A. O. PITTENGER[1]

University of Maryland Baltimore County

A random time R is called a regular birth time for a Markov Process if (i) the R-past and R-future are conditionally independent with respect to $X(R)$ and (ii) the post-R process evolves as a Markov process, perhaps with different probability laws. In this paper we characterize each regular birth time in terms of an earlier, coterminal time L. It is shown (Theorem 4.2) that to the post-L process R appears as an optional time, perhaps with dependency on pre-L information and on a certain invariant set.

1. Introduction. The essential feature in the definition of a Markov process is the Markov property and that has two distinct aspects. First, at a constant time t, past and future are conditionally independent given the present. Second, the process viewed from time t onward behaves like a Markov process following the same transition probabilities. As the study of Markov processes progressed, it was noticed that certain random times, called optional or stopping times, also have both of these properties, and this feature was isolated as the strong Markov property. It is not true that all Markov processes possess the strong Markov property, but virtually all of any interest are strong Markov.

Now an important subclass of stopping times, the terminal times, can be viewed roughly as the first entrance times of sets, and it was natural to investigate properties of analogous times which can be viewed as last exit times from sets. These investigations were spurred by Chung's work on boundary problems of Markov chains [2] and by Meyer, Smythe and Walsh's paper [9] on coterminal times (L) in which it was shown that the post-L process evolves according to a new semigroup. In [10], and subsequently in [5] and [8] by completely different techniques, it was shown that coterminal times also possess the property of conditional independence of past and future given the present.

In [7] Jacobsen and Pitman obtained for discrete time and space Markov chains a characterization of all the random times R possessing both properties described above: conditional independence of past and future given present; and for which the post-R process evolves according to a semigroup, not necessarily the original one. In essence, they showed that preceding any such time R there is a coterminal time L so that, from the perspective of the post-L process, R looks like an optional time with possible dependence on pre-L information and on a certain invariant set. Since post-L processes are strong Markov, it is to be expected that their positive optional times would have both properties, but it is surprising that these are essentially the only such times.

In this paper we extend that characterization to general strong Markov processes in the form stated in Theorem 4.2. As the reader may suspect, there is a bit of σ-algebraing necessary to state and prove Theorem 4.2, but most of the key ideas are fairly intuitive.

To facilitate reading this paper, we give a brief outline here. In Section 2 we specify the underlying process and give some basic terminology. Regular birth times are defined as are entrance laws and probability measures whose definition involves entrance laws.

Since the proof of Theorem 4.2 relies heavily on some splitting properties of coterminal

Received April 25, 1979; revised July 14, 1980.

[1] Written while on sabbatical leave at the University of British Columbia.

AMS 1970 *Subject Classifications.* Primary 60J25; secondary 60G40.

Key words and phrases. Strong Markov processes, strong Markov property, regular birth times, coterminal times.

times, these results are included in Section 3. However, the reader is advised to skip lightly through this section on a first reading as the arguments involved are rather technical. With that machinery in place, we establish in Section 4 the easy half of Theorem 4.2 by starting with a coterminal time L and defining different kinds of new random times based on L which also possess the strong Markov property. That these modifications are exhaustive is the theme of the remainder of the paper.

If R is a regular birth time, then associated with R is a semigroup $\{H_t, t \geq 0\}$. In Section 5 we define A_0 as the set of x for which H^x, the probability on path space generated by the H-semigroup, is absolutely continuous with respect to P^x, the probability on path space generated by the P-semigroup. It is then possible to define a Radon-Nikodym derivative M, which turns out to be a multiplicative martingale, and which can be used to construct a coterminal time L preceding R. In essence, L marks the time at which a path enters A_0 and stays in A_0 thereafter.

The tools of Sections 3 and 5 are combined in Section 6 to complete the proof. The details are involved but the idea is to use the strong Markov property at L and at R to establish properties of $\rho = R - L$ as given in Lemma (3.8).

Section 7 puts the key result of [7] into the context of this paper. Finally, we present in the Appendix an extension of Walsh's result on perfecting a multiplicative functional, a result which was used in Section 5.

2. Notation. Let X be a right Markov process with Borel semigroup $(P_t, t \geq 0)$ and which is defined on a Borel subset of a compact metric space $(E, \mathbb{E})$. We assume an isolated absorbing point Δ in E. The assumption means that for each initial distribution μ, there exists a right continuous, strong Markov process $(X_t^\mu, \mathbb{F}_t)$ on E with semigroup (P_t). The σ-fields $\mathbb{F}_t$ are the usual right continuous completions of the minimal σ-fields $\mathbb{F}_t^0 = \sigma(X_t, 0 \leq s \leq t)$, and it is assumed that $P^\cdot(\Lambda)$ is Borel measurable for $\Lambda \in \mathbb{F}_\infty^0$. We do not require left limits in this paper, but branching points may exist. However, all initial measures will put zero weight on the set of branching points, so that $P^\mu(X_{0+} \in A) = \mu(A)$. For a complete discussion of these matters see [4].

The "past plus present" of a random time R is defined as the σ-field $\mathbb{F}(R)$ generated by $\mathbb{F}(S) \cap \{S \leq R\}$ for all stopping times S. ($\mathbb{F}(S)$ is the usual σ-field associated with stopping times, and the notation is consistent with the definition given here.) $\mathbb{F}(R)$ was introduced in [10], and Getoor and Sharpe [5] noted that $\mathbb{F}(R)$ measurable random variables coincide with functions of the form $Z(R)$, where Z is an *optional* process—i.e. $Z(t, \omega)$ is measurable with respect to the σ-field on $[0, \infty) \times \Omega$ generated by $\{(t, \omega): S \leq t < T, S \text{ and } T \text{ stopping times}\}$. (See [3] for the exposition of these ideas.) It is easy to see [10] that if $R \leq S$ then $\mathbb{F}(R) \subset \mathbb{F}(S)$ iff $R \in \mathbb{F}(S)$, and from Getoor and Sharpe's observation, or from [10], $X(R) \in \mathbb{F}(R)$.

The R-future of a process will be denoted as $\mathbb{G}(R)$ and is defined as $\theta_R^{-1}\mathbb{F}$.

Terminal times T are stopping times ($\{T < t\} \in \mathbb{F}$) with the property that $T = t + T \circ \theta_t$ on $\{T > t\}$. *Coterminal times* L were defined originally by Meyer, Smythe and Walsh [9] using killing operators. However, the more appropriate random time is what was called in [9] an exact coterminal time. In this paper we will use the latter time, dropping the adjective exact and using Getoor and Sharpe's definition [6]:

$$L = \sup\{t: (t, \omega) \in H\}, \tag{2.1}$$

where H is an optional, homogeneous set. Optional means $Z_t(\omega) = 1_H(t, \omega)$ is optional, and homogeneous means $Z_{s+t}(\omega) = Z_t(\theta_s\omega)$. (See [6] for a discussion of this point.)

We should note here that in [6] Getoor and Sharpe are concerned with the harder problem of characterizing random times R which possess only the property of conditional independence, and their methodology is quite different.

