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THE PROPORTION OF BROWNIAN SOJOURN OUTSIDE A MOVING BOUNDARY

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Let B_t be a d-dimensional Brownian motion starting at zero and $h(t)$ a positive nondecreasing function of $t > 0$. It is shown that $\lim\sup_{s\downarrow 0}(1/s)\text{meas}\{t \in (0, s]: |B_t| > h(t)\sqrt{t}\} = 1 - e^{-4(q-1)}$; where q is defined as a simple functional of h and, if we take $h_c(t) = c\sqrt{(2\log\log(1/t))}$ for h and if $0 < c \leqq 1$, $q = 1/c^2$. We also investigate $X^* = \lim\sup_{s\downarrow 0}(1/s)\text{meas}\{t \in (0, s]: |B_t| < (\sqrt{t})/h(t)\}$ and find upper and lower bounds of X^*, which indicate in particular that if $h = h_c (c > 0)$, $X^* = p_c$ (say) is positive and less than one and tends to zero (one) as $c \uparrow \infty$ (respectively, $c \downarrow 0$). The problem for the case $s \to \infty$ is also treated.

0. Introduction and results. Let B_t be a standard d-dimensional Brownian motion starting at zero and $h(t)$ be a positive function of $t > 0$ and consider the proportion

$$X_s = \frac{1}{s}\cdot\text{meas}\{t: 0 < t < s, |B_t| > h(t)\sqrt{t}\}$$

where meas$\{\cdot\}$ denotes Lebesgue measure and $|x|$ Euclidean length in $\mathbb{R}^d$. The function $h(t)$ is assumed throughout this paper to be nonincreasing in $0 < t < 1$ and nondecreasing in $t > 1$. It is shown by Strassen [6] that if we take $h_c(t) = c\sqrt{(2\log(|\log t| + 2))}$ for $h(t)$ and if $0 < c \leqq 1$, then $\lim\sup_{s\to\infty} X_s = 1 - \exp[-4(c^{-2} - 1)]$ (a.s.). His proof can be modified to verify

$$\lim_{s\downarrow 0}\sup X_s = 1 - \exp\left[-4\left(\frac{1}{c^2} - 1\right)\right] \qquad \text{a.s.}$$

for the same h. In the following theorems these are slightly improved. First let

$$q = \sup\left\{r \geqq 0: \int_0^1 \exp\left[-r\cdot\frac{h(t)^2}{2}\right]\frac{dt}{t} = \infty\right\}.$$

THEOREM 1. *If $q \geqq 1$, $\lim_{s\downarrow 0}\sup X_s = 1 - e^{-4(q-1)}$* (a.s.).

Next let

$$q' = \sup\left\{r \geqq 0: \int_1^\infty \exp\left[-r\cdot\frac{h(t)^2}{2}\right]\frac{dt}{t} = \infty\right\}.$$

THEOREM 1′. *If $q' \geqq 1$, $\lim_{s\to\infty}\sup X_s = 1 - e^{-4(q'-1)}$* (a.s.).

The proof offered in the present paper is different from Strassen's; it is based on Motoo's proof of Komogorov's test for Brownian motion (cf. [5]) and strongly depends on the Markov property of Brownian motion. Just as Motoo's method can be applied to find a lower modulus of continuity for d-dimensional Brownian paths, our method is effective in treating

$$\lim_{s\downarrow 0}\inf\frac{1}{s}\cdot\text{meas}\left\{t: 0 < t < s, |B_t| > \frac{\sqrt{t}}{h(t)}\right\},$$

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or, equivalently, in treating $\limsup_{s\downarrow 0} X_s^*$, where

$$X_s^* = \frac{1}{s} \cdot \text{meas}\left\{t : 0 < t < s, |B_t| < \frac{\sqrt{t}}{h(t)}\right\}.$$

Let β_0 be the smallest positive root of $J_{d/2-1}(\beta) = 0$ (J_ν is a Bessel function).

THEOREM 2. *There exist continuous functions $\theta_*(u)$ and $\theta^*(u)$ of $u \geqq 0$ with the following properties*:

(1) $\theta_*(0) = \theta^*(0) = 0$
(2) $\theta^*(u) \geqq \theta_*(u) \geqq (2/\beta_0^2)u$ *for* $u \geqq 0$
(3) $\limsup_{u\uparrow\infty} \theta^*(u)/u < \infty$
(4) $\lim_{u\downarrow 0} \theta_*(u)/u = \infty$ *if* $d = 1$ *or* 2
(4)′ $\limsup_{u\downarrow 0} \theta^*(u)/u < \infty$ *if* $d = 3, 4, \cdots,$

and with probability one,
(5) $1 - e^{-\theta_*(q)} \leqq \lim_{s\downarrow 0}\sup X_s^* \leqq 1 - e^{-\theta^*(q)}$.

A corresponding result for $\limsup_{s\uparrow\infty} X_s^*$ holds, though it is not stated. It should be also remarked that in the definition of X_s or X_s^* above we can replace $|B_t|$ by a Bessel process starting at zero with parameter $\mu > 0$ (it coincides in law with $|B_t|$ if μ is an integer and $\mu = d$), with statements of Theorems 1, 1′ and 2 remaining valid, unchanged except that in (4) and (4)′ of Theorem 2 "$d = 1$ or 2" and "$d = 3, 4, \cdots$" should be replaced respectively by "$0 < \mu \leqq 2$" and by "$\mu > 2$".

The value of $\limsup_{s\downarrow 0(\uparrow\infty)} X_s^*$ is of course a (sure) functional of $h(t)$. Theorem 2 suggests that the functional differs markedly among different dimensions of Brownian motions. An explicit form of it should be found; it is still open whether the functional is reduced to a function of q (or q').

There are a few related works. D. Geman [2a] and [2b] studied a similar problem for a class of stochastic processes including a wide class of Gaussian processes; he gave sufficient conditions on $h(t)$ for $\limsup_{s\downarrow 0} X_s = 0$ or for $\limsup_{s\downarrow 0} X_s^* = 0$ (B_t is replaced by a process in the class and $\sqrt{t}$ by the variance of its value at t in the definitions of X_s and X_s^*), by applying a real variable lemma. N. Kôno recently obtained the equalities in Theorems 1 and 1′ for Gaussian processes with index $\alpha > 0$ (with the same interpretation of X_s as above), by improving Strassen's method.

In Section 1 we shall introduce a diffusion process obtained from a d-dimensional Brownian motion through a well-known transformation, which will be used throughout the paper. Theorem 1 will be proved in Sections 2 and 3. Theorem 1′ can be proved in an analogous way. A brief sketch of its proof will be given at the end of Section 3. The proof of Theorem 2 will be given in Sections 4 to 6. In the Appendix we shall prove several lemmas which are used in the proof of Theorem 1.

