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THE INTEGRAL OF THE ABSOLUTE VALUE OF THE PINNED WIENER PROCESS — CALCULATION OF ITS PROBABILITY DENSITY BY NUMERICAL INTEGRATION

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L. A. Shepp [1] has studied the distribution of the integral of the absolute value of the pinned Wiener process, and has expressed the moment generating function in terms of a Laplace transform. Here we apply Shepp's results to obtain an integral for the density of the distribution. This integral is then evaluated by numerical integration along a path in the complex plane.

1. Introduction. Let $\tilde{W}_t$, $0 \le t \le 1$, be the pinned Wiener process, i.e., a standard Wiener process starting from 0 at $t = 0$ and conditioned to pass through zero at $t = 1$. Let

$$\xi = \int_0^1 |\tilde{W}|\, dt, \tag{1}$$

$$\phi(r) = Ee^{-\xi r}, \tag{2}$$

$$p(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi r}\phi(r)\, dr \tag{3}$$

where $p(\xi)$ is the probability density of the random variable ξ.

Shepp [1] has shown that

$$\phi(\sqrt{2}s^{3/2})/\sqrt{\pi s} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du\, e^{us}Ai(u)/Ai'(u) \tag{4}$$

where $Ai(u)$ is the Airy function [2] and $Ai'(u)$ is its derivative, and he has pointed out that $p(\xi)$ can be calculated from (3) and (4).

Here we shall use (4) and the first five moments of $p(\xi)$ to calculate $\phi(r)$, and then determine $p(\xi)$ by numerically integrating (3) along a suitable path in the complex r-plane.

2. Statement of Results. The following table of ξ, $p(\xi)$ lists some values of $p(\xi)$ that have been calculated from (3) by numerical integration:

.100	0.089	.25	3.732	.6	0.3785	1.0	0.00725
.125	0.757	.30	3.008	.7	0.1649	1.1	0.00204
.150	2.001	.40	1.614	.8	0.0650	1.2	0.00051
.200	3.753	.50	0.803	.9	0.0230	1.3	0.00011

The largest value of $p(\xi)$ that occurred in our calculations was $p(.225) = 3.8993$, and the smallest was $p(1.5) = 3.7 \cdot 10^{-6}$.

Fig. 1 shows $p(\xi)$ plotted as a function of ξ. From results in [1], the mean value of ξ is 0.3133 and its standard deviation is 0.1382.

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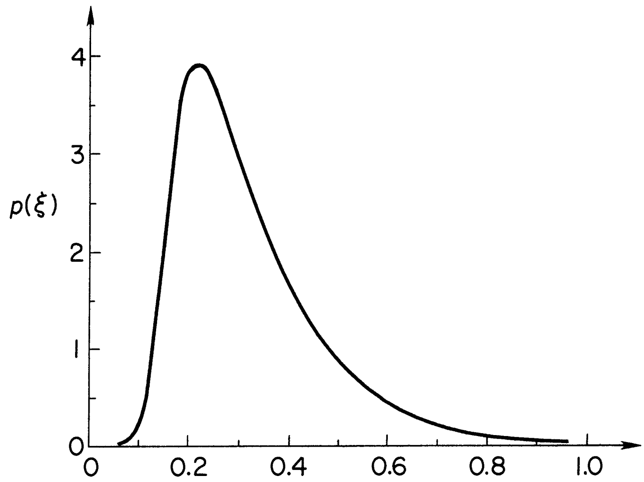

Fig. 1. Probability density $p(\xi)$ as a function of ξ.

When ξ tends to zero, $p(\xi)$ is given by the saddle point approximation (see Section 5 below)

(5) $$p(\xi) \approx .2285\xi^{-3}\exp(-.07833\xi^{-2}), \quad \xi \text{ small.}$$

This approximation is in error by about two percent at $\xi = 0.1$.

When the calculated values of $\ln p(\xi)$ are plotted against ξ^2 for the range $1 \le \xi \le 1.5$, they are found to lie close to a straight line, and the results suggest the approximations

$$p(\xi) \approx 4.1 \exp(-6.3\xi^2), \qquad \xi \text{ large,}$$

$$\phi(r) \approx O[2.8 \exp(r^2/27.2)], \quad r \to -\infty.$$

3. Calculation of $\phi(r)$. The power series

(6) $$\phi(r) = \textstyle\sum_{n=0}^{\infty} (-)^n m_n r^n/n!$$

can be obtained by expanding expression (2) for $\phi(r)$ and setting $m_n = E(\xi^n)$. Shepp [1] has given a recurrence relation for m_n and shows that

(7) $$m_0 = 1 \qquad m_2 = 7/60 \qquad m_4 = 19/720$$
$$m_1 = (1/4)\sqrt{\pi/2} \qquad m_3 = 21\, m_1/128 \qquad m_5 = 101\, m_1/2048.$$

A second expression for $\phi(r)$ can be obtained from the integral (4) by noting that (i) $Ai(u)$ and $Ai'(u)$ are integral functions of u, (ii) the zeros of $Ai'(u)$ are simple and lie on the negative real u-axis, and (iii) from [2], as $|u| \to \infty$

$$Ai(u)/Ai'(u) \sim -u^{-1/2} + O(u^{-2}), \quad |\arg u| < \pi.$$

Therefore when $Re(s) > 0$ the path of integration in the integral (4) can be displaced to the left. The contributions of the residues of the poles of $1/Ai'(u)$ then give

