The Annals of Probability
1982, Vol. 10, No. 1, 199–206

ADDITIVE AMARTS[1]

BY G. A. EDGAR

The Ohio State University

Multi-parameter martingales and amarts can be studied using methods developed for amarts defined on a directed set by A. Millet and L. Sucheston. To study an amart indexed by $\mathbb{N} \times \mathbb{N}$, we use an associated process indexed by the "lower layers" of $\mathbb{N} \times \mathbb{N}$. J. B. Walsh's convergence theorem for two-parameter strong martingales is recovered as a special case. Vector-valued versions of some of the results are also stated.

The work in this paper began as an attempt to understand some of the recent work on multi-parameter martingales (e.g. [13]) from the point of view of amarts. It was motivated partly by the modern trend to generalize everything. But most of the motivation was the experience from the one-parameter case: often results are equally easy in the amart setting compared to the martingale setting, and occasionally the situation is even clearer for amarts.

There have already been a few papers that treat multi-parameter martingales using amarts: [11] [7] [12]. The approach used there seems to be different from the one used in the present paper.

In this paper, an amart version of Walsh's discrete-parameter convergence theorem [13, Theorem 3.5] is proved. The present paper closes with the version of this convergence theorem for processes with values in a Banach space.

In this paper the "stopping domain" is identified as a special case of the "stopping time" with values in a directed set. This makes it possible to apply the theory of amarts indexed by a directed set, as developed by Astbury [1], Edgar-Sucheston [5], Millet-Sucheston [10], and others.

No matter what index set is used, I consider a *martingale* to be a process that satisfies

(1) $$E[X_t \mid \mathscr{F}_s] = X_s \quad \text{for} \quad s \le t.$$

Again, an *amart* is a process such that

(2) $$E[X_\tau] \quad \text{converges}$$

as τ runs through the directed set of all simple stopping times.

Even though the parameter set $I = \mathbb{N} \times \mathbb{N}$ was the original motivation for this paper, mention of that case has been postponed to later in the paper, and more general versions of the main results have been placed first.

This paper deals with the "discrete-parameter" case. Investigation of "continuous-parameter" processes from this point of view is left for the future.

1. General directed set.

We begin with a few basic definitions concerning amarts on a directed set. The main references are [5], [1], [9], [10].

Let $(\Omega, \mathscr{F}, P)$ be a fixed probability space. Let J be a directed set. A family $(\mathscr{F}_t)_{t\in J}$ of σ-algebras contained in $\mathscr{F}$ satisfying $\mathscr{F}_s \subseteq \mathscr{F}_t$ if $s \le t$ is called a *stochastic basis*. A family

Received May 1980; revised December 1980.

[1] Research supported in part by N.S.F. grant MCS 77-04049 A01.

AMS (1980) *subject classification*. 60G48.

Key words and phrases. Amart, additive amart, strong martingale, stopping time, stopping domain.

$(X_t)_{t\in J}$ of random variables is called a *stochastic process*. The process (X_t) is said to be *adapted* to the basis $(\mathscr{F}_t)$ iff X_t is $\mathscr{F}_t$-measurable for all $t \in J$. A *simple stopping time* of the basis $(\mathscr{F}_t)$ is a function $\tau:\Omega \to J$ with finitely many values such that $\{\omega:\tau(\omega) = t\} \in \mathscr{F}_t$ for all $t \in J$. The set of all simple stopping times of $(\mathscr{F}_t)$ will be denoted

$$\Gamma(J; (\mathscr{F}_t)_{t\in J});$$

this will be shortened to $\Gamma(J)$ or Γ when the missing symbols are to be deduced by the reader. A stochastic process (X_t) is an *amart* for $(\mathscr{F}_t)$ iff

(i) (X_t) is adapted to $(\mathscr{F}_t)$,
(ii) $E|X_t| < \infty$ for all $t \in J$,
(iii) the net $(E[X_\tau])_{\tau\in\Gamma}$ of real numbers converges.

It is known [5, page 199] that the process (X_t) is a martingale (that is, conditions (i) and (ii) hold and $E[X_t|\mathscr{F}_s] = X_s$ for $s \leq t$) if and only if conditions (i) and (ii) hold and the net $(E[X_\tau])_{\tau\in J}$ is constant. It has therefore been suggested that a synonym for "martingale" is "exact amart."

We begin with a technical but useful result. Millet and Sucheston [10] have proved a similar result concerning martingales. Note that uniform integrability is not postulated.