1. Preliminaries. We shall make use of the following facts. Let B_t be a d-dimensional Brownian motion starting at zero; the probability measure is denoted by P and the associated expectation by E. Then the process defined by

$$N(t) = e^{t/2}|B(e^{-t})| \qquad t \geqq 0 \tag{1.1}$$

(we write $B(t)$ for B_t) is a diffusion process with initial distribution $P[N(0) > x] = P[|B_1| > x]$; the backward equation associated with it is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{d-1}{x} - x\right)\cdot\frac{\partial u}{\partial x} \qquad x > 0 \tag{1.2}$$

with boundary condition

$$\lim_{x\downarrow 0} x^{d-1}\cdot\frac{\partial u}{\partial x} = 0 \tag{1.2′}$$

(cf. Ito-McKean (1965) pages 162–163). According to the usual way of describing Markov processes let us denote by P_x^N the probability measure on $W = C([0, \infty))$ (the space of all continuous functions of $t \geqq 0$) which is induced by the process $N(t)$ starting at x. Thus for example $P_x^N[w(t) > y] = P[N(t) > y \mid N(0) = x]$. We shall denote by m_b the first passage time to b, i.e.,

$$m_b = \inf\{t \geqq 0 : w(t) = b\} \qquad w \in W.$$

Then it is a standard exercise to see $E_x^N[m_b] < \infty$ for $x > 0$ and $b > 0$.

2. Proof of Theorem 1 (I). Using the process $N(t)$ defined by (1.1), we have

$$X_s = e^T \int_T^\infty I(N(t) - g(t)) e^{-t} dt \qquad T = -\log s$$

(2.1) where $\quad I(x) = 0$ or 1 according as $x \leqq 0$ or > 0,

and $\quad g(t) = h(e^{-t}).$

Let $0 \leqq \tau_0 < \tau_1 < \cdots$ be the successive passage times of $N(t)$ to $x = 1$ via $x = 2$. Then excursions $\{N(t): \tau_{n-1} \leqq t < \tau_n\}$ are independent and identical in law. Especially $\tau_n - \tau_{n-1}$, $n = 1, 2, \cdots$ are i.i.d. random variables with finite mean: $\gamma \equiv E[\tau_1 - \tau_0] < \infty$, and therefore the strong law of large numbers implies that

$$\frac{n}{2} \cdot \gamma < \tau_{n-1} < \tau_n < 2n\gamma \quad (n \uparrow \infty) \qquad \text{a.s.} \tag{2.2}$$

(where $A_n (n \uparrow \infty)$ means that statements A_n are true for all sufficiently large n). Writing $X(s)$ for X_s, let

$$U_n = \begin{cases} \sup\{X(e^{-T}) : \tau_{n-1} < T < \tau_n\} & \text{if } N(t) > g(t) \text{ for some } t \in (\tau_{n-1}, \tau_n), \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly

$$\lim_{s \downarrow 0} \sup X_s = \lim_{n \uparrow \infty} \sup U_n.$$

The rest of this section will be devoted to showing

$$\lim_{s \downarrow 0} \sup X_s \geqq 1 - e^{-4(\bar{q}-1)}. \tag{2.3}$$

The opposite inequality will be established in the next section.

Given $a > 0$, $0 < \delta < 1$, letting $b = a + \delta$ and $c = a + 2\delta$, we set

$$\sigma^{n,1} = \inf\{t > \tau_{n-1} : N(t) = b\}$$

$$\sigma^{n,2} = \inf\{t > \sigma^{n,1} : N(t) = a \text{ or } c\}$$

$$Z_n(a) = \begin{cases} \sigma^{n,2} - \sigma^{n,1} & \text{if } \sigma^{n,1} < \tau_n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $U_n \geqq 1 - \exp[-Z_n(a)]$ if $a > g(\tau_n)$. Let Ω_1 be the event described in (2.2). Since $g(2\gamma n) > g(\tau_n)(n \uparrow \infty)$ for each path belonging to Ω_1, we have for each $u > 0$

$$\begin{aligned} &\{U_n \geqq 1 - e^{-u} \quad \text{for infinitely many } n\} \\ &\supset \{1 - \exp[-Z_n(g(2\gamma n))] \geqq 1 - e^{-u} \quad \text{for infinitely many } n\} \cap \Omega_1 \\ &= \{Z_n(g(2\gamma n)) \geqq u \quad \text{for infinitely many } n\} \cap \Omega_1. \end{aligned} \tag{2.4}$$

Let us calculate a lower bound of $P[Z_n(a) > u]$. By the strong Markov property of $N(t)$

$$P[Z_n(a) > u] = P_2^N[m_b < m_1] P_b^N[m_a \wedge m_c > u]. \tag{2.5}$$

Since a canonical scale associated with $N(t)$ is given by

$$s(x) = \int_1^x \exp\left[-\int_1^z \left(\frac{d-1}{y} - y\right) dy\right] dz \tag{2.6}$$
$$= \int_1^x z^{1-d}\exp(z^2/2)\, dz,$$

$$P_2^N[m_b < m_1] = \frac{s(2) - s(1)}{s(b)} \sim C\cdot b^d \exp(-b^2/2) \qquad \text{as} \quad b \to \infty \tag{2.7}$$

where $C = \int_1^2 z^{1-d} \exp(z^2/2)\, dz$. ($F(b) \sim G(b)$ as $b \to r$ means $\lim_{b\to r} (F(b)/G(b)) = 1$.) The following estimate will be established in the appendix (Lemma A.1):

$$P_b^N[m_a \wedge m_c > u] \geqq \exp\left(-\frac{a^2u}{8} - \frac{\pi^2 u}{8\delta^2} + O(a)\right) \tag{2.8}$$

where $O(a)$ is uniform in u and δ (this uniformity is not needed for the proof of Theorem 1; see Remark 1 which follows). From (2.7), (2.8) and (2.5) it follows that

$$P[Z_n(a) > u] \geqq \exp\left[-\frac{a^2}{2}\left(1 + o(1)\right) - \frac{\pi^2 u}{8\delta^2}\right] \tag{2.9}$$

where $o(1) \to 0$ as $a \to \infty$ (uniformly in $u > 0$ and in $0 < \delta < 1$).

Now the proof of (2.3) is easy. Let $q > 1$ and take any u such that $q > 1 + u/4$. Then, by (2.9) and by $g(t) = h(e^{-t})$,

$$\sum_{n=1}^\infty P[Z_n(g(2\gamma n)) > u] \geqq \text{const.} \int_0^\infty \exp\left[-\frac{g(2\gamma t)^2}{2}\left(1 + \frac{u}{4}\right)(1 + o(1))\right] dt$$
$$= \text{const.} \int_0^1 \exp\left[-\frac{h(t)^2}{2}\left(1 + \frac{u}{4}\right)(1 + o(1))\right] \frac{dt}{t}$$
$$= +\infty.$$

Since $Z_n(g(2\gamma n))$, $n \geqq 1$, are independent, an application of the Borel Cantelli lemma combined with the relation (2.4) implies $\limsup_{n\uparrow\infty} U_n \geqq 1 - e^{-u}$. Thus (2.3) has been proved.

