*The Annals of Probability*
1982, Vol. 10, No. 1, 247–251

# ON A CONJECTURE IN GEOMETRIC PROBABILITY REGARDING ASYMPTOTIC NORMALITY OF A RANDOM SIMPLEX

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A conjecture in geometric probability about the asymptotic normality of the $r$-content of the $r$-simplex, whose $r + 1$ vertices are independently uniformly distributed random points of which $p$ are in the interior and $r + 1 - p$ are on the boundary of a unit $n$-ball, is proved by Ruben (1977). In this article it is shown that the exact density of the random $r$-content is available in the most general case. The technique of inverse Mellin transform is used to get the exact density, thus requiring the knowledge of the $k$th moment of the $r$-content for all real $k$. This $k$th moment is already available in the literature. Approximations and asymptotic results as well as a simpler alternate proof for the conjecture are also given.

**1. Introduction.** Consider a set of $r + 1$ independently identically uniformly distributed random points $(1 \le r \le n)$ of which $p$ are in the interior $(p \ge 0)$ and $q$ are on the boundary $(q \le r + 1, p + q = r + 1)$ of a unit $n$-ball. Let $\Delta_n$ denote $r!$ times the $r$-content of the $r$-simplex generated by these points. Let $\Delta_n^* = (2n/r)^{1/2}(\Delta_n - (r + 1)^{1/2})$. The exact density of $\Delta_n$ for $r = 1$, $p = 2$, $q = 0$ is obtained by Hammersley (1950). Miles (1971) obtained the exact density of $\Delta_n$ for $r = 1$, $p = 1$, $q = 1$ and $r = 1$, $p = 0$, $q = 2$ and he conjectured that $\Delta_n^*$ is asymptotically normally distributed with mean value zero and standard deviation unity. Ruben (1977) showed that the conjecture was correct.

In this article it is shown that the exact density of $\Delta_n$ in the most general case is available. The method is illustrated for some particular values of $p$, $q$ and $r$. It is shown that this problem is connected to a wide class of problems associated with the distributions of multivariate test statistics as well as to generalized special functions such as $G$ and $H$ functions. A method is given for getting approximations and asymptotic results for $\Delta_n$. This also leads to a simpler proof for the conjecture of Miles (1971).

**2. Exact densities of random $r$-contents.** The exact densities will be obtained by using the technique of inverse Mellin transforms with the help of the moment expressions. The $k$th moment of $\Delta_n$ is given by Miles (1971) and Ruben (1977). Rewriting equation (14) of Ruben (1977) one has

$$\begin{aligned} E(\Delta_n^2)^h = {} & \Gamma[(r + 1)(n/2) + (r + 1)h - r + p] \textstyle\prod_{j=0}^{r-1} \Gamma(h + (n - j)/2)[\Gamma(1 + n/2)]^p \\ & \cdot[\Gamma(n/2)]^{r+1-p}/\{\Gamma[(r + 1)(n/2) + rh - r + p] \textstyle\prod_{j=0}^{r-1} \Gamma((n - j)/2) \\ & \cdot[\Gamma(h + 1 + n/2)]^p[\Gamma(h + n/2)]^{r+1-p}\} \end{aligned} \tag{2.1}$$

which is also valid for complex values of $h$ where $E$ denotes the expected value. For convenience these gammas will be rewritten in a slightly different form. With the help of the multiplication formula for gamma functions, namely,

$$\Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-1/2} \textstyle\prod_{j=0}^{m-1} \Gamma(z + j/m), \qquad m = 1, 2, \cdots \tag{2.2}$$

Received July 21, 1980; revised December 2, 1980.
[1] This research is supported by the Natural Sciences and the Engineering Research Council of Canada.
*AMS* 1980 *subject classifications.* 60D05, 33A35.
*Key words and phrases.* Random volumes, exact density, asymptotic normality.

one can expand the gamma functions $\Gamma[(r+1)(n/2)+(r+1)h-r+p]$ and $\Gamma[(r+1)(n/2)+rh-r+p]$ by taking $m=(r+1), r$ and $z=(h+n/2)-(r-p)/(r+1), h+((n/2)(r+1)/r)-(r-p)/r$ respectively and simplify the $h$th moment of $\Delta_n^2$ to the following:

$$E(\Delta_n^2)^h = CA(h) \tag{2.3}$$

where

$$\begin{aligned} A(h) = & \textstyle\prod_{j=0}^{r} \Gamma(h+n/2-(r-p)/(r+1)+j/(r+1)) \\ & \cdot \textstyle\prod_{j=1}^{r} \Gamma(h+n/2-(j-1)/2)[(r+1)^{r+1}/r^r]^h / \{\prod_{j=0}^{r-1} \Gamma(h+(n/2) \\ & \cdot (r+1)/r-(r-p)/r+j/r)[\Gamma(h+1+n/2)]^p[\Gamma(h+n/2)]^{r+1-p}\} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} C = & (2\pi)^{-1/2}(r+1)^{(r+1)(n/2)-r+p-1/2}[\Gamma(1+n/2)]^p \\ & \cdot [\Gamma(n/2)]^{r+1-p}/[r^{(r+1)(n/2)-r+p-1/2} \textstyle\prod_{j=1}^{r} (n/2-(j-1)/2)]. \end{aligned} \tag{2.5}$$