There are some random variables frequently associated with a coterminal time L. First are the random times $L(t) \in \mathbb{F}_t$ defined by

$$L(t) = \lim_{u \downarrow t} \sup\{s \leq u: (s, \omega) \in H\}, \tag{2.2}$$

so that $L = L(t)$ on $L \le t$. Next is the terminal time

$$T = \inf\{t > 0: (t, \omega) \in H\},$$

and the key relation is $L \le s$ iff $L(t) \le s < t$ and $T \circ \theta_t = \infty$.

NOTATION 2.3. To conserve letters, the usual expectation of a function on a set, $E^\mu[g(\omega), \Lambda]$, will be denoted as $P^\mu(g, \Lambda)$. The reader should have no difficulty distinguishing between functions and sets.

Next we have

DEFINITION 2.4. Suppose $(H_t,\ t \ge 0)$ is a nearly-Borel semigroup on $(E, \mathbb{E})$ which supports a right continuous, strong Markov process with respect to $(\mathbb{F}_t)$ for each initial μ. A family of probabilities $(Q_s,\ s > 0)$ is called an *entrance law* (of mass one) with respect to H if $Q_s \cdot H_t = Q_{s+t}$.

Just as a probability P^x is defined on path space by the semigroup P_t, so a probability Q can be defined on path space using an entrance law Q_s and the associated semigroup H_t. If the entrance law is indexed by x, we denote the probability on path space by Q^x.

Now let R be a random time with $\mathbb{F}(R)$ and $\mathbb{G}(R)$ defined as above. Then

DEFINITION 2.5. R is a *regular birth time* if

(a) $\mathbb{F}(R)$ and $\mathbb{G}(R)$ are conditionally independent on $\{R \le \infty\}$ given $X(R)$ for each P^x, $x \in E$.

(b) there exists a family $(Q_s^x,\ s \ge 0, x \in E)$ of entrance laws with respect to H such that $Q^x(\Lambda)$, $\Lambda \in \mathbb{F}_\infty^0$, is nearly Borel measurable in x and

$$P^x[\theta_R^{-1}\Gamma \mid \mathbb{F}(R)] = Q^{X(R)}(\Gamma) \qquad \text{a.s. } P^x, x \in E.$$

To simplify notation we will call (R, Q, H) a *regular birth system.* If $Q_s^x = H_s^x$, we will simply write (R, H).

As an example, (T, P) is a regular birth system for stopping times. A more esoteric example is a coterminal time L; in [10], see also [5] and [8], it is shown that there exist a semigroup K and entrance laws D^x so that (L, D, K) is a regular birth system. We should note that (2.5b) was expressed in [10] in the equivalent form

$$(2.6) \qquad P^\mu[\Lambda, L < t, \theta_t^{-1}\Gamma] = P^\mu[\Lambda, L < t, \int D^{X(L)}(t - L, dy) H^y(\Gamma)],$$

where $\Lambda \in \mathbb{F}(L)$.

One further comment about (L, D, K) is necessary to verify that the assumption in (2.4) of nearly Borel measurability is no restriction in the context of this paper. It was shown in [9] and [10] that $K_t^x(A) = P^x(X_t \in A \mid T = \infty)$, where T is the terminal time associated with L. Then $g(x) = 1 - P^x(T = \infty)$ is excessive, since $\{T = \infty\} = \lim[\{T \cdot \theta_s = \infty\}, s \downarrow 0]$, and the assumptions of strong Markov and P Borel suffice for g to be nearly Borel measurable ([4, Section 9]). To show K is nearly Borel, it thus suffices to show that $P^x(\Pi_1^n f_i(X_{r_i}), T = \infty)$ is nearly Borel, where the f_i are continuous. To show this, we set $n = 1$ for simplicity and define $h(x) = \int \alpha e^{-\alpha t} p^x[f(X_{t+r}), T \cdot \theta_t = \infty]\, dt$. Then it is easy to check that h is α-excessive and hence nearly Borel. Since $P^x[f(X_r), T = \infty) = \lim[h(x), \alpha \uparrow \infty]$, the verification is complete.

We include one other observation. If (R, Q, H) is a regular birth system, then by assumption the probabilities Q support a right continuous, strong Markov process $(X_t, t > 0)$, where the restriction $t > 0$ is essential, since the Blumenthal $0 - 1$ law may fail with respect to Q. However, it is true that paths are right continuous at zero:

LEMMA 2.7. *Suppose (R, Q, H) is a regular birth system. Then for each $x \in E$* a.s. P^x *on $\{R < \infty\}$ $Q^{X(R)}[\omega' : \lim X_\varepsilon(\omega') = X(R(\omega))] = 1$.*

3. Measurability Results. Before beginning the proof of (4.2), we need to develop some results on splitting σ-algebras at coterminal times. These results are essential for the proofs in Sections 4 and 6, but are a bit unmotivated at this point. The reader probably is well advised to skip the proofs the first time through.

LEMMA 3.1. *Let L be a coterminal time and R any random time. Then*

$$\{L \le R\}\mathcal{F}(L) \subset \{L \le R\}\mathcal{F}(R).$$

PROOF. Suppose S is a stopping time. Then

$$\mathcal{F}(S) \cap \{S \le L\} \cap \{L \le R\} = \mathcal{F}(S) \cap \{S \le R\} \cap \{S \le L \le R\},$$

and it suffices to prove the last set coincides with an $\mathcal{F}(R)$ set on $\{L \le R\}$. But that is immediate from $\{S \le R\} \in \mathcal{F}(R)$ and

$$\{L < r < S \le R\} = \{L_r < r < S \le R < T_r\} \cap \{L \le R\},$$

since $T_r = r + T\circ\theta_r$ is a stopping time, and thus the first set on the right is in $\mathcal{F}(R)$.

There is an analogous result whose proof we leave to the reader: Suppose T is a terminal time and R any random time. Then $\{R < T\}\mathcal{G}(R) \supset \{R < T\}\mathcal{G}(T)$.

REMARK 3.2. R. Getoor has observed that (3.1) holds for L the end of an optional set. All of the following also hold for such times, but our implicit interpretation of L shall be as a coterminal time.

We now define a post-L σ-algebra as follows. Let U be a random time. Then

DEFINITION 3.3. $\mathcal{F}(L, U) = \sigma(\theta_L^{-1}\mathcal{F}_s \cap \{s \le U\}$, all $s \ge 0)$ and

$$\mathcal{F}(L, U+) = \cap_{\varepsilon>0} \mathcal{F}(L, U + \varepsilon)$$

We can now split $\mathcal{F}_t$:

LEMMA 3.4. $\{L \le t\}\mathcal{F}_t = \{L \le t\}(\mathcal{F}(L) \vee \mathcal{F}(L, (t - L) +))$.