REMARK 1. If we take $h_c(t) = c\sqrt{(2 \log(|\log t| + 2))}$ for $h(t)$, then $g(2a) - g(a/2) = O(1/g(a))$ as $a \to \infty$. Let $\eta(t) \downarrow 0$ as $t \downarrow 0$ and assume

$$h_c(t)\eta(t) \to \infty \qquad \text{as} \quad t \downarrow 0. \tag{2.10}$$

Then, by setting $\delta(t) = \eta(e^{-t})$, it occurs w.p. 1 that

$$\cap_{\tau_{n-1}<t<\tau_n}(g(t), g(t) + \delta(t)) \supset (g(2\gamma n), g(2\gamma n) + \tfrac{1}{2}\, \delta(2\gamma n))$$

for all large enough n. By this relation and the inequality (2.9), the argument made above shows in fact that under (2.10)

$$\lim_{s\downarrow 0} \sup \frac{1}{s} \cdot \text{meas}\{t: 0 < t < s,\, h_c(t)\sqrt{t} < |B_t| < (h_c(t) + \eta(t))\sqrt{t}\}$$
$$\geqq 1 - e^{-4(q-1)} \qquad\qquad (q = 1/c^2).$$

(The opposite inequality is trivial by what we shall prove in the next section.)

3. Proof of Theorem 1 (II). To complete the proof of Theorem 1 we prove

$$\lim_{s\downarrow 0} \sup X_s \leqq 1 - e^{-4(q-1)} \qquad \text{a.s.} \tag{3.1}$$

Given real numbers $b > 2$, $L > 0$ and a positive integer n, let $\sigma_0 < \pi_0 < \sigma_1 < \pi_1 \cdots$ be the successive passage times to b and to 1, alternately, after τ_{n-1}; i.e.,

$$\sigma_0 = \inf\{t > \tau_{n-1} : N(t) = b\}$$
$$\pi_0 = \inf\{t > \sigma_0 : N(t) = 1\}$$
$$\sigma_1 = \inf\{t > \pi_0 : N(t) = b\}$$

etc., and let

$$Y_i = \text{meas}\{t : \sigma_i < t < \pi_i, N(t) > b\} \qquad i = 0, 1, 2 \cdots,$$
$$\nu = \min\{i \geqq 0 : \sigma_{i+1} - \pi_i > L\}.$$

Finally let

$$Z'_n(b) = \begin{cases} \sum_{i=0}^{\nu} Y_i & \text{if } \sigma_0 < \tau_n \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see

$$U_n \leqq 1 - \exp[-Z'_n(b)] + e^{-L} \quad \text{if} \quad b < g(\tau_{n-1}). \tag{3.2}$$

Clearly $Y_0, Y_1, \cdots$ are i.i.d. random variables, which are independent of ν.

In the Appendix we shall prove the following estimates (Lemmas A.3 and A.4):

$$P[Y_0 > t] \leqq C_1 e^{b/2} \exp(-tb^2/8) \quad \text{for} \quad 0 < t < u \tag{3.3}$$

$$P_1^N[m_b < L] \leqq C_2 e^L \exp\left(-\frac{b^2}{2} + C_3 b\right) \tag{3.4}$$

where C_1 depends on d and u only; C_2 and C_3 depends on d only. By (3.3) and by Lemma 3.1 below, we have

$$P[\textstyle\sum_{i=0}^{k} Y_i > u] \leqq K(b)^{k+1} \exp(-ub^2/8), \tag{3.5}$$

where $K(b) = (1 + ub^2/8)C_1 e^{b/2}$. Since $P[\nu = k] \leqq P[\nu \geqq k] = (P_1^N[m_b < L])^k$, it follows from (3.4) and (3.5) that

$$\begin{aligned} P[\textstyle\sum_{i=0}^{\nu} Y_i > u] &= \textstyle\sum_{k=0}^{\infty} P[\nu = k] P[\sum_{i=0}^{k} Y_i > u] \\ &\leqq \left\{\sum_{k=0}^{\infty} \left[K(b) C_2 e^L \exp\left(-\frac{b^2}{2} + C_3 b\right)\right]^k\right\} K(b) \exp(-ub^2/8) \\ &\leqq \exp\left[-\frac{ub^2}{8}(1 + o(1))\right] \end{aligned}$$

where $o(1) \to 0$ as $b \to \infty$ for each $u > 0$ and $L > 0$. By the equality $P[Z'_n(b) > u] = P_2^N[m_b < m_1] P[\sum_{i=0}^{\nu} Y_i > u]$ and by (2.7), we have that

$$P[Z'_n(b) > u] \leqq \exp\left[-\left(\frac{u}{4} + 1\right)\frac{b^2}{2}(1 + o(1))\right]$$

and then as before that if $u/4 + 1 > q$,

$$\textstyle\sum_{n=1}^{\infty} P[Z'_n(g(\gamma n/2)) > u] < \infty$$

which implies, by applying the Borel-Cantelli lemma,

$$Z'_n(g(\gamma n/2)) < u \ (n \uparrow \infty) \qquad \text{a.s.}$$

Consequently, by (2.2) and (3.2), $\limsup U_n \leqq 1 - e^{-u} + e^{-L}$ a.s. if $u/4 + 1 > q$, so that (3.1) has been established.

Lemma 3.1. *Let $\xi_1, \xi_2, \cdots$ be independent and strictly positive random variables. If for positive constants α, u and A*

$$P[\xi_i > t] \leqq A e^{-\alpha t} \quad \text{for} \quad 0 < t < u, i = 1, 2, \cdots,$$

then for $i = 1, 2, \cdots$,

$$P[\textstyle\sum_{i=1}^{k} \xi_i > t] \leqq (1 + \alpha u)^{k-1} A^k e^{-\alpha t} \qquad 0 < t < u.$$

PROOF. For $k = 2$ the inequality is obtained as follows.

$$P[\xi_1 + \xi_2 > t] = \int_0^\infty P[\xi_1 > t - s]\, d_s P[\xi_2 \leqq s]$$

$$\leqq -A(e^{-\alpha(t-s)} \wedge 1)P[\xi_2 > s] \Big|_{s=0}^{\infty} + A\alpha \int_0^t e^{-\alpha(t-s)} P[\xi_2 > s]\, ds$$

$$= Ae^{-\alpha t} + A^2 \alpha t e^{-\alpha t} \leqq (1 + \alpha t) A^2 e^{-\alpha t}.$$

By induction, the same calculation verifies the inequality for all $k \geqq 1$.