1. THEOREM. *Let J be a countable directed set, let $(\mathscr{F}_t)_{t\in J}$ be a stochastic basis, and let $J_o \subseteq J$ be a subset cofinal in J. For each $u \in J_o$, let $\mathscr{C}_u$ be a class of stochastic processes indexed by $J(u)$, where $\{t \in J_o: t \geq u\} \subseteq J(u) \subseteq \{t \in J: t \geq u\}$. Suppose*:

(*i*) *If $(X_t) \in \mathscr{C}_u$, then (X_t) is an L^1-bounded amart for $(\mathscr{F}_t)_{t\in J(u)}$.*
(*ii*) *If $(X_t)_{t\in J(u)} \in \mathscr{C}_u$, $u_1 \in J_o$, $u_1 \geq u$, and $\Lambda \in \mathscr{F}_{u_1}$, then $(1_\Lambda(X_t - X_{u_1}))_{t\in J(u_1)} \in \mathscr{C}_{u_1}$.*
(*iii*) *There is constant C so that for all $u \in J_o$, all $(X_t) \in \mathscr{C}_u$, and all $a > 0$,*

$$aP\{\sup_{t\in J_o\cap J(u)}|X_t| > a\} \leq C \sup_{\tau\in\Gamma(J(u))} E|X_\tau|.$$

Then for all $(X_t) \in \mathscr{C}_u$, the net $(X_t)_{t\in J_o\cap J(u)}$ converges a.s.

Before proving this, we separate out a lemma. Compare [13, Lemma 3.6] and [10, Theorem 2.5].

2. LEMMA. (*a*) *Let J be a directed set, $(\mathscr{F}_t)$ a stochastic basis, and (X_t) an L^1-bounded amart. Let $\varepsilon, \eta > 0$. Then there exist $t_o \in J$ and $\Lambda \in \mathscr{F}_{t_o}$ such that $P(\Lambda) > 1 - \eta$ and*

$$\int_\Lambda |X_\tau - X_{t_o}|\, dP < \varepsilon$$

for all simple stopping times $\tau \geq t_o$. (*b*) *If, in addition, $X_t \to 0$ in probability, then there exist $t_o \in J$ and $\Lambda \in \mathscr{F}_{t_o}$ such that $P(\Lambda) > 1 - \eta$ and $\int_\Lambda |X_\tau|\, dP < \varepsilon$ for all $\tau \geq t_o$.*

PROOF OF LEMMA. The process (X_t) converges in probability [5, Theorem 2.10], say to X_∞. (In case (b), $X_\infty = 0$.) Then $\tilde{X}_t = X_t - E[X_\infty|\mathscr{F}_t]$ is an L^1-bounded amart, converging to 0 in probability. By the "lattice property" for amarts [1, Corollary 2.1], the positive and negative parts $\tilde{X}_t^+$, $\tilde{X}_t^-$ are also L^1-bounded amarts, still converging to 0 in probability. Apply the amart "Riesz decomposition" [1, Theorem 2.1] to each one separately:

$$\tilde{X}_t^+ = Y_t^{(+)} + Z_t^{(+)},$$

$$\tilde{X}_t^- = Y_t^{(-)} + Z_t^{(-)},$$

and $(Y_t^{(+)})$, $(Y_t^{(-)})$ are martingales, $(Z_\tau^{(+)})$ and $(Z_\tau^{(-)})_{\tau\in\Gamma}$ converge to 0 in L^1. An examination of the proof of the Riesz decomposition shows that $Y_t^{(+)} \geq 0$, $Y_t^{(-)} \geq 0$. Also, $Y_t^{(+)}$ and $Y_t^{(-)}$ converge to 0 in probability. Thus, we can write

$$X_t = Y_t^{(+)} - Y_t^{(-)} + Z_t,$$

where $Z_t = Z_t^{(+)} - Z_t^{(-)} + E[X_\infty \mid \mathscr{F}_t]$, so that $Y_t^{(+)}$ and $Y_t^{(-)}$ are martingales converging to 0, and $(Z_\tau)_{\tau\in\Gamma}$ converges to X_∞ in L^1-norm.