PROOF. From (3.1) $\{L \le t\}\mathcal{F}(L) \subset \{L \le t\}\mathcal{F}_t$, and it follows that $L \wedge t$ is $\mathcal{F}_t$ measurable. The sets $\theta_L^{-1}\mathcal{F}_s \cap \{s \le t + \varepsilon - L\}$, $0 < \varepsilon$, are generated by right continuous functions of the form $g(u, \omega) = f(X(L + u))$, f continuous, on the set $\{u \le s \le t + \varepsilon - L\}$. Taking $2^{-n} < \varepsilon$ and L_n as the usual 2^{-n} discrete skeleton of L, g can be written as the limit of $f(X(t_k + u))$ on $\{L_n = t_k\}$, i.e. of $\{L \le t\}\mathcal{F}_{t+2\varepsilon}$ measurable functions, proving the inclusion in one direction.
Conversely, $\mathcal{F}_u \cap \{u \le L\} \subset \mathcal{F}(L)$ and $f(X(u))$, $L < u \le t + \varepsilon$, is the limit of $\mathcal{F}(L, t + 2\varepsilon - L)$ measurable functions $f(X(u - t_{k-1} + L))$ on $\{L < u \le t + \varepsilon;\ L_n = t_k < t + \varepsilon\}$.

An analogous assertion holds for $\mathcal{F}(L + t)$:

LEMMA 3.5. $\mathcal{F}((L + t) +) = \mathcal{F}(L) \vee \mathcal{F}(L, t +)$.

PROOF. Again $\mathcal{F}(L) \subset \mathcal{F}(L + t)$, and for one inclusion it suffices to examine $f(X(L + u)) = \lim f(X(t_k + u))$ on $\{L_n = t_k,\ u + t_k < t + 2\varepsilon\}$. This shows the right hand side of (3.5) is in $\mathcal{F}(L + t + 2\varepsilon)$.
Conversely if S is a stopping time, $\mathcal{F}(S) \cap \{S \le L + t + \varepsilon\} = (\mathcal{F}(S) \cap \{S \le L\}) \vee (\mathcal{F}(S) \cap \{L < S \le L + t + \varepsilon\})$, and we need only consider the second part. But with $\Lambda \in \mathcal{F}(S)$, L_n as before and r rational, the proof reduces to observing that the sets $\Lambda \cap \{L < r < S \le L + t + \varepsilon\}$ can be expressed as limits of sets of the form $\Lambda_r \cap \{L < r < S\} \cap \{S < t_k + t + \varepsilon,\ L_n = t_k\}$, $\Lambda_r \in \mathcal{F}_r$, which are in $\mathcal{F}(L) \vee \mathcal{F}(L, t + 2\varepsilon)$ by (3.4).

The last σ-field result necessary for Theorem 4.2 is

LEMMA 3.6. *Suppose $R = L + \rho$, where $\{\rho < t\} \in \mathbb{F}(L + t)$, all t. Then*

$$\mathbb{F}(T) \cap \{L = R\} = \mathbb{F}(L) \cap \{L = R\}$$

and

$$\mathbb{F}(R+) \cap \{L < R\} = (\mathbb{F}(L) \vee \mathbb{F}(L, \rho+)) \cap \{L < R\}.$$

PROOF. The first assertion is immediate from the definitions. For the second, $\mathbb{F}(L) \subset \mathbb{F}(R)$ and again we assume f continuous and examine $f(X(L + s))$ on $\{s \le \rho + \varepsilon, 0 < \rho\}$. This will be the limit of $f(X(t_k + s))$ on $\{t_k + s < L + \rho + 2\varepsilon, L_n = t_k, 0 < \rho\}$, and since $\rho \in \mathbb{F}(R)$, the limit will be in $\mathbb{F}(R + 2\varepsilon)$.

The proof in the other direction reduces to sets of the form $\Lambda \cap \{L < S \le L + \rho + \varepsilon, 0 < \rho\}$, with $\Lambda \in \mathbb{F}(S)$ and S a stopping time. The argument this time is based on

$$\begin{aligned}
&\Lambda_r \cap \{L < S < r < t_k + \rho + \varepsilon, L_n = t_k < r\} \\
&\quad \in \mathbb{F}(L + r + \varepsilon - t_k) \cap \{L_n = t_k < r\} \cap \{r - t_k - \varepsilon < \rho\} \\
&\quad \subset \mathbb{F}(L) \vee \mathbb{F}(L, r + \varepsilon - t_k)) \cap \{r - t_k - \varepsilon < \rho\} \\
&\quad \subset \mathbb{F}(L) \vee \mathbb{F}(L, \rho + 2\varepsilon).
\end{aligned}$$

The next set of results concerns representations of random times in terms of pre- and post-L dependence. These results are necessary for the statement and proof of Theorem 4.2, but again the reader might defer their proofs until later.

LEMMA 3.7. *Suppose H is a bounded $(\mathbb{F}(L) \vee \mathbb{F}(L, t - L))$ measurable function. Then for each initial measure μ there exists an $\mathbb{F} \times \mathbb{F}$ measurable function G such that*

(*a*) $H(\omega) = G(\omega, \theta_L\omega)$ a.s. P^μ

(*b*) *$G(\cdot, \omega')$ is $\mathbb{F}(L)$ measurable*

(*c*) *if $L(\omega) < t$, $G(\omega, \cdot)$ is $\mathbb{F}(t - L(\omega))$ measurable.*

PROOF. On $\{t \le L\}$ the function $G(\omega, \omega') = H(\omega)$ will do, and we restrict our attention to $\{L < t\}$. On this set H is $\{L < t\}\mathbb{F}_t$ measurable by (3.4). Using a monotone class argument and functions of the form

$$G(\omega, \omega') = \textstyle\sum_{i=0}^{n} 1_{[t_i, t_{i+1})}(L) \prod_{j=0}^{i} 1_{A_j}(X_{t_j}) \prod_{k=i+1}^{n} 1_{A_k}(X(t_k - L(\omega), \omega')),$$

we will have $H(\omega) = G(\omega, \theta_L\omega)$ for a determining class of $(\sigma(L) \vee \mathbb{F}_t^0) \cap \{L < t\}$ measurable functions. Suppose for such H_n, $H_n \uparrow H$. Then using $\tilde{G}_n(\omega, \omega') = \max(G_k(\omega, \omega'), k \le n)$ we have $H_n(\omega) = \tilde{G}_n(\omega, \theta_L\, \omega)$ and $\tilde{G}_n(\omega, \omega') \uparrow G(\omega, \omega')$. Since all three properties are preserved under monotone limits, we have the lemma for $\sigma(L) \vee \mathbb{F}_t^0$ measurable functions. The assertion for general H is immediate, although the G obtained is not necessarily unique.

Finally, we come to the last, and essential, measurability result.