(8) $$\phi(\sqrt{2}s^{3/2})/\sqrt{\pi s} = -\textstyle\sum_{n=1}^{\infty} \dfrac{Ai(a_n')}{Ai''(a_n')} \exp(a_n' s)$$
$$= \textstyle\sum_{n=1}^{\infty} \exp(a_n' s)/(-a_n'), \quad Re(s) > 0$$

where $u = a_n'$ is the nth zero of $Ai'(u)$. In going from the first to the second equation in (8), we have used

$$Ai''(u) - uAi(u) = 0.$$

The values of a'_n for $n = 1$ to 10 are listed on page 478 of [2]. For $n > 10$ we can use the approximation

$$a'_n \approx -y^{2/3}\left(1 - \frac{7}{48y^2} + \frac{35}{288y^4}\right),$$
$$y = \frac{3\pi}{2}\left(n - \frac{3}{4}\right),$$

which follows from equations (10.4.95) and (10.4.105) of [2].

The expression we seek for $\phi(r)$ can now be obtained by setting

(9) $$\sqrt{2}s^{3/2} = r, \qquad s = (r/\sqrt{2})^{2/3},$$

in (8). It is

(10) $$\phi(r) = \sqrt{\pi}(r/\sqrt{2})^{1/3} \textstyle\sum_{n=1}^{\infty} \exp[a'_n(r/\sqrt{2})^{2/3}]/(-a'_n), \quad |\arg r| < 3\pi/4.$$

In our calculations we obtain $\phi(r)$ for $|r| \le 1/2$ from the first six terms in the power series (6) and the moments (7). For $|r| > 1/2$ and $|\arg r| < 3\pi/4$, we used the series (10). It turned out that we never had to use more than 40 terms in (10) if the series was truncated when the terms became less than 10^{-6}.

4. Numerical Integration of the Integral (3) for $p(\xi)$. The symmetry of $\phi(r)$ about the real r-axis allows us to write (3) as

(11) $$p(\xi) = \text{Real}\,\frac{1}{\pi i}\int_0^{i\infty} e^{\xi r}\phi(r)\,dr.$$

We seek a path of integration in the complex r-plane such that (i) the integrand in (11) will decrease rapidly, and (ii) $\phi(r)$ can be calculated by one or the other of the two methods described in Section 3.

Some experimentation suggests the path

(12) $$r = v^{1/3} - v + iv$$

where v is real and runs from 0 to ∞. This path starts out along the positive r-axis and then curves upwards and to the left. It crosses the imaginary r-axis at $v = 1$ and then tends to run parallel to the ray $\arg r = 3\pi/4$ (so that we can still use the series (10) for $\phi(r)$). When v is large the absolute value of $\exp(\xi r)$ decreases as $\exp(-\xi v)$ while the individual terms in the series (10) remain at $O(1)$.

The further change of variable [3, Sec. VI]

(13) $$v = (1/\xi)\exp[x - 3e^x]$$

increases the rate of convergence, and (11) becomes

(14) $$p(\xi) = \text{Real}\,\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{\xi r}\phi(r)\,\frac{dr}{dv}\frac{dv}{dx}\,dx$$

where dr/dv and dv/dx can be obtained from (12) and (13), respectively.

The trapezoidal rule with truncation at $x = \pm 3.5$ was used to evaluate (14). It was found that, as ξ decreased, more points were required in the numerical integration to achieve the desired accuracy. In our calculations, several trials led us to use $\Delta x = 0.075$ (94 points) for $\xi \ge 0.4$ and $\Delta x = 0.025$ (280 points) for $0.1 \le \xi < 0.4$.

5. Behavior of $p(\xi)$ when ξ is small. When r tends to $+\infty$, the leading term in (10) shows that

$$\phi(r) \to Cr^{1/3}\exp[-\alpha r^{2/3}]$$

where

$$\alpha = -a_1'/2^{1/3}, \quad C = (1/\alpha)\sqrt{\pi/2}.$$

The integrand in the integral (3) for $p(\xi)$ tends to

$$Cr^{1/3}\exp[\xi r - \alpha r^{2/3}]. \tag{15}$$

Setting the derivative of the exponent equal to zero and solving for r shows that the exponential term in (15) has a saddle point at

$$r_0 = (2\alpha/3)^3\xi^{-3}.$$

When ξ is small, r_0 is large and the leading term in the asymptotic series for $p(\xi)$ given by the method of steepest descent (from r_0) leads to

$$p(\xi) \sim C(2\pi f_2)^{-1/2} r_0^{1/3} e^{f_0} = \xi^{-3}\sqrt{\beta/3}\exp(-\tfrac{1}{2}\beta\xi^{-2}) \tag{16}$$

where

$$f_0 = \xi r_0 - \alpha r_0^{2/3} = -(\alpha/3) r_0^{2/3},$$
$$f_2 = (2\alpha/9) r_0^{-4/3},$$
$$\beta = (2\alpha/3)^3.$$

From $a_1' = -1.0188$ we get $\alpha = 0.8086$, $\beta = 0.1567$, and finally, from (16), the approximation (5) for $p(\xi)$.

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