Now choose $t_o \in J$ so that

(i) $E|Z_\tau - Z_{t_o}| < \dfrac{\varepsilon}{5}$ for all $\tau \geq t_o$, and

(ii) $P\left\{Y_{t_o}^{(+)} < \dfrac{\varepsilon}{5},\ Y_{t_o}^{(-)} < \dfrac{\varepsilon}{5}\right\} > 1 - \eta.$

Let $\Lambda = \{Y_{t_o}^{(+)} < \varepsilon/5,\ Y_{t_o}^{(-)} < \varepsilon/5\} \in \mathscr{F}_{t_o}$. Now for $\tau \in \Gamma$, $\tau \geq t_o$, we have (by the martingale property)

$$0 \leq \int_\Lambda Y_\tau^{(\pm)}\, dP = \int_\Lambda Y_{t_o}^{(\pm)}\, dP < \frac{\varepsilon}{5}.$$

So

$$\int_\Lambda |X_\tau - X_{t_o}|\, dP$$

$$\leq \int_\Lambda Y_\tau^{(+)}\, dP + \int_\Lambda Y_{t_o}^{(+)}\, dP + \int_\Lambda Y_\tau^{(-)}\, dP + \int_\Lambda Y_{t_o}^{(-)}\, dP + \int |Z_\tau - Z_{t_o}|\, dP < \varepsilon.$$

The argument for (b) is similar. □

PROOF OF THEOREM. Let $(X_t)_{t\in J}$ belong to the class $\mathscr{C}_u$. Then (X_t) is an L^1-bounded amart, and hence [5, Theorem 2.10] converges in probability, say to X_∞. Choose $t_1 < t_2 < \cdots$ in J_o and $\Lambda_k \in \mathscr{F}_{t_k}$ recursively so that

(i) $X_{t_k} \to X_\infty$, a.s.

(ii) $P[\Lambda_k] > 1 - 2^{-k}$,

(iii) $\displaystyle\int_{\Lambda_k} |X_\tau - X_{t_k}|\, dP < 4^{-k}$ for $\tau \geq t_k$.

This is possible by the Lemma and the fact that J_o is cofinal in J. Then for each k, the process $(1_{\Lambda_k}(X_t - X_{t_k}))_{t\in J(t_k)}$ belongs to $\mathscr{C}_{t_k}$. By the maximal inequality (iii),

$$P\{\sup_{t\geq t_k, t\in J_o} |X_t - X_{t_k}| > 2^{-k}\} \leq \frac{C}{2^{-k}} \sup_{\tau\geq t_k} E\,|1_{\Lambda_k}(X_\tau - X_{t_k})| + P(\Lambda_k^c)$$

$$\leq \frac{C}{2^{-k}} 4^{-k} + 2^{-k} = (C+1)2^{-k}.$$

So by the Borel-Cantelli Lemma, $(X_t)_{t\in J_o}$ converges a.s. □

A similar theorem for uncountable directed sets can be obtained by requiring essential convergence rather than pointwise convergence.

2. Lower layers. We now change to a slightly different setting. Let I be a directed set with least element 0, and *locally finite* in the sense that all intervals $[0, t] = \{s \in I : 0 \leq s \leq t\}$ are finite. A subset $S \subseteq I$ is a (*lower*) *layer* iff from $s \leq t$ and $t \in S$ it follows that $s \in S$. We will write $\mathscr{L}(I)$ or $\mathscr{L}$ for the set of all nonvoid finite layers of I. Clearly the union of two layers is a layer, so $\mathscr{L}(I)$ is a directed set when ordered by inclusion. The directed set I is canonically identified with the cofinal subset of $\mathscr{L}$ consisting of the *lower intervals* $[0, t]$.

If $(\mathscr{F}_t)_{t\in I}$ is a stochastic basis indexed by I, the *associated stochastic basis* $(\mathscr{H}_S)_{S\in\mathscr{L}(I)}$ indexed by the layers is defined by

(3) $$\mathscr{H}_S = \bigvee_{s\in S} \mathscr{F}_s,$$

the least σ-algebra containing $\bigcup_{s\in S} \mathscr{F}_s$. If $(X_t)_{t\in I}$ is a stochastic process indexed by I that has the form

$$X_t = \sum_{s\leq t} Y_s$$

for some *difference process* $(Y_s)_{s\in I}$, the *associated process* $(F_S)_{S\in\mathscr{L}(I)}$ indexed by the layers is defined by

(4) $$F_S = \sum_{s\in S} Y_s.$$

The associated process has these two properties:

(5) $$F_{[0,t]} = X_t$$

(6) $$F_{S_1\cup S_2} + F_{S_1\cap S_2} = F_{S_1} + F_{S_2}.$$

A process indexed by $\mathscr{L}(I)$, or a cofinal subset of $\mathscr{L}(I)$, that satisfies (6) will be called an *additive* process. If (F_s) is additive, then (5) defines the associated process (X_t). Of course, our definitions have been made so that if (Y_t) is adapted to $(\mathscr{F}_t)$, then (F_S) defined by (4) is adapted to $(\mathscr{H}_S)$ defined by (3).