One can look upon (2.3) as the Mellin transform of the density of $\Delta_n^2$ by taking $h=s-1$. Then from the theory of inverse Mellin transform the density of $x=\Delta_n^2$, denoted by $f(x)$, is as follows where $\alpha=(n/2)+h-(r-1)/2$.

$$\begin{aligned} f(x) &= (x^{-1})(2\pi i)^{-1}\int_L CA(h)x^{-h}\,dh \\ &= x^{(n/2)-1-(r-1)/2}C'(2\pi i)^{-1}\int_{L'} A(\alpha-n/2+(r-1)/2)x^{-\alpha}\,d\alpha \end{aligned} \tag{2.6}$$

where $i=(-1)^{1/2}$, $L$, $L'$ are suitable contours and $C'=C[(r+1)^{r+1}/r^r]^{(r-1)/2-(n/2)}$. From the structure of the gamma products in (2.4) it is easy to see that $L$ and $L'$ can always be found and that $f(x)$ is uniquely determined.

*Case* 1. $r=1, p=2, q=0$.

In this case $C'$ reduces to the form $C'=n\Gamma(1+n/2)/(4\pi)^{1/2}$ and

$$f(x) = n\Gamma(1+n/2)x^{n/2-1}(4\pi)^{-1/2}\int_{L'} \{\Gamma(\alpha+1/2)/[\alpha\Gamma(\alpha+1+n/2)]\}(x/4)^{-\alpha}\,d\alpha.$$

The integrand has poles of order unity at $\alpha=0$, $\alpha=-j-1/2, j=0,1,\cdots$. The residue at $\alpha=0$ is $\Gamma(1/2)/\Gamma(1+n/2)$ and the residue at $\alpha=-j-1/2$ is $(-1)^j(x/4)^{j+1/2}/[j!(-j-1/2)\Gamma(-j+1/2+n/2)]$. Simplifying this with the help of the conversion formula

$$\Gamma(\beta-j) = (-1)^j\Gamma(\beta)\Gamma(1-\beta)/\Gamma(1-\beta+j) \tag{2.7}$$

where the gamma ratios are interpreted as

$$\Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1) = (a)_k,$$

one gets the density as follows.

$$\begin{aligned} f(x) = & (n/2)x^{n/2-1}\{1-[\Gamma(1+n/2)/\Gamma(3/2)\Gamma((n+1)/2)](x/4)^{1/2} \\ & \cdot {}_1F_1(1-(n+1)/2; 3/2; x/4)\} \end{aligned} \tag{2.8}$$

where ${}_1F_1(\cdot)$ is a hypergeometric series. A general hypergeometric series of the form ${}_pF_q(\cdot)$ is defined as follows.

$$ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x) = \textstyle\sum_{k=0}^{\infty} \{(a_1)_k\cdots(a_p)_k/[(b_1)_k\cdots(b_q)_k]\}(x^k/k!) \tag{2.9}$$

which is convergent for all $x$ if $q \geq p$ and for $|x|<1$ if $p=q+1$. If one of the $a_j$'s is a negative integer then ${}_pF_q(\cdot)$ is a polynomial. If a $b_j$ is a negative integer the series is not defined unless there is an $a_i$ such that $(a_i)_k=0$ before $(b_j)_k=0$, $k=0,1,\cdots$.

Miles (1971) has given (2.8) in the form of an incomplete beta function. But it is not difficult to see that the expressions are one and the same. Miles' density in this case, denoted by $f_{2,0}(\Delta)$ is as follows.

$$
\begin{aligned}
f_{2,0}(\Delta) &= n\Delta^{n-1}\{\Gamma(1+n/2)/\Gamma(1/2)\Gamma((n+1)/2)\} \\
&\quad\cdot \int_0^{1-(\Delta/2)^2} t^{(n-1)/2}(1-t)^{-1/2}\,dt \\
&= n\Delta^{n-1}\{1-[\Gamma(1+n/2)/\Gamma(1/2)\Gamma((n+1)/2)] \\
&\quad\cdot \int_0^{(\Delta/2)^2} t^{(1/2)-1}(1-t)^{(n+1)/2-1}\,dt\}
\end{aligned}
\tag{2.10}
$$

which is obtained by changing $t$ to $1-t$ and integrating out by using the beta integral. Now expand $(1-t)^{(n+1)/2-1}$ by using the binomial expansion, convert the gammas by using (2.7) and transform $\Delta$ to $x=\Delta^2$; then (2.10) agrees with (2.8).