A comment on $\limsup_{s\uparrow\infty} X_s$: To treat it we consider the process

$$\hat{N}(t) = e^{-t/2} |B(e^t)| \qquad t > 0,$$

which is a diffusion process with the same law as $N(t)$. Using this we have

$$X_s = e^{-T} \int_0^T I(\hat{N}(t) - \hat{g}(t)) e^t\, dt + O(1/s) \qquad T = \log s,$$

where $\hat{g}(t) = h(e^t)$. The inequality $\limsup_{s\uparrow\infty} X_s \geqq 1 - e^{-4(q'-1)}$ can be proved in the very same way as in Section 2. For the proof of the opposite inequality the arguments of Section 3 apply, if we notice the following. For given $u > 0$ and $L > 0$ let us define $\hat{\tau}_n$, $\hat{U}_n$ and $\hat{Z}'_n(b)(b > 2)$ similarly as τ_n, U_n and $Z'_n(b)$. Then the inequality $0 < \hat{Z}'_n(b) < u$ implies $\text{meas}\{t : \hat{\tau}_{n-1} - (L + u) < t < \hat{\tau}_n, \hat{N}(t) > b\} < u$, which in turn implies $\hat{U}_n < 1 - e^{-u} + e^{-L}$, provided $b < g(\hat{\tau}_{n-1} - (L + u))$. Therefore, by noting $\hat{\tau}_{n-1} - (L + u) > (1/2)\gamma n\,(n \uparrow \infty)$ a.s., we have the inclusion $\{\hat{U}_n \leqq 1 - e^{-u} + e^{-L} (n \uparrow \infty)\} \supset \{\hat{Z}'_n(g(\gamma n/2)) \leqq u\,(n \uparrow \infty)\}$ a.s.

4. Proof of Theorem 2 (Lower bound).

For the proof of Theorem 2, we shall follow arguments similar to those made in preceding sections. Thus we begin with

$$X_s^* = e^T \int_T^\infty I(g^*(t) - N(t)) e^{-t}\, dt \qquad T = -\log s,$$

where $g^*(t) = 1/h(e^{-t})$. Letting $\tau_0^* < \tau_1^* < \cdots$ be the successive passage times of $N(t)$ to 1 via 1/2 and

$$(4.1) \qquad U_n^* = \sup\{X^*(e^{-T}) : \tau_{n-1}^* < T < \tau_n^*\},$$

we see $\limsup U_n^* = \limsup_{s\downarrow 0} X_s^*$.

Let us write for simplicity $X^* = \limsup_{s\downarrow 0} X_s^*$. In this section we prove the part of Theorem 2 concerning the lower estimate of X^*, i.e., the existence of a function $\theta_*(u)$ asserted in Theorem 2.

First let us prove $X^* \geqq 1 - e^{-q/\lambda_0}$, where $\lambda_0 = \beta_0^2/2$. Let $0 < u < q$ be given. It suffices to show

$$(4.2) \qquad X^* \geqq 1 - e^{-u/\lambda_0}.$$

We consider a new process $N^b(t)$ defined by

$$(4.3) \qquad N^b(t) = \frac{1}{b} \cdot N(tb^2) \qquad t > 0$$

for each $b > 0$. $N^b(t)$ is a diffusion on $x \geqq 0$ and the generator associated with it is

$$(4.4) \qquad \mathbf{G}^{(b)} = \frac{1}{2}\left(\frac{d^2}{dx^2} + \left(\frac{d-1}{x} - b^2 x\right)\frac{d}{dx}\right)$$

with the same boundary condition (1.2)′. We denote by $P_x^{(b)}$ the probability measure associated with $N^b(t)$ starting at x (for $b > 0$) and define $P_x^{(0)}$ as one corresponding to $\mathbf{G}^{(0)}$.

Then for $0 < a < b$

$$P_a^N[m_b < u] = P_{a/b}^{(b)}[m_1 < u/b^2].$$

Note that $u(t, x) \equiv P_x^{(b)}[m_1 > t]$ is a nonincreasing solution of the following parabolic equation

$$\frac{\partial u}{\partial t} = \mathbf{G}^{(b)} u \qquad (t > 0, 0 < x < 1) \tag{4.5}$$

with boundary conditions

$$\lim_{x \downarrow 0} x^{d-1}(\partial u/\partial x) = 0 \quad \text{and} \quad \lim_{u \uparrow 1} u(t, x) = 0. \tag{4.6}$$

Then by a comparison theorem based on the maximum principle for parabolic equations we see

$$P_x^{(b)}[m_1 > t] \geqq P_x^{(0)}[m_1 > t] \qquad 0 < x < 1 \tag{4.7}$$

(cf. [7], Lemma 4). It is easy to see that the spectrum of the differential operator $\mathbf{G}^{(0)}$ restricted on $0 < x < 1$ with the boundary condition (4.6) is discrete. The first eigenvalue is equal to λ_0, because the function $w(x) = x^{1-d/2} J_{d/2-1}(\beta x)$ solves $\mathbf{G}^{(0)} w + (\beta^2/2) w = 0 (x > 0)$ and satisfies the first condition in (4.6). Now from the eigenfunction expansion of solutions of (4.5-6) we deduce $\lim_{t \to \infty} P_x[m_1 > t] e^{\lambda_0 t} > 0$ $(0 < x < 1)$. This combined with (4.7) implies that

$$P_{b/2}^N[m_b > u] \geqq C \exp(-\lambda_0 u / b^2) \tag{4.8}$$

for $b > 0$ small enough. The deduction of (4.2) from (4.8) is the same as that of (2.3) from (2.9).

What we have in addition to show in this section is that if $d = 2$, $\theta_*(u)$ (in (5)) can be chosen to satisfy (4). The rest of this section is devoted to its proof.

Let $d = 2$. Fix an integer $n > 0$ and two numbers $0 < a < b < 1/2$ for a moment and set

$$e(0) = \inf\{t > \tau_{n-1}^* : N(t) = a\}$$

and for $k = 1, 2, \cdots$, inductively,

$$f(k) = \inf\{t > e(k-1) : N(t) = b\}$$

$$e(k) = \inf\{t > f(k) : N(t) = a\}.$$

And then set for $k = 1, 2, \cdots$

$$Q_k = f(k) - e(k-1), \qquad R_k = e(k) - f(k)$$

and

$$\zeta = \min\{k \geqq 0 : e(k) > \tau_n^*\}.$$

Finally set

$$G = \textstyle\sum_{i=1}^{\zeta} Q_i, \qquad H = \sum_{i=1}^{\zeta-1} R_i$$

($\sum_{i=1}^{k}$ is interpreted as zero if $k \leqq 0$). Then $U_n^* \geqq \exp(e(0)) \int_{e(0)+H}^{e(0)+G+H} e^{-t}\, dt$ if $g^*(\tau_n^*) > b$, i.e.,

$$U_n^* \geqq e^{-H}(1 - e^{-G}) \qquad \text{if} \quad g^*(\tau_n^*) > b. \tag{4.9}$$

Since $P_b^N[m_a < u]$ lies between $P_b^N[m_a < u \mid m_a < m_1]$ and $P_b^N[m_a < u \mid m_1 < m_a]$ and the latter is less than $P_1^N[m_a < u] \leqq P_b^N[m_a < u]$, we see

$$P_b^N[m_a < u] \leqq P_b^N[m_a < u \mid m_a < m_1],$$

and then for $i = 1, 2, \cdots, k-1$,

$$
\begin{aligned}
P[R_i < u \mid \zeta = k] &= P_b^N[m_a < u \mid m_a < m_1] \\
&\geqq P_b^N[m_a < u] \\
&= P_{b/a}^{(a)}[m_1 < u/a^2] \\
&\geqq P_{b/a}^{(0)}[m_1 < u/a^2],
\end{aligned} \tag{4.10}
$$

where the first equality follows from the fact that the excursions $N(t): f(j) \leqq t < e(j)$ $(j = 1, 2 \cdots)$ are independent and ζ depends on them only.