LEMMA 3.8. *Suppose $L \le R$ and $\{R < t\}$ is in $\mathbb{F}_t \cap \{L < t\}$ for all $t > 0$. Then for each μ there exists a non-negative $\mathbb{F} \times \mathbb{F}$ measurable function $\rho(\omega, \omega')$ such that*

(*a*) $R(\omega) = L(\omega) + \rho(\omega, \theta_L\omega)$ a.s. P^μ

(*b*) $\rho(\cdot, \omega') \in \mathbb{F}(L)$

(*c*) *$\rho(\omega, \cdot)$ is an optional time.*

PROOF. Let R_n be the 2^{-n} skeleton of R and define $h_{n,k}(\omega)$ as 1 on $\{R_n = t_k\}$ and 0 elsewhere. Using (3.4) and (3.7) we can find a $\{0, 1\}$ valued function $g_{n,k}(\omega, \omega')$ such that

$$h_{n,k}(\omega) = g_{n,k}(\omega, \theta_L\omega) \qquad \text{a.s. } P^\mu. \tag{3.9}$$

Since $L \leq R$, we can modify $g_{n,k}(\omega, \omega')$ by setting it equal to 0 on $\{t_k < L(\omega)\}$ and without affecting (3.7 b) or (3.7 c). Similarly, we can assume that $g_{n,i}(\omega, \omega') = 1$ implies $g_{n,j}(\omega, \omega') = 0, j > i$, without changing (3.9) or the measurability properties.

Define

$$G_n(\omega, \omega') = \begin{cases} \infty & g_{n,k}(\omega, \omega') = 0, \quad \text{all } k \\ \sum_{k=1}^{\infty} t_k g_{n,k}(\omega, \omega') & \text{otherwise} \end{cases}$$

and note $L(\omega) \leq G_n(\omega, \omega')$. We then have $R_n(\omega) = G_n(\omega, \theta_L\omega)$, $G_n(\cdot, \omega') \in \mathbb{F}(L)$, and, by virtue of the modifications of the $g_{n,k}$,

$$\{\omega' : G_n(\omega, \omega') < t\} \in \mathbb{F}(t - L(\omega)).$$

Then $G(\omega, \omega') \equiv \liminf G_n(\omega, \omega')$ gives $R(\omega) = G(\omega, \theta_L\,\omega)$ and $\rho(\omega, \omega') = G(\omega, \omega') - L(\omega)$, with $\infty - \infty = \infty$, is the ρ of the assertion: (a) and (b) are immediate and (c) follows from

$$\{\omega' : \rho(\omega, \omega') < t\} = \{\omega' : G(\omega, \omega') < t + L(\omega)\}$$
$$\in \mathbb{F}(t + L(\omega) - L(\omega)) = \mathbb{F}_t.$$

4. Regular Birth Systems. Suppose L is a fixed coterminal time and (L, D, K) its regular birth system as given in Section 2. Suppose $\rho(\omega, \omega')$ is a positive $\mathbb{F} \times \mathbb{F}$ measurable function such that $\rho(\cdot, \omega') \in \mathbb{F}(L)$ and $\rho(\omega, \cdot)$ is optional. Then (R, K) is also a regular birth system, where $R(\omega) = L(\omega) + \rho(\omega, \theta_L\omega)$. To see this we note that $\rho(\omega, \theta_L\omega)$ satisfies the hypotheses of (3.6), and we need only use sets in $\mathbb{F}(R)$ of the form $\Lambda \cap \theta_L^{-1}\Lambda_s \cap \{s \leq \rho(\omega, \theta_L\omega\})$, $\Lambda \in \mathbb{F}(L)$. Then

$$\begin{aligned} P[\Lambda, \theta_L^{-1}\Lambda_s \cap \{s \leq \rho(\omega, \theta_L\omega)\}, R < \infty, \theta_R^{-1}(\Gamma)] \\ &= P[\Lambda, L < \infty, D^{X(L)}[\Lambda_s, \{s \leq \rho(\omega, \omega') < \infty\}, \theta_{\rho(\omega,\omega')}^{-1}\Gamma]] \\ &= P[\Lambda, L < \infty, D^{X(L)}[\Lambda_s, \{s \leq \rho(\omega, \omega') < \infty\}, K^{X(\rho(\omega,\omega'))}(\Gamma)]] \\ &= P[\Lambda, \theta_L^{-1}\Lambda_s \cap \{s \leq \rho(\omega, \theta_L\omega)\}, \{R < \infty\}, K^{X(R)}(\Gamma)]. \end{aligned}$$

We should comment that a monotone class argument is necessary to justify the last equality; the proof, however, is omitted (cf. [1, I.8.16]).

Suppose another R is defined by setting $R = L$ on $\Lambda_0 \cap \theta_L^{-1}\Gamma_0$, where $\Lambda_0 \in \mathbb{F}(L)$ and $\Gamma_0 \in \mathbb{F}_0$, and equal to infinity otherwise. There then exists a $\rho(\omega, \theta_L\omega)$ with the properties of (3.6) such that $\Lambda_0 \cap \theta_L^{-1}\Gamma_0 = \{\omega : \rho(\omega, \theta_L\omega) = 0\}$. Hence with $\Lambda \in \mathbb{F}(R) \cap \{\mathrm{R} = \mathrm{L}\} = \mathbb{F}(L) \cap \{R = L\}$, we have

$$P[\Lambda, R < \infty, \theta_L^{-1}\Gamma] = P[\Lambda, L < \infty, \Lambda_0, D^{X(L)}[\Gamma_0, \Gamma]].$$

If we define

$$Q^x(\Gamma) = \frac{D^x[\Gamma_0, \Gamma]}{D^x[\Gamma_0]} = D^x(\Gamma \mid \Gamma_0),$$

with $0/0 = 0$, then it is easy to see that

$$P[\Lambda, R < \infty, \theta_R^{-1}\Gamma] = P[\Lambda, R < \infty, Q^{X(R)}(\Gamma)]$$

and that the $Q^{X(R)}$ define entrance laws with respect to the semigroup K of L.

There is one further possibility. Suppose $\Gamma_1 \subset \{L < \infty\}$ is also an invariant set. Let R_1 be defined to be zero on Γ_1 and infinite elsewhere; let $\bar{R}$ be one of the preceding examples; and set $R = \max(R_1, \bar{R})$. Then $\{R < \infty\} = \Gamma_1 \cap \{\bar{R} < \infty\}$, and the above analysis works for $\bar{R}$, provided we condition K and D:

$$(4.1) \qquad \begin{cases} \bar{K}(x, t, f) = P^x[f(X_t) \mid (\Gamma_1 \cap \{T = \infty\})] \\ \bar{Q}^x(\Gamma) = D^x[\Gamma \mid (\Gamma_1 \cap \Gamma_0)]. \end{cases}$$

$\bar{K}$ and $\bar{Q}$ will also be nearly-Borel measurable.

The foregoing completes half of the proof of the following:

THEOREM 4.2. *Let R be a random time. Then R is a regular birth time, and (R, Q, H) a regular birth system, for X iff there exists an invariant set Γ_1 and a coterminal time $L \le R$, and thus a regular birth system (L, D, K), such that $\Gamma_1 \subset \{L < \infty\}$ and for each μ there exists a $\rho(\omega, \omega') \ge 0$, a $\Lambda_0 \in \mathbb{F}(L)$ and a $\Gamma_0 \in \mathbb{F}_0$ so that*

(a) $R = \infty$ a.s. P^μ *on* Γ_1^c

(b) $R(\omega) = L(\omega) + \rho(\omega, \theta_L\omega)$ a.s. P^μ *on* Γ_1

(c) $\rho(\cdot, \omega') \in \mathbb{F}(L)$

(d) $\rho(\omega, \cdot)$ *is an optional time*

(e) $\Lambda_0 \cap \theta_L^{-1}\Gamma_0 = \{\omega: \rho(\omega, \theta_L\omega) = 0\}$ a.s. P^μ.