By using the same procedure as described in (2.6) to (2.8) one can easily verify all the known cases. Now consider the following case $r=2, p=3, q=0$ which is not obtained by others.

*Case* 2. $r=2, p=3, q=0$.

In this case (2.6) reduces to the following.

$$
\begin{aligned}
f(x) = C'x^{(n-3)/2}(2\pi i)^{-1} \int &\{\Gamma(\alpha+5/6)\Gamma(\alpha+7/6)\Gamma(\alpha)/[(\alpha+1/2) \\
&\cdot\Gamma(\alpha+3/2)\Gamma(\alpha+1+n/4)\Gamma(\alpha+3/2+n/4)]\}(4x/27)^{-\alpha}\,d\alpha.
\end{aligned}
$$

The poles are of order unity at $\alpha=-1/2$, $\alpha=-v-5/6$, $v=0,1,\cdots$, $\alpha=-j-7/6$, $j=0,1,\cdots$, $\alpha=-k$, $k=0,1,\cdots$. The residues are the following.

$(4x/27)^{1/2}\Gamma(1/3)\Gamma(2/3)\Gamma(-1/2)/[\Gamma(1/2+n/4)\Gamma(1+n/4)]$; $(-1)^v(4x/27)^{v+5/6}\Gamma(-v+1/3)\Gamma(-v-5/6)/[v!(-v-1/3)\Gamma(-v+2/3)\Gamma(-v+1/6+n/4)\Gamma(-v+2/3+n/4)]$, $v=0,1,\cdots$; $(-1)^j(4x/27)^{j+7/6}\Gamma(-1/3-j)\Gamma(-j-7/6)/[j!(-j-2/3)\Gamma(-j+1/3)\Gamma(-j-1/6+n/4)\Gamma(-j+1/3+n/4)]$, $j=0,1,\cdots$; $(-1)^k(4x/27)^k\Gamma(-k+5/6)\Gamma(-k+7/6)/[k!(-k+1/2)\Gamma(-k+3/2)\Gamma(-k+1+n/4)\Gamma(-k+3/2+n/4)]$, $k=0,1,\cdots$.

Converting the gammas by using (2.7) and summing up by using (2.9) one has the following.

$$
\begin{aligned}
f(x) = C'\,x^{(n-3)/2}&\{\Gamma(1/3)\Gamma(2/3)\Gamma(-1/2)(4x/27)^{1/2}/[\Gamma(1/2+n/4)\Gamma(1+n/4)] \\
&-\Gamma(-5/6)\Gamma(1/3)\Gamma(1/3)(4x/27)^{5/6}\,{}_4F_3(1/3,1/3,5/6-n/4,1/3 \\
&-n/4;2/3,11/6,4/3;4x/27)/[\Gamma(1/6+n/4)\Gamma(2/3+n/4)\Gamma(2/3)\Gamma(4/3)] \\
&-\Gamma(2/3)\Gamma(-1/3)\Gamma(-7/6)(4x/27)^{7/6}\,{}_4F_3(2/3,2/3,7/6-n/4, \\
&2/3-n/4;4/3,13/6,5/3;4x/27)/[\Gamma(5/3)\Gamma(1/3)\Gamma(n/4-1/6)\Gamma(n/4+1/3)] \\
&-\Gamma(-1/2)\Gamma(5/6)\Gamma(7/6)\,{}_4F_3(-1/2,-1/2,-n/4,-n/4 \\
&-1/2;1/6,-1/6,1/2;4x/27)/[\Gamma(1/2)\Gamma(3/2)\Gamma(1+n/4) \\
&\cdot\Gamma(3/2+n/4)]\} \qquad \text{for } 0<4x/27<1.
\end{aligned}
\tag{2.11}
$$

There are several other cases where the poles of the integrand in (2.6) are simple. For example, consider the cases $r=2, p=2$; $r=2, p=1$; $r=2, p=0$. In all those cases the technique of Case 2 above can be applied. When the poles are of higher orders the general techniques discussed in the next section are needed.