In the following we let

$$b = 2a.$$

Let $\{\xi_i\}_{i=1}^{\infty}$ and $\{\eta_i\}_{i=1}^{\infty}$ be two sequences of i.i.d. random variables, whose common distributions are

$$P[\xi_i < u] = P_1^{(0)}[m_2 < u] \qquad \text{and} \qquad P[\eta_i < u] = P_2^{(0)}[m_1 < u].$$

It is not difficult to see that the random variables $Q_1, \cdots, Q_k; R_1, \cdots, R_{k-1}$ conditioned on $\zeta = k$ are independent. By noting this it follows from (4.10) and the inequality $P[Q_1 > u \mid \zeta = k] \geqq P_1^{(0)}[m_2 > u/a^2]$ that

$$
\begin{aligned}
J_a &\equiv P[e^{-H}(1 - e^{-G}) > e^{-1}(1 - e^{-u})] \\
&\geqq P[H < 1, G > u] \\
&\geqq P[\zeta = k] P[\textstyle\sum_{i=1}^{k-1} R_i < 1, \sum_{i=1}^{k} Q_i > u \mid \zeta = k] \\
&\geqq P[\zeta = k] P[\textstyle\sum_{i=1}^{k-1} \eta_i < 1/a^2] P[\sum_{i=1}^{k} \xi_i > u/a^2]
\end{aligned}
$$

for each $k > 1$. To estimate the last factor above we set

$$\rho_1(x) = \sup_{\alpha>0}\{x\alpha - \log E[e^{\alpha\xi_1}]\}.$$

Then, fixing $y > 0$, we have

$$
\begin{aligned}
&\lim_{a\downarrow 0} a^2 \log P[\textstyle\sum_{i=1}^{k} \xi_i > u/a^2] \,|_{k=[y/a^2]} \\
&\quad = \lim_{k\uparrow\infty} \frac{y}{k} \log P\left[\textstyle\sum_{i=1}^{k} \xi_i > k \cdot \frac{u}{y}\right] \\
&\quad = -y\rho_1(u/y)
\end{aligned}
$$

(see Theorem 1 of [1]). Similarly

$$\lim_{a\downarrow 0} a^2 \log P[\textstyle\sum_{i=1}^{k-1} \eta_i < 1/a^2] \,|_{k=[y/a^2]} = -y\rho_2(1/y)$$

where

$$\rho_2(x) = \sup_{\alpha<0}\{x\alpha - \log E[e^{\alpha\eta_1}]\}.$$

The condition $d = 2$ is used for the estimate of $P[\zeta = k]$ as is done below. Since $s(x) = \int_1^x z^{-1} \exp(z^2/2)\, dz = \log x + \text{const} + o(1)$ (as $x \downarrow 0$), $P_{2a}^N[m_a < m_1] \sim (\log 2a)/\log a \sim 1$ $(a \downarrow 0)$. From the identity

$$P[\zeta = k] = P_{1/2}^N[m_a < m_1] P_b^N[m_1 < m_a] (P_b^N[m_a < m_1])^{k-1}$$

it follows that

$$\lim_{a\downarrow 0} a^2 \log P[\zeta = k] \,|_{k=[y/a^2]} = 0.$$

Consequently for all $y > 0$

$$\lim_{s\downarrow 0} \inf a^2 \log J_a \geqq -y[\rho_1(u/y) + \rho_2(1/y)].$$

Let $M(\alpha) = \log E[e^{\alpha\xi_1}]$. Since $0 < M'(0) = E[\xi_1] < \infty$ and $\rho_1(M'(0)) = 0$, we have, by taking

$y = u/M'(0)$ in the above,

$$J_a \geqq \exp[-D(u)(1+o(1))/a^2] \qquad (a \downarrow 0)$$

where

$$D(u) = \frac{u}{M'(0)} \cdot \rho_2\left(\frac{M'(0)}{u}\right).$$

This combined with (4.9) and $a = b/2$ shows as before that

$$X^* \geqq e^{-1}(1 - \exp[-D^{-1}(q/4)]).$$

Since $\lim_{x\uparrow\infty} \rho_2(x) = 0$, we see $\lim_{u\downarrow 0} D^{-1}(u)/u = \infty$. To construct a function $\theta_*(u)$, choose $\varepsilon > 0$ so that $e^{-1}(1 - e^{-3x}) \geqq 1 - e^{-x}$ for $0 < x < \varepsilon$, and then $\delta > 0$ so that $\delta/\lambda_0 < \varepsilon$ and $D^{-1}(u/4) \geqq 4u/\lambda_0$ for $0 < u < \delta$. Clearly there exists a continuous increasing function $g(u)$ on $[0, \delta]$ such that

$$g(\delta) = \frac{3\delta}{\lambda_0}, \qquad \lim_{u\downarrow 0}\frac{g(u)}{u} = \infty \qquad \text{and} \qquad D^{-1}(u/4) \geqq g(u) \geqq \frac{3u}{\lambda_0}.$$

Since $X^* \geqq 1 - e^{-q/\lambda_0}$(a.s.) as has been shown, if we set

$$\theta_*(u) = \frac{g(u)}{3} \quad \text{if} \quad 0 \leqq u \leqq \delta \qquad \text{and} \qquad = \frac{u}{\lambda_0} \quad \text{if} \quad u > \delta,$$

$\theta_*(u)$ fulfills all the requirements in Theorem 2.

5. Proof of Theorem 2 (Upper bound I). In this section we prove that if $d \geqq 3$ there exists a constant $\lambda > 0$ such that

(5.1) $$X^* \leqq 1 - e^{-q/\lambda}.$$

The proof of the remaining part of Theorem 2 will be given in the next section.