The post R process evolves according to $\bar{K}$, the K semigroup conditioned on Γ_1 as in (4.1). *On $\{L < R < \infty\} Q^{X(R)} = \bar{K}^{X(R)}$ and on $\{L = R\} Q^{X(R)}$ is a $D^{X(R)}$ conditional probability as in* (4.5).

As an example of the need for conditional entrance laws, suppose X behaves like two dimensional Brownian motion on $(-1, 1) \times (-\infty, \infty)$ and elsewhere like a compatible diffusion with drift away from the y-axis. Let L be the last hit of the positive y-axis. Define $\theta_L^{-1}\Gamma_0 = \{$at time L, X exits to the right from the positive y-axis$\}$ and $\Gamma_1 = \{X$ ultimately stays in the positive right half plane$\}$. If $R = L$ on $\Gamma_1 \cap \theta_L^{-1}\Gamma_0$ and is otherwise infinite, then R will be a regular birth time whose entrance law is that of L conditioned by $\Gamma_0 \cap \Gamma_1$ as above.

If the post-R process evolves according to the original semigroup, then, as the reader will learn in the next section, $L = 0$ and by the theorem, R must be an optional time. (Γ_1 has full measure.) We record this as

COROLLARY 4.3. *Suppose R is a regular birth time. Then R evolves according to $(P_t, t \ge 0)$ iff R is an optional time.*

Finally, it follows from (4.2) that the only semigroups associated with regular birth times are those obtained by conditioning the original semigroup on an invariant set and on the set $\{T = \infty\}$, where T is a terminal time and, by Getoor and Sharpe's result [6], the first hit of an optional, homogeneous set. (The reader might compare this with Theorem 2.3 in [7]. See Section 7 below.)

5. Construction of the coterminal time L. Suppose now that R is a regular birth time with entrance laws $Q^{X(R)}$ and semigroup H. The idea behind the construction below is that if H is related to a regular birth time, then H must be absolutely continuous with respect to P, and the Radon-Nikodym derivative can then be used to define a coterminal time L which will precede R. Intuitively, L is the first regular birth time compatible with the semigroup H.

Recall that P is assumed Borel and H nearly Borel. Let $\{\Lambda_n\}$ be a measure-determining, countable collection of $\mathbb{F}^0$ sets and let $\varepsilon = \varepsilon(r)$, $\delta = \delta(m)$ be sequences of decreasing positive numbers. Define

$$S(\varepsilon, \delta, n) = \{x: H^x(\Lambda_n) \ge \varepsilon, P^x(\Lambda_n) < \delta\}$$

$$S(\varepsilon, \delta) = \cup_n S(\varepsilon, \delta, n), \qquad S(\varepsilon) = \cap_m S(\varepsilon, \delta(m)),$$

$$S = \cup_r S(\varepsilon(r)).$$

Then it is easy to show the following:

LEMMA 5.1. $A_0 = S^c = \{x: H^x \ll P^x\}$. *Note that* A_0 *is a nearly Borel set.*

We wish to show that if $x \in A_0$, then H^x puts full measure on paths which remain in A_0. This is reminiscent of [1, III], but the hypotheses are different, and we provide details. Since S is nearly Borel, we can approximate its hitting time by hitting times of compact sets. Let $K \subset S$ be compact, $x \in A_0$, and $T = \inf\{t > 0, x_t \in K\}$. Hence $X(T) \in K$ and $T > 0$ a.s. P^x for $x \in A_0$. Suppose for some $\varepsilon > 0$ that $H^x(T < \infty, X(T) \in S(\varepsilon)) > 0$. Define

$$(5.2)\qquad \begin{cases} A_{(m,n)} = \{\omega : n \text{ is the smallest index such that } X(T) \in S(\varepsilon, \delta(m), n)\} \\ A(m) = \cap_{m=M}^{\infty} \cap_n A(m, n)\theta_T^{-1}(\Lambda_n). \end{cases}$$

It then follows easily that

$$\lim_M H^x(t < \infty, A(M)) \geq \varepsilon\, H^x(T < \infty, X(T) \in S(\varepsilon))$$

and

$$\lim_M P^x(T < \infty, A(m)) \leq \lim_M \delta(M) P^x(T < \infty, X(T) \in S(\varepsilon)).$$

Thus,

LEMMA 5.3. *For every* $x \in A_0$, $H^x(T_s < \infty) = 0$.

Define M_∞ as dH^x/dP^x, so that M_∞ is a P^x equivalent version of a limit of sums of terms of the form $(H^{X(0)}(\Lambda_n)/P^{X(0)}(\Lambda_n))1_{\Lambda_n}(\omega)$. That is, there exists an $\mathbb{F}$ measurable random variable M which coincides P^x a.s. with M_∞ on $\{\omega : X(0) = x\}$, all $x \in A_0$. It is easy to check that $M_t = P^x(M_\infty \mid \mathbb{F}_t)$ is a Radon-Nikodym derivative of H^x with respect to P^x on $\mathbb{F}_t$ and using (5.3) that for $\Lambda_s \in \mathbb{F}_s$, $\Lambda_t \in \mathbb{F}_t$

$$P^x[\Lambda_s, \theta_s^{-1}\Lambda_t, M_s \cdot (M_t \circ \theta_s)] = P^x[\Lambda_s, \theta_s^{-1}\Lambda_t, M_{s+t}].$$

Thus, we can take a P^x version of (M_t) which is a non-negative, right-continuous uniformly integrable martingale and which is also a multiplicative functional. Furthermore, $M_t = 0$ if $t > T_S$ since

$$0 = H^x(T_S < \infty) = P^x(T_S < \infty, M_\infty) = P^x(T_S < \infty, M_S)$$

implies $M_S = 0$, P^x a.s. on $\{T_S < \infty\}$, and once a non-negative martingale hits zero it stays there.

The function M above can be defined for all ω, but we can modify it by setting $M_t(\omega) = 0$ for all $t \geq 0$ when $X_0(\omega) \notin A_0$. This definition is compatible with the M_t used above and obviously gives us a multiplicative functional:

$$(5.4)\qquad M_t = M_s \cdot M_{t-s} \circ \theta_s \qquad \text{a.s.}$$

We would like to perfect the multiplicative functional by invoking [12]; unfortunately those results require $0 < M_t \leq 1$ and in fact need not be valid in our case. By modifying the proof, however, we can obtain Lemma 5.5 below. To avoid too much dancing on the head of a null set we relegate the proof to an appendix.

LEMMA 5.5. *There exist a set of* $\Gamma \in \mathbb{F}$ *and* $\mathbb{F}_t$ *measurable functions* $\bar{M}_t$ *such that for all initial measures* μ, $P^\mu(\Gamma) = 1$ *and on* Γ

$$\bar{M}_t = \bar{M}_s \cdot \bar{M}_{t-s} \circ \theta_s, \qquad 0 < s < t \leq \infty$$

Furthermore $\bar{M}_s$ *is right continuous except possibly at* $s = 0$. *For Lebesgue almost all* $0 < s < t$, $\bar{M}_{t-s} \circ \theta_s = M_{t-s} \circ \theta_s$. *Finally* $\bar{M}_{t-r}(\theta_r \omega) = 0$ *if* $0 < r < t$ *and* $x_t(\omega) \notin A_0$.