**3. General case.** When poles of the integrand in (2.6) are of higher orders the residues will contain psi and zeta functions. From the structure of the gamma product in (2.4) it may be seen that $f(x)$ in (2.6) is a Meijer's $G$-function. The theory, applications and computable representations of a $G$-function are available from Mathai and Saxena (1973). The details won't be given here due to the fact that the expressions occupy too much space. One can also invert (2.1) directly. This leads to an $H$-function. The theory, applications and computable representation of an $H$-function are available from Mathai and Saxena (1978). From the structure in (2.4) it is easy to note that $\Delta_n^2$ can be considered as a constant times the product of independent real scalar beta random variables of the first kind. This structure appears in a wide class of likelihood ratio tests associated with multinormal populations. For details see Mathai and Saxena (1973).

**4. Approximations and asymptotic results.** A number of approximations for the density of $\Delta_n$ will be derived here. The method will also give a simpler proof for Miles' conjecture about the asymptotic normality of $\Delta_n^*$. Consider the following asymptotic formula for gamma functions.

$$\begin{aligned}\log \Gamma(x+h) &= \log(2\pi)^{1/2} + (x+h-1/2)\log x - x \\ &\quad - \textstyle\sum_{r=1}^{m} (-1)^r B_{r+1}(h)/[r(r+1)x^r] + R_{m+1}(x)\end{aligned} \tag{4.1}$$

where $R_{m+1}(x) = O(x^{-(m+1)})$ and $B_r(h)$ is the Bernoulli polynomial of degree $r$ and order unity which is defined by

$$ue^{hu}/(e^u - 1) = \textstyle\sum_{r=0}^{\infty} u^r B_r(h)/r!$$

where for example, $B_1(h) = h - 1/2$, $B_2(h) = h^2 - h + 1/6$, $B_3(h) = h^3 - 3h^2/2 + h/2$. This asymptotic formula is valid for $x \to \infty$ when $h$ is bounded. Various orders of approximations are available from (4.1). A first approximation to the order of $O(x^{-1})$ is available by taking

$$\log \Gamma(x+h) \approx \log(2\pi)^{1/2} + (x+h-1/2)\log x - x \tag{4.2}$$

where $\approx$ denotes "approximately equal to." A second approximation is available by taking

$$\log \Gamma(x+h) \approx \log(2\pi)^{1/2} + (x+h-1/2)\log x - x + B_2(h)/(2x). \tag{4.3}$$

Under these two approximations it will be shown that $\Delta_n$ is approximated to a degenerate and a normal random variable respectively. Replace $h$ by $h/2$ and take the resulting expression in (2.1), namely, $E(\Delta_n^h)$. Approximate all the gammas by using (4.2). Then one gets

$$E(\Delta_n^h) \approx (r+1)^{h/2} \tag{4.4}$$

which means that $\Delta_n$ degenerates into $(r+1)^{1/2}$ with probability one. Using (4.3) if the gammas in (2.1) are approximated then one gets the following.

$$E(\Delta_n^h) \approx (r+1)^{h/2} \exp\{(1/n)[rh^2/(4(r+1)) - (r^2+4p+1)rh/(4(r+1))]\}. \tag{4.5}$$

But $E(\Delta_n^h) = Ee^{h\log\Delta_n}$. Hence when $h = it$, $i = (-1)^{1/2}$ and $t$ is a real arbitrary parameter, one has the characteristic function of $\log\Delta_n$. By comparing the characteristic function of a normal random variable it is easy to see from (4.5) that $\log[\Delta_n/(r+1)^{1/2}]$ is approximately normally distributed with mean value $\mu = -r(r^2+4p+1)/(4n(r+1))$ and variance $r/(2n(r+1))$. Hence $n^{1/2}\log[\Delta_n/(r+1)^{1/2}]$ is approximately normal with mean value $\mu n^{1/2}$ and variance $r/(2(r+1))$. Since $\mu n^{1/2} \to 0$ as $n \to \infty$ it is seen that $[n(r+1)/r]^{1/2}$ $\log[\Delta_n/(r+1)^{1/2}]$ is asymptotically normal with mean value zero and variance unity. But the function $\log y$ is such that its first derivative at $y = (r+1)^{1/2}$ is nonzero. Hence from the result of Rao (1973) pages 385–386, it follows that $\Delta_n^* = (2n/r)^{1/2}[\Delta_n - (r+1)^{1/2}]$ goes to standard normal as $n \to \infty$. This is Miles' conjecture which was proved by Ruben (1977) by using a slightly lengthier procedure. By taking successive terms from (4.1) one can get successive approximations for the density of $\Delta_n$. By rearranging the terms and using a

Box's type procedure, see Anderson (1958) pages 203–207 for details, one can get a Chi squared approximation.

**Acknowledgement.** The author would like to thank the referee for his comments and suggestions which enabled him to improve the presentation of the material in this article.

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