The argument is very similar to that of Section 3. For $0 < a < ¼$, $L > 0$ and an integer $n > 0$, let, as in Section 3,

$$\sigma_0^* = \inf\{t > \tau_{n-1}^* : N(t) = a\}$$

$$\pi_0^* = \inf\{t > \sigma_0^* : N(t) = 1\}$$

$$Y_0^* = \text{meas}\{t : \sigma_0^* < t < \pi_0^*, N(t) < a\}$$

and similarly define Y_i^* for $i \geqq 1$ and ν^* (τ_i^* is defined in Section 4). Then

(5.2) $$X^* \leqq 1 - \exp[-\textstyle\sum_{i=1}^{\nu^*} Y_i^*] + e^{-L}.$$

Let us compute an upper bound of $\phi(\alpha) \equiv E[e^{\alpha Y_0^*}]$. For this purpose we let $b = 2a$ and make use of the notations G, ζ and $P_x^{(b)}$ defined in the previous section. Clearly for $\alpha > 0$,

$$\phi(\alpha) \leqq E[e^{\alpha G} \mid G > 0]$$
$$= \textstyle\sum_{k=1}^{\infty} P[\zeta = k \mid G > 0](E_a^N[e^{\alpha m_b}])^k.$$

Since $d \geqq 3$, we have $\lim_{a\downarrow 0} P_{2a}^N[m_a < 1] = (½)^{d-2}$ and therefore with $(½)^{d-2} < \mu < 1$ (taken arbitrarily)

$$P[\zeta = k \mid G > 0] \leqq (P_b^N[m_a < m_1])^{k-1}$$
$$\leqq \text{const.}\ \mu^{k-1} \qquad (\text{for all } a \text{ and } k).$$

Let $\psi(\alpha) = E_{1/2}^{(0)}[e^{\alpha m_1}]$ which is finite for $\alpha < \lambda_0$. Then, noting

$$E_a^N[\exp(\alpha m_b/a^2)] = E_{1/2}^{(b)}[e^{4\alpha m_1}] \to \psi(4\alpha) \qquad \text{as} \quad a \downarrow 0$$

and taking $\lambda > 0$ so small that

$$\mu\psi(4\lambda) < 1,$$

we obtain for sufficiently small a

$$\phi(\lambda/a^2) \leqq \text{const.}\frac{\psi(4\lambda)}{1-\mu\psi(4\lambda)}.$$

Let $D_1(\lambda)$ denote the quantity in the right side above and take a positive δ so small that $\delta D_1(\lambda) < 1$. Since $P[\nu^* = k] \leqq (P^N_{1/2}[m_a < L])^k$ and $\lim_{a\downarrow 0} P^N_{1/2}[m_a < L] = 0$, we have

$$\begin{aligned} P[\textstyle\sum_{i=0}^{\nu^*} Y_i^* > u] &\leqq e^{-\alpha u} E[\exp(\alpha \textstyle\sum_{i=0}^{\nu^*} Y_i^*)] \qquad (\alpha = \lambda/a^2) \\ &\leqq \textstyle\sum_{k=0}^{\infty} \exp(-\lambda u/a^2)\phi(\lambda/a^2)^{k+1} P[\nu^* = k] \\ &\leqq \exp(-\lambda u/a^2)\frac{D_1(\lambda)}{1-\delta D_1(\lambda)} \qquad (a \downarrow 0). \end{aligned}$$

As before this together with (5.2) implies (5.1) by the use of the Borel-Cantelli lemma.

6. Proof of Theorem 2 (Upper bound II). To complete the proof of Theorem 2, we must find a continuous function $\theta^*(u)$, when $d = 1$ or 2, such that $\theta^*(0) = 0$

(6.1) $$X^* \leqq 1 - e^{-\theta^*(q)}$$

(6.2) and $$\lim_{u\uparrow\infty} \sup \theta^*(u)/u < \infty.$$

Since the assertion for $d = 2$ follows from that for $d = 1$, we assume $d = 1$. The proof is a refinement of that given in the previous section. We use the notations g^*, τ_n^*, U_n^* etc. introduced in Section 4.

Given $0 < a < b < 1/2$, $L > 0$ and $n > 0$, Let $e(k)$, $f(k)$, Q_k and R_k be the same as in Section 4 and let

$$\nu_1 = \min\{j: \textstyle\sum_{i=1}^{j} R_i > L\}; \qquad S_1 = \textstyle\sum_{i=1}^{\nu_1+1} Q_i,$$

$$\nu_2 = \min\{j: \textstyle\sum_{i=\nu_1+2}^{j} R_i > L\}; \qquad S_2 = \textstyle\sum_{i=\nu_1+2}^{\nu_2+1} Q_i$$

and in general

$$\nu_{k+1} = \min\{j: \textstyle\sum_{i=\nu_k+2}^{j} R_i > L\}; \qquad S_{k+1} = \textstyle\sum_{i=\nu_k+2}^{\nu_{k+1}+1} Q_i$$

In the definition of S_k the additional term Q_{ν_k+1} is added so that the sequence $\{S_k\}_{k=1}$ becomes independent of the following random variable

$$\zeta^* = \min\{k: N(t) = 1 \text{ for some } t \in (f(\nu_k+1), e(\nu_k+2))\}.$$

Now let $$Z_n^*(a) = \max_{1\leqq i\leqq \zeta^*}(S_i + S_{i+1}).$$

Then from the inequality

$$U_n^* \leqq \max_j \min_k\{1 - \exp[-\textstyle\sum_{i=j}^{k} Q_i] + \exp[-\textstyle\sum_{i=j}^{k} R_i]\}, \qquad \text{if } g^*(\tau_{n-1}^*) \leqq a,$$

where $\min_k$ and $\max_j$ are taken under restrictions $k \geqq j$ and $1 \leqq j \leqq \nu_{\zeta^*} + 1$, respectively, it follows that

(6.3) $$U_n^* \leqq 1 - \exp[-Z_n^*(a)] + e^{-L}, \qquad \text{if } g^*(\tau_{n-1}^*) \leqq a.$$

Clearly

$$\begin{aligned} P[Z_n^*(a) > u] &= \textstyle\sum_{k=1}^{\infty} P[\zeta^* = k] P[\max_{1\leqq i\leqq k}(S_i + S_{i+1}) > u] \\ &\leqq \textstyle\sum_{k=1}^{\infty} P[\zeta^* = k] \sum_{i=1}^{k} P[S_i + S_{i+1} > u] \\ &= P[S_1 + S_2 > u] E[\zeta^*]. \end{aligned}$$

Now we let $b = 2a$. Since $d = 1$, we have $P_b^N[m_a > m_1] = [s(a) - s(b)]/s(a) \sim a/(-s(0))$ and therefore as $a \downarrow 0$,

$$E[\zeta^*] = P_b^N[m_1 > m_a]/P_b^N[m_a > m_1] \sim \frac{-s(0)}{a}.$$

On the other hand, by Lemma 6.1 below, there exists a positive constant κ (depending on L/u only) such that

$$P\left[S_1 > \frac{u}{2}\right] = O(\exp(-\kappa u/a^2)) \quad \text{as} \quad a \downarrow 0 \tag{6.4}$$

which implies

$$P[S_1 + S_2 > u] \leqq 2P\left[S_1 < \frac{u}{2}\right] = O(\exp(-\kappa u/a^2)).$$