An *additive amart* is a process $(F_S)_{S\in\mathscr{L}(I)}$, indexed by the layers, that is additive and is an amart (formula (2) above; the directed set is $\mathscr{L}(I)$). Similarly, an *additive martingale* is defined by (6) and (1) for index set $\mathscr{L}(I)$.

The following is a special case of Theorem 1.2 of [9] (take $J = \mathscr{L}(I)$, $p = \infty$).

3. THEOREM. *Let $(X_t)_{t\in I}$ and $(F_S)_{S\in\mathscr{L}}$ be associated processes.*

(*i*) *If* $\sup_{t\in I} E|X_t| < \infty$ *and* $\sup_{\tau\in\Gamma(\mathscr{L})} |E[F_\tau]| < \infty$, *then* $\sup_{\tau\in\Gamma(\mathscr{L})} E|F_\tau| < \infty$.

(*ii*) *If $(F_S)_{S\in\mathscr{L}}$ is an L^1-bounded amart, then $(|F_S|)_{S\in\mathscr{L}}$ is also an L^1-bounded amart.*

If $(F_S)_{S\in\mathscr{L}}$ is an L^1-bounded additive amart, the process $(|F_S|)_{S\in\mathscr{L}}$ need not be additive. In other words, if (X_t) and (F_S) are associated process, then $(|X_t|)$ and $(|F_S|)$ are usually not associated.

We now consider the special case $I = \mathbb{N} \times \mathbb{N}$ with the usual ordering: $(i, j) \leq (m, n)$ if and only if $i \leq m$ and $j \leq n$. In that case every process (X_t) can be obtained by adding a difference process,

$$X_t = \sum_{s\leq t} Y_s;$$

namely, define

(7) $$Y_{mn} = \begin{cases} X_{mn} - X_{m-1,n} - X_{m,n-1} + X_{m-1,n-1}, & m > 0, \quad n > 0 \\ X_{on} - X_{o,n-1}, & m = 0, \quad n > 0 \\ X_{mo} - X_{m-1,o}, & m > 0, \quad n = 0 \\ X_{oo}, & m = 0, \quad n = 0. \end{cases}$$

In the case $I = \mathbb{N} \times \mathbb{N}$, an additive martingale is the same as a "strong martingale" in the terminology of [13]. If $(\mathscr{F}_{m,n})_{(m,n)\in I}$ is a stochastic basis, we write $\mathscr{F}_{m,\infty} = \bigvee_{n\in\mathbb{N}} \mathscr{F}_{m,n}$ and similarly for $\mathscr{F}_{\infty,n}$.

4. PROPOSITION. *Let $I = \mathbb{N} \times \mathbb{N}$, and let $(\mathscr{F}_{m,n})_{(m,n)\in I}$ be a stochastic basis. Suppose $(X_t)_{t\in I}$ is an adapted, integrable process, and $(F_S)_{S\in\mathscr{L}}$ the associated process. Then the following are equivalent.*

(a) (F_S) *is an (additive) martingale*

(b) *The net* $(E[F_\tau])_{\tau\in\Gamma(\mathscr{L})}$ *is constant*

(c) $E[X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1} + X_{i,j} | \mathscr{F}_{i,\infty} \vee \mathscr{F}_{\infty j}] = 0$

$E[X_{i+1,o} - X_{i,o} | \mathscr{F}_{i,\infty}] = 0$

$E[X_{o,j+1} - X_{o,j} | \mathscr{F}_{\infty,j}] = 0$

for all $i, j \in \mathbb{N}$.

PROOF. (a) ⇔ (b) holds for any directed set in place of $\mathscr{L}(\mathbb{N} \times \mathbb{N})$.