We drop the bar and simply refer to M_t in the following discussion. Let

$$H = \{(t, \omega): X_t \notin A_0 \text{ or } \lim_{s\uparrow t} M_{t-s}\circ\theta_s = 0\},$$

where the second condition implies the limit exists. Since $X_{t+u}(\omega) = X_t(\theta_u\omega)$ and, using $s = u + r$, $M_{u+t-s}\circ\theta_s\omega = M_{t-r}\circ\theta_r\circ\theta_u\omega$, H is an optional, homogeneous set. Then by the results cited in Section 2, $L = \sup\{t: (t, \omega) \in H\}$ is a coterminal time. If $L' = \sup\{t: M_\infty\circ\theta_t = 0\}$, we have

LEMMA 5.6. *$L' = L$ on $L' < \infty$.*

PROOF. Suppose $r < L$. Then either there is a $t > r$ such that $X_t(\omega) \notin A_0$ or $M_{t-s}\circ\theta_s \to 0$ as $s \uparrow t$. In the former case $M_{t-r}\circ\theta_r = 0$ and $M_\infty\circ\theta_r = (M_{t-r}\circ\theta_r)(M_\infty\theta_t) = 0$, while in the latter case

$$M_\infty\circ\theta_r = (M_{s-r}\circ\theta_r)(M_{t-s}\circ\theta_s)(M_\infty\circ\theta_t) \to 0$$

as $s \uparrow t$. Either way $L' \geq L$.

If $L < r < L' < t$, then $M_{t-r}\circ\theta_r = 0$ but $M_{s-r}\circ\theta_r \to 1$ as $s \downarrow r$. Thus, let $t_0 = \inf\{s > r: M_{s-r}\circ\theta_r = 0\} \leq t$, and $r < s < t_0$. It follows that $0 = M_{t_0-r}\circ\theta_r = (M_{s-r}\circ\theta_r)(M_{t_0-s}\circ\theta_s)$, implying $M_{t_0-s}\circ\theta_s = 0$, and thus $r < L$.

6. Completion of the proof of Theorem 4.2. Assume (R, Q, H) is a regular birth system and let $P = P^\mu$ be fixed. We can use the ideas of Lemma 3.4, a monotone class argument and the strong Markov property at R to show

LEMMA 6.1. *Suppose $\Lambda \in \mathbb{F}_t \vee \mathbb{F}(R)$. Then*

$$P(\Lambda, R < t, \theta_t^{-1}\Gamma) = P(\Lambda, R < t, H^{X_t}(\Gamma)).$$

Now the observation which justifies Section 5 is in

LEMMA 6.2. *$P(X(R + t) \notin A_0$, any $t > 0) = 0$.*

PROOF. By virtue of (5.3) once the process is in A_0, it stays there according to the H laws. Hence if suffices to show $P(R < t, X_t \in A_0^c) = 0$. But by (6.1) it follows that with $\Lambda \in \mathbb{F}_t$

$$P(\Lambda, R < t, H^{X_t}(\Gamma)) \leq P(\Lambda, \theta_t^{-1}(\Gamma)) = P(\Lambda, P^{X_t}(\Gamma)).$$

Using the construction of (5.2), with $X(t)$ replacing $X(T)$, it is then easy to see

$$\varepsilon P(R < t, X(t) \in S(\varepsilon)) \leq \lim_M \delta(M)P(X(t) \in S(\varepsilon)) = 0,$$

completing the proof.

Now bring in the multiplicative martingale M, the coterminal time L and the time L' of Section 5. Since

$$P[R < r < L'] = P[R < r, \theta_r^{-1}(M_\infty = 0)]$$
$$= P[R < r, H^{X(r)}(M_\infty = 0)] = 0,$$

we can conclude $L \leq R$ a.s. on $R < \infty$.

Choose an $\mathbb{F}_\infty^0$ measurable version which is P equivalent to R and which we denote as R. Then if R_n is the 2^{-n} skeleton of R, R_n is also $\mathbb{F}_\infty^0$ measurable. Consequently for fixed $t_k = k2^{-n}$ there is a $\{0, 1\}$ valued function f with domain the usual product of countably many copies of E and such that for a given sequence of times $r_1, r_2, \cdots, R_n(\omega) = t_k$ iff $f(X_{r_1}, X_{r_2}, \cdots) = 1$. Writing f as $f(X_r, r < t_k; X_u, u \geq t_k)$ it follows P a.s. that

$$P[R_n = t_k \mid \mathbb{F}_{t_k}] = P^{X(t_k)}(C_k(\omega))$$

where

$$C_k(\omega) = \{\omega' : f(X_r(\omega), r < t_k; X_{u-t_k}(\omega'), u \geq t_k) = 1\}.$$

Using (6.1) we have for $\Lambda \in \mathbb{F}_{t_k}$

$$\begin{aligned} P[\Lambda, R_n = t_k, \theta_{t_k}^{-1}(\Gamma)] &= P[\Lambda, R_n = t_k, H^{X(t_k)}(\Gamma)] \\ &= P[\Lambda, P^{X(t_k)}(C_k(\omega)) \cdot H^{X(t_k)}(\Gamma)] \\ &= P[\Lambda, P^{X(t_k)}(C_k(\omega), \Gamma)]. \end{aligned} \tag{6.3}$$

From the last equality and $\Lambda \in \mathbb{F}_{t_k}$, it follows that P a.s.

$$H^{X_{t_k}(\omega)}(\Gamma) = P^{X_{t_k}(\omega)}(\Gamma \mid C_k(\omega)).$$

Since $X(t_k) \in A_0$ a.s., for P almost all $X(t_k)$

$$1_{C_k(\omega)}(\omega') = P^{X_{t_k}(\omega)}(C_k(\omega)) \cdot M_\infty(\omega') \qquad \text{a.s.} \quad P^{X(t_k)}.$$

Substituting this into the last line of (6.3) gives

$$\begin{aligned} P[\Lambda, \theta_{t_k}^{-1}(\Gamma), R_n = t_k] &= P[\Lambda, P^{X(t_k)}(C_k(\omega)) \cdot P^{X(t_k)}(M_\infty, \Gamma)] \\ &= P[\Lambda, \theta_{t_k}^{-1}(\Gamma), P^{X(t_k)}(C_k(\omega)) \cdot M_\infty \circ \theta_{t_k}\omega] \end{aligned}$$

and P a.s.

$$1_{[t_k]}(R_n) = P^{X_{t_k}(\omega)}(C_k(\omega)) \cdot M_\infty \circ \theta_{t_k}\omega.$$

We may assume, by a redefinition of the t_k and R_n if necessary, that $P[L' = t_k$, some k and $n] = 0$. If $\Lambda(k, n) = \{\omega : P^{X(t_k)}(C_k) > 0\}$, then we can rewrite the last equality as

$$\{R_n = t_k\} = \Lambda(k, n) \cap \{L' \leq t_k\} \; P \text{ a.s.},$$

where the restriction on the t_k avoids the issue of whether $M_\infty(\theta_t\omega) > 0$ when $L' = t$. Suppose we could define $\bar{R}_n$ such that $R_n = \bar{R}_n$ on $L' < \infty$, $\bar{R}_n \downarrow \bar{R}$ and $\{\bar{R} < t\} \in \mathbb{F}_t \cap \{L < t\}$. Lemma 3.8 then establishes the existence of the ρ of Theorem 4.2, except for (4.2.e). The invariant set Γ_1 is $\lim[\theta_t^{-1}(\Gamma \cap \{M_\infty > 0\}), t \uparrow \infty]$, Γ the set described in (5.5) and the Appendix, and R is indeed infinite on Γ_1^c almost surely P.