Thus $P[Z_n^*(a) > u] \leqq$ const. $a^{-1}\exp(-\kappa u/a^2)$, from which together with (6.3) it follows, as before, that

$$X^* \leqq 1 - e^{-u} + e^{-L} \qquad \text{if} \quad q < \kappa u.$$

For each $n = 1, 2, \cdots$, let $\varepsilon_n = 1 - e^{-1/n}$ and choose L_n, $u_n > 0$ so that $\exp(-L_n) + 1 - \exp(-u_n) < \varepsilon_{n+1}$. By denoting by κ_n the corresponding κ's, let $\delta_n = \kappa_n u_n$. Then

$$X^* \leqq \varepsilon_{n+1} \qquad \text{if} \quad q \leqq \delta_n.$$

We can assume that δ_n is decreasing. Let $\theta^*(u)$ be any continuous function on $[0, \delta_1]$, which is decreasing and takes values $1/n$ at δ_n. Then (6.1) holds for $0 \leqq u \leqq \delta_1$ and $\theta^*(0) = 0$. To obtain the relation (6.2), we let $L = 2u$ in the above. Then κ in (6.4) becomes independent of u and therefore it follows that if $q < \kappa u$, $X^* \leqq 1 - e^{-u} + e^{-2u}$, or equivalently that $X^* \leqq 1 - e^{-q/\kappa} + e^{-2q/\kappa}$. This guarantees that $\theta^*(u)$ can be chosen so that $\lim_{u\uparrow\infty}\theta^*(u)/u = 1/\kappa$. The proof of Theorem 2 is complete if we prove the following lemma.

LEMMA 6.1. *Let $b = 2a$ in the definition of S_1. Then for each $K > 0$, there exist positive constants κ, a_0 and M such that $P[S_1 > u] \leqq M \cdot \exp(-\kappa u/a^2)$ if $0 < a < a_0$ and $L/u < K$ $(0 < u, 0 < L)$.*

PROOF. Let

$$\phi^a(\alpha) = E_{1/2}^{(2a)}[e^{\alpha m_1}] \qquad \text{and} \qquad \psi^a(\alpha) = E_1^{(2a)}[e^{\alpha m_{1/2}}]$$

where $E_x^{(a)}$ is defined in Section 4. Since

$$\begin{aligned} P[\nu_1 = k] \leqq P[\nu_1 \geqq k] &= P[\textstyle\sum_{i=1}^{k-1} R_i \leqq L] \\ &\leqq \exp[\beta L/(2a)^2](\psi^a(-\beta))^{k-1} \qquad \text{for} \quad \beta > 0 \end{aligned}$$

and $E[e^{\alpha Q_1}] = \phi^a((2a)^2\alpha)$, we have for $\alpha > 0$

$$\begin{aligned} P[S_1 > u] &\leqq e^{-\alpha u}E[e^{\alpha S_1}] \\ &= e^{-\alpha u} \textstyle\sum_{k=1}^{\infty} P[\nu_1 = k](E[e^{\alpha Q_1}])^k \qquad (6.5) \\ &\leqq e^{-\alpha u}e^{\beta L/(2a)^2}\phi^a((2a)^2\alpha) \textstyle\sum_{k=1}^{\infty} [\psi^a(-\beta)\phi^a((2a)^2\alpha)]^{k-1}. \end{aligned}$$

Now we set $\alpha = \lambda/(2a)^2$ and for a given $K > 0$ take $\lambda > 0$ and $\beta > 0$ so small that $\lambda > \beta K$ and

$$\phi^a(\lambda)\psi^a(-\beta) < 1 \qquad \text{for} \quad 0 < a < a_0/2 \tag{6.6}$$

with some constant a_0. This is possible, because $\phi^{0\prime}(0) < \infty$, $\psi^{0\prime}(0-) = \infty$ and, as $a \downarrow 0$, $\phi^a(\lambda)$ and $\psi^a(-\beta)$ decreasingly approach $\phi^0(\lambda)$ and $\psi^0(-\beta)$, respectively (for $\lambda > 0, \beta > 0$).

From (6.5) and (6.6) we deduce that if $L < Ku$,

$$P[S_1 > u] \leqq \exp\left[-\frac{(\lambda - \beta K)u}{4a^2}\right] \cdot \frac{\phi^a(\lambda)}{1 - \phi^a(\lambda)\psi^a(-\beta)}.$$

Consequently if we put $\kappa = (\lambda - \beta K)/4$, we obtain the required estimate.

APPENDIX

Here are given several lemmas which are used in the proof of Theorem 1.

LEMMA A.1. *For $a > [(d^2-1)/2d]^{1/2}$, $0 < \delta < 1$, $b = a + \delta$, $c = a + 2\delta$ and $t > 0$*

$$P_b^N[m_a \wedge m_c > t] \geqq C\cdot\exp\left[-\frac{c^2t}{8} - \frac{\pi^2 t}{8\delta^2} - \frac{c\delta}{2}\right]$$

where C depends on d only.

PROOF. Through the change of dependent variable

$$u(t, x) = v(t, x)Q(x), \qquad Q(x) = x^{(1-d)/2}\exp(x^2/4)$$

the equation (1.2) is transformed to

$$\text{(A.1)} \qquad \frac{\partial v}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{1}{8}\cdot\left[2d - (d^2-1)\frac{1}{x^2} - x^2\right]v.$$

Since $u(t, x) = P_x^N[m_a \wedge m_c > t]$ is a unique solution of (1.2), in the domain $t > 0$, $a < x < c$, with $u(0, x) = 1$ and $u(t, a) = u(t, c) = 0$ and since $Q'(x) > 0$ for $x > \sqrt{(2(d-1))}$, *after a simple comparison argument we see that for $a > \sqrt{(2(d-1))}$*

$$P_b^N[m_a \wedge m_c > t] \geqq \frac{Q(b)}{Q(c)}\exp\left[-\frac{t}{8}\left(c^2 + \frac{d^2-1}{a^2} - 2d\right)\right]P[|B_s^{(1)}| < \delta \quad \text{for} \quad 0 < s < t]$$

where $B_t^{(1)}$ is a standard one-dimensional Brownian motion starting at 0. Now the lemma follows from

$$\begin{aligned} &P[|B_s^{(1)}| < \delta \quad \text{for} \quad 0 < s < t] \\ &= \int_{-\delta}^{\delta}\left\{\sum_{k=0}^{\infty}\frac{1}{\delta}\cdot\exp\left[-\left(\frac{\pi(2k+1)}{2\delta}\right)^2\cdot\frac{t}{2}\right]\cos\frac{\pi(2k+1)y}{2\delta}\right\}dy \\ &\geqq \frac{4}{\pi}\cdot\left(\exp\left[-\left(\frac{\pi}{2\delta}\right)^2\cdot\frac{t}{2}\right] - \frac{1}{3}\exp\left[-\left(\frac{3\pi}{2\delta}\right)^2\cdot\frac{t}{2}\right]\right) \\ &\geqq \frac{8}{3\pi}\cdot\exp\left(-\frac{\pi^2}{8\delta^2}t\right). \end{aligned}$$

LEMMA A.2. *For $b > a > 0$ and $t > 0$*

$$P_b^N[m_a > t] \leqq C\exp\left[-\frac{t}{8}(a^2 - 2d) + \frac{b\delta}{2}\right]$$

where $\delta = b - a$ and C depends on d only.

PROOF. By the same consideration as in the proof of Lemma A.1 we have

$$P_b^N[m_a > t] \leqq \frac{Q(b)}{Q(a)}\cdot\exp\left[-\frac{t}{8}(a^2 - 2d)\right]P[B_s^{(1)} > -\delta \text{ for } 0 < s < t]$$

which implies the required inequality.

Let $S = \text{meas}\{t: 0 < t < m_1, w(t) > b\}$ ($w \in W$). Then the distribution of Y_0 in Section 3 is given by $P[Y_0 > t] = P_b^N[S > t]$.

LEMMA A.3. *For each $\varepsilon > 0$ and $u > 0$ there exists a constant C_1 such that*

$$P_b^N[S > t] \leqq C_1 \exp\left[-\frac{t}{8} b^2 + \varepsilon b\right] \qquad \text{for} \quad b > 2, 0 < t < u.$$

(C_1 depends on u, ε and d only.)

PROOF. Let us take $0 < \delta < 1$ and $a = b - \delta$. Let $e_0 = 0$ and $f_1 < e_1 < f_2 < \cdots$ be successive passage times of $w \in W$ to a and to b, alternately: $f_k = \inf\{t > e_{k-1}: w(t) = a\}$, $e_k = \inf\{t > f_k: w(t) = b\}$ $(k = 1, 2, \cdots)$, and let

$$\zeta = \min\{i \geqq 1;\, e_i > m_1\}.$$

Clearly

$$S \leqq \textstyle\sum_{k=1}^{\zeta} (f_k - e_{k-1}). \tag{A.2}$$

Since $f_k - e_{k-1}$ $(k = 1, 2, \cdots,)$ and ζ are mutually independent and $P_b^N[f_k - e_{k-1} > t] = P_b^N[m_a > t]$, from Lemma 3.1 and Lemma A.2 it follows that

$$P_b^N[\textstyle\sum_{i=1}^{k} (f_i - e_{i-1}) > t] \leqq \left(1 + \frac{a^2}{8} t\right)^{k-1} \left[C \exp\left(\frac{a\delta}{2}\right)\right]^k \exp\left[-\frac{t}{8}(a^2 - 2d)\right].$$

Since $P_b^N[\zeta = k] \leqq (P_b^N[m_b < m_1])^{k-1}$ and for large a

$$P_a^N[m_b < m_1] = \frac{s(a)}{s(b)} \leqq 2\exp[-(a\delta + \delta^2/2)]$$

where $s(x)$ is a canonical scale associated with $N(t)$ (see (2.6)), we obtain

$$\begin{aligned} &P_b^N[\textstyle\sum_{k=1}^{\zeta} (f_k - e_{k-1}) > t] \\ &= \textstyle\sum_{k=1}^{\infty} P_b^N[\zeta = k] P_b^N[\sum_{i=1}^{k} (f_i - e_{i-1}) > t] \\ &\leqq \textstyle\sum_{k=1}^{\infty} \left\{\left(1 + \frac{a^2}{8} t\right) 2C \exp\left(-\frac{a\delta}{2}\right)\right\}^{k-1} C e^{a\delta/2} \exp\left[-\frac{t}{8}(a^2 - 2d)\right] \\ &= \frac{C}{1-\beta} \exp\left[-\frac{t}{8}(a^2 - 2d) + \frac{a\delta}{2}\right] \qquad (a \uparrow \infty) \end{aligned}$$

where β stands for the quantity enclosed by braces in the third line above and is smaller than ½ for large enough a. By (A.2) this completes the proof of the lemma.

LEMMA A.4. *For $L > 0$ and $b > 1$*

$$P_1^N[m_b < L] \leqq C' e^L \exp\left(-\frac{b^2}{2} + Cb\right)$$

where C and C' depend on d only.

PROOF. First let $d = 1$ and set for $\alpha > 0$

$$\phi_\alpha(x) = E_x^N[\exp(-\alpha m_b)] \qquad 0 < x < b.$$

Then, as is well known, $\phi_\alpha(x)$ is a unique solution of

$$\frac{1}{2} \cdot \phi'' - \frac{x}{2} \cdot \phi' - \alpha\phi = 0 \tag{A.3}$$

with $\phi'(0) = 0$ and $\phi(b) = 1$ and expressed as

$$\phi_\alpha(x) = \frac{v(x) + v(-x)}{v(b) + v(-b)} \qquad 0 < x < b$$

where $v(x)$ $(x \geqq -b)$ is an increasing solution of (A.3) with $v(-b) > 0$. When $\alpha = 1$, such a solution is given by

$$v^*(x) = \int_0^\infty \exp\left(xt - \frac{t^2}{2}\right) t\, dt$$

$$= \exp\left(\frac{x^2}{2}\right) \int_{-x}^\infty \exp\left(-\frac{u^2}{2}\right)(x + u)\, du,$$

for which we see

$$v^*(x) \begin{cases} \leqq (1 + x\sqrt{(2\pi)})\exp(x^2/2) \\ \geqq \tfrac{1}{2} (1 + x\sqrt{(2\pi)})\exp(x^2/2). \end{cases}$$

From these it follows that

$$\phi_1(x) \leqq 2 \frac{v^*(x)}{v^*(b)} \leqq 4 \frac{1 + x\sqrt{(2\pi)}}{1 + b\sqrt{(2\pi)}} \exp\left(-\frac{b^2 - x^2}{2}\right).$$

Therefore for $0 < x < b$, we have,

$$P_x^N[m_b < L] \leqq e^L \phi_1(x) \leqq 4e^L \exp\left(-\frac{b^2 - x^2}{2}\right)$$

as desired.

The general case $d > 1$ is reduced to the one-dimensional case as follows. Take a constant $C = C(d) \geqq 1$ so large that $(\tfrac{1}{2})$ $((d - 1)/x - x) < -(\tfrac{1}{2})$ $(x - C)$ for $x > C$, and consider a diffusion process on $x \geqq C$ associated with the generator $\mathbf{G}$: $\mathbf{G}u = (\tfrac{1}{2})u'' - (\tfrac{1}{2}) \cdot (x - C)u'$ with $u'(C) = 0$. Then by the usual comparison argument we have

$$P_x^N[m_b < L] \leqq P_{x-C}^{N^*}[m_{b-C} < L] \qquad (C \leqq x < b)$$

where $P_x^{N^*}$ is P_x^N defined for one dimensional Brownian motion. Thus $P_1^N[m_b < L] \leqq P_C^N[m_b < L] \leqq$ const. $e^L \exp[-((b - C)^2 - C^2)/2]$.

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