Since R agrees with $\bar{R}$ on Γ_1, the rest of (4.2) follows once we define $\bar{R}_n$ and establish (4.2.e). Recalling the $L(t)$ from (2.2) we define $\Gamma(1, n) = \Lambda(1, n)$, $\Gamma(k + 1, n) = \Lambda(k + 1, n) - \cup[\Lambda(j, n) \cap \{L(t_{k+1}) \leq t_k\}, 1 \leq j \leq k]$,

$$\bar{R}_n' = \begin{cases} t_k & \omega \in \Gamma(k, n) \cap \{L \leq t_k\} \\ \infty & \text{if no such } t_k \end{cases}$$

and

$$\bar{R}_n = \min\{\bar{R}_i', 1 < i < n\}.$$

Since $L = L(t)$ on $L \leq t$, it is easy to see $\bar{R}_n'$ is well-defined and $R_n = \bar{R}_n'$ a.s. on $L' < \infty$. Since R_n decreases with n, $R_n = \bar{R}_n$, thus defining the desired $\bar{R} = \lim \bar{R}_n$.

It remains to show ρ satisfies (4.2.e.), and in the discussion below it suffices to assume P-equivalent, $\mathbb{F}_\infty^0$ measurable versions of R, L and ρ. From (3.1) we have $\{L = R\} \in \mathbb{F}(R)$, and thus for $\Lambda \in \mathbb{F}(L)$

$$\begin{aligned} P[\Lambda, L = R < \infty, \theta_L^{-1}\Gamma] &= P[\Lambda, L = R < \infty, Q^{X(R)}(\Gamma)] \\ &= P[\Lambda, L < \infty, D^{X(L)}(\{\omega' : \rho(\omega, \omega') = 0\}, \Gamma_1, \Gamma)]. \end{aligned} \tag{6.4}$$

If follows that P a.s. on $L < \infty$

$$Q^{X(L)}(\Gamma) \cdot D^{X(L)}(\Gamma_1, \{\rho = 0\}) = D^{X(L)}(\Gamma, \Gamma_1, \{\rho = 0\}) \tag{6.5}$$

for all $\Gamma \in \mathbb{F}_\infty^0$. Let $\Lambda_0 = \{\omega : D^{X(L)}(\Gamma_1, \rho = 0) > 0\}$, which is in $\mathbb{F}(L)$. From (6.5) it is easy to show that for fixed x the D^x measure of $\Gamma_1 \cap \{\omega' : \rho(\omega, \omega') = 0\}$ is the same for almost all $\omega \in \Lambda_0$ which satisfy (6.5) and have $X(L(\omega)) = x$. Since $\{\rho = 0\}$ is in $\mathbb{F}^0_{0+}$, it follows that for each $\varepsilon > 0$ there exists a countable family $\{\Gamma_n, \Gamma_n \in \mathbb{F}^0_\varepsilon\}$ such that for each such ω and x we can find a Γ_n so that (6.5) holds with an error of at most ε when Γ_n replaces $\{\omega' : \rho(\omega, \omega') = 0\}$. But the same Γ_n then works for all such ω associated with a common x. Hence using (2.7) and a union of sets of the form $\{X_0 \in A_n\} \cap \Gamma_n$, it is possible to define a $\Gamma(\varepsilon)$ in $\mathbb{F}^0_\varepsilon$ so that there is at most an ε error in (6.5) for almost all ω in $\{L < \infty\} \cap \Lambda_0$. Finally $\Gamma_0 = \liminf \Gamma(\varepsilon)$, the limit along a sequence $\varepsilon \downarrow 0$, gives an $\mathbb{F}^0_{0+}$ set which has no error. Reverting back to (6.4)

$$P[\Lambda, \theta_L^{-1}\Gamma, L = R < \infty] = P[\Lambda, L < \infty, \Lambda_0, D^{X(L)}(\Gamma, \Gamma_1, \Gamma_0)]$$
$$= P[\Lambda, \theta_L^{-1}\Gamma, L < \infty, \Gamma_1 \cap \Lambda_0 \cap \theta_L^{-1}\Gamma_0]$$

or, P a.s., $\{\omega : L = R < \infty\} = \Gamma_1 \cap \Lambda_0 \cap \theta_L^{-1}\Gamma_0$ on $L < \infty$. We can then redefine ρ by setting $\rho(\omega, \omega') = 0$ for $(\omega, \omega') \in \Lambda_0 \times \Gamma_0$ and equal to its original value elsewhere, thus completing the proof of (4.2).

7. The Markov Chain Case. The inspiration for the preceding was [7]. The key result there was expressed somewhat differently, and it is only fitting that we indicate how that expression can be obtained from the proof here. We have in mind [7, 2.3] in which it was shown that the only conditioning sets C which can support a Markov chain are of the form $C_0 \cap C_* \cap C_\infty$. C_0 denotes an initial condition: $C_0 = \{\omega : X_0(\omega) \in A_0\}$. C_* is defined by $C_* = \{\omega : (X_k, X_{k+1}) \in V, 0 \le k < \infty\}$, $V \in \mathbb{E} \times \mathbb{E}$, and C_∞ is invariant: $C_\infty = \theta^{-1}C_\infty$. Following an approach using a multiplicative martingale M, one can show that

$$P(C \mid \mathbb{F}_n) = P^{X_0}(C)M_n(\omega), \qquad P \text{ a.s.}$$

Then $M_n(\omega) = \prod_1^n M_1(\theta_{k-1}\omega)$, or, since $M_1 \in \sigma(X_0, X_1)$, $M_n(\omega) = \prod_{k=1}^n u(X_{k-1}, X_k)$. It turns out that $C_0 = \{\omega : P^{X_0}(C) > 0\}$, $C_* = \{\omega : u(X_{k-1}, X_k) > 0$, all $k \ge 1\}$, and $C_\infty = \{\omega : \lim_n M_\infty \circ \theta_n = 1\}$.

Acknowledgements. This paper was written during a sabbatical visit to the University of British Columbia, and the author is most grateful to the Department of Mathematics for its hospitality and support. The author would also like to thank the referee for a number of constructive suggestions.

APPENDIX

Proof of Lemma 5.5. We follow the notation of Walsh [12], and the reader is advised to have a copy available. (See [11] for a discussion of essential limits.) Let M_t, $0 \le t \le \infty$, be a right continuous version of the function defined in Section 5 and define $\bar{M}_t(\omega) = \text{ess lim sup}\{M_{t-s}(\theta_s\omega), s \downarrow 0\}$ with $\bar{M}_0(\omega) = \text{ess lim sup}\{\bar{M}_s(\omega), s \downarrow 0\}$. Using m to denote Lebesgue measure, we can modify the Λ of [12] by defining:

$\Lambda = \{\omega$: for m almost all s $M_s(\omega) < \infty$, $M_{t-s}(\theta_s\omega) < \infty$, and $M_t(\omega) = M_s(\omega) \cdot M_{t-s}(\theta_s\omega)$, $s \le t \le \infty$. $M_{0+}(\theta_s\omega) = 1$ if $X_s(\omega) \in A_0$ and $M_t(\omega) = 0$ for all $t \ge D_0 = \inf\{r : M_r = 0\}$.$\}$.

The Fubini-type arguments of [12] show that for any initial measure μ, $P^\mu(\Lambda) = 1$ and also $P^\mu(\Gamma) = 1$ where $\Gamma = \{\omega : \theta_s\omega \in \Lambda, m \text{ a.a. } s \ge 0\}$. As before $\omega \in \Gamma$ implies $\theta_t\omega \in \Gamma$.

Suppose $\omega \in \Gamma$. Then it is possible to define a set of full m-measure, with elements denoted by r, such that for m-almost all s and all t:

(A.1) $$M_{t-r}(\theta_r\omega) = M_{s-r}(\theta_r\omega) \cdot M_{t-s}(\theta_s\omega), \qquad r < s < t,$$

and

(A.2) $$M_{t-s}(\theta_s\omega) = M_{r-s}(\theta_s\omega) \cdot M_{t-r}(\theta_r\omega), \qquad s < r < t.$$

Suppose $M_{t-r}(\theta_r\omega) = 0$. Then in (A.2) $M_{t-s}(\theta_s\omega) = 0$ and $\bar{M}_t = 0$. If $M_{t-r}(\theta_r\omega) > 0$ we can take essential limits in (A.2) to obtain, in either case,

$$\bar{M}_t(\omega) = \bar{M}_r(\omega)\cdot M_{t-r}(\theta_r\omega), \qquad m \text{ a.a. } r < t, \tag{A.3}$$

provided $0 = 0.\infty$ should that case occur.

If $M_{t-r}(\theta_r\omega) = 0$, then $M_{u-r}(\theta_r\omega) = 0$, all $u \geq t$, and $\bar{M}_u(\omega) = 0 = \bar{M}_t(\omega)$. If $M_{t-r}(\theta_r\omega) > 0$, the right continuity of $\bar{M}_u(\omega)$ at t follows from (A.3), and hence we have right continuity of $\bar{M}_t(\omega)$ for all $t > 0$.

If both $\bar{M}_u(\omega)$ and $\bar{M}_{t-u}(\theta_u\omega)$ are positive, we can use the right continuity of $\bar{M}_u(\omega)$ and take essential limits in (A.3) to obtain

$$\bar{M}_t(\omega) = \bar{M}_u(\omega)\cdot\bar{M}_{t-u}(\theta_u\omega), \qquad 0 < u < t. \tag{A.4}$$

If $\bar{M}_u(\omega) = 0$, then using u in place of t in (A.2) gives for m a.a. $r < u$ either $\bar{M}_r(\omega) = 0$ or $M_{u-r}(\theta_r\omega) = 0$. Either condition gives $\bar{M}_t(\omega) = 0$ and (A.4) holds in that case. If $\bar{M}_{t-u}(\theta_u\omega) = 0$, taking $r < u < s < t$ in (A.1) gives $M_{t-r}(\theta_r\omega) = M_{u-r}(\theta_r\omega)\cdot\bar{M}_{t-u}(\theta_u\omega) = 0$. Again $\bar{M}_t(\omega) = 0$ and (A.4) is established in all cases. Note that if $0 < \bar{M}_u(\omega) < \infty$ (A.4) holds for $t = u$ and $\bar{M}_{0+}(\theta_u\omega) = 1$.

Now assume the m almost all r in (A.1) do not include the countable times when $X_r(\omega)$ is in $A_0^c \cap \{x : x \text{ irregular for } A_0^c\}$. If $X_r(\omega) \in A_0$, then $M_{s-r}(\theta_r\omega) \to 1$ as $s \downarrow r$ and

$$M_{t-r}(\theta_r\omega) = \bar{M}_{t-r}(\theta_r\omega). \tag{A.5}$$

If $X_r(\omega) \in A_0^c$, then $X_u(\omega) \in A_0^c$ for a sequence of $u \downarrow r$, and it follows that for all s near r $M_{t-s}(\theta_s\omega) = M_{t-r}(\theta_r\omega) = 0$, establishing (A.5) for m almost all $r < t$. It follows from (A.5) that $\bar{M}_t = M_t$ a.s. p^x for all $x \in A_0$.

The last assertion of (5.5) is immediate from the definition of Λ and $\bar{M}_{t-r}(\theta_r\omega)$.

REFERENCES

[1] Blumenthal, R. M., Getoor, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.

[2] Chung, K. L. (1966). On the boundary theory for Markov chains II. *Acta. Math.* **115** 111–163.

[3] Dellacherie, C. (1972). *Capacités et processus stochastiques.* Springer, Berlin.

[4] Getoor, R. K. (1975). Markov Processes: Ray Processes and Right Processes. Lecture Notes in Math. 440. Springer, Berlin.

[5] Getoor, R. K. and Sharpe, M. J. (1973). Last exit decompositions and distributions. *Indiana Univ. Math. J.* **23** 377–404.

[6] Getoor, R. K. and Sharpe, M. J. (1979) The Markov property at co-optional times. Zeit Wahrsch. verw Gebiete **48** 201–211.

[7] Jacobsen, M. and Pitman, J. W. (1977). Birth, death, and conditioning of Markov chains. *Ann. Probability* **5** 430–450.

[8] Maisonneuve, B. (1975). Exit Systems. *Ann. Probability* **3** 399–411.

[9] Meyer, P. A., Smythe, R. T. and Walsh, J. B. (1972). Birth and death of Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **3** 295–305. Univ. of Cal. Press.

[10] Pittenger, A. O. and Shih, C. T. (1973). Coterminal families and the strong Markov property. *Trans. AMS* **182** 1–42.

[11] Walsh, J. B. (1971). Some topologies connected with Lebesgue measure. Séminaire de Probabilitiés V, Univ. de Strasbourg. Lec. Notes in Math 191. Springer, Berlin.

[12] Walsh, J. B. (1972). The perfection of multiplicative functionals. Séminaire de Probabilitiés VI, Univ. de Strasbourg, Lecture Notes in Math. 258, 233–242.

Department of Mathematics
University of Maryland Baltimore County
Baltimore, Maryland 21228