(b) ⇒ (c). I will prove the first equation; the other two are similar. Let $(i, j) \in I$ be given. For $m \geq i$ and $n \geq j$, consider the layer $S = \{t \in I: t \leq (i, n) \text{ or } t \leq (m, j)\}$. Then $\mathscr{H}_S = \mathscr{F}_{i,n} \vee \mathscr{F}_{m,j}$. Let $\Lambda \in \mathscr{H}_S$. Define $\tau: \Omega \to \mathscr{L}$ by

$$\tau(\omega) = \begin{cases} S, & \text{if } \omega \notin \Lambda \\ S \cup \{(i+1, j+1)\}, & \text{if } \omega \in \Lambda. \end{cases}$$

Then $\tau \in \Gamma(\mathscr{L})$. So by (b), $E[F_\tau] = E[F_S]$. Thus

$$\int_\Lambda (X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} + X_{i,j})\, dP = \int (F_\tau - F_S)\, dP = 0.$$

This holds for all $\Lambda \in \mathscr{F}_{in} \vee \mathscr{F}_{mj}$, for all m, n so it must also hold for $\Lambda \in \mathscr{F}_{i\infty} \vee \mathscr{F}_{\infty j}$.

(c) ⇒(b). Let $\tau \in \Gamma(\mathscr{L})$. Since (F_S) is additive

(8) $$F_\tau = \textstyle\sum_{t\in I} 1_{\{t\in\tau\}} Y_t,$$

where (Y_t) is the difference process defined by (7). But

$$\{(i, j) \in \tau\} = \{(i, j) \notin \tau\}^c = (\cup\{\tau = S\})^c$$

where the union is over the (countable) collection of all $S \in \mathscr{L}$ with $(i, j) \notin S$. Thus $\{(m+1, n+1) \in \tau\} \in \mathscr{F}_{m\infty} \vee \mathscr{F}_{\infty n}$, $\{(m+1, 0) \in \tau\} \in F_{m\infty}$, and $\{(0, n+1) \in \tau\} \in \mathscr{F}_{\infty n}$. Combining this with the hypothesis (c), yields $E[1_{\{t\in\tau\}} Y_t] = 0$ for t of any of the above forms. Thus, by (8), $E[F_\tau] = E[Y_o]$. □

The argument appearing in the last part of the above proof shows that every "stopping domain" in the sense of [13] belongs to $\Gamma(\mathscr{L})$. The converse is false. However, the stopping times τ_1 and τ_2 appearing in Theorem 5, below, are (usually) counterexamples.

It should be noted that if the three equations in part (c) are replaced by inequalities $E[\cdots] \geq 0$, then it is equivalent to the condition that the net $(E[F_\tau])$ be increasing. If the process (X_t) vanishes on the axes, this is equivalent to the "strong submartingale" in a paper of A. Millet ("Colloque ENST CNET sur les processus à deux indices 1980", *Lecture Notes in Mathematics,* **863).**

If $(F_S)_{S\in\mathscr{L}(I)}$ is an additive amart, $u \in I$, $\Lambda \in \mathscr{F}_u$, then the process $(Y_t)_{t\geq u}$ defined by $Y_t = (F_{[o,t]} - F_{[0,u]})1_\Lambda$ is associated with an additive amart $G_S = F_S - F_{[0,u]}$, where S runs through those layers in I corresponding to arbitrary layers in $\{t \in I: t \geq u\}$. If (F_S) is L^1-bounded, then so is (G_S), by the Proposition above. So in order to apply Theorem 1 to obtain a pointwise convergence theorem, it only remains to establish a maximal inequality. This is done using the method of Walsh [13, Theorem 3.3].

5. THEOREM. *Let* $I = \mathbb{N} \times \mathbb{N}$, *and suppose* $(F_S)_{S\in\mathscr{L}(I)}$ *is an additive process. Then for all* $a > 0$,

$$aP\{\sup_{t\in I} |F_{[o,t]}| > a\} \leq 6 \sup_{\tau\in\Gamma(\mathscr{L})} E|F_\tau|.$$

PROOF. Fix $u = (u_1, u_2) \in I$, and consider $\Lambda_u = \{\sup_{t\leq u} |F_{[o,t]}| > a\}$. For $t = (t_1, t_2)$ $[o, u]$, let $U_t = \{s = (s_1, s_2) \in [o, u]: s_1 \leq t_1 \text{ or } s_2 \leq t_2\}$.

$$\tau_o = \bigcap U_t,$$

where the intersection is over all $t \in [o, u]$ with $|F_{[o,t]}| > a$ (and $\tau_o = [o, u]$ if there is no such t). Then $\tau_o \in \Gamma(\mathscr{L})$. Next, define

$$\tau_1 = \bigcap (\tau_o \cap U_{(i,o)})$$

where the intersection is over all i such that $|F_{\tau_o \cap U_{(i,o)}}| > a/3$ (and $\tau_1 = \tau_o$ if there is no such i); and

$$\tau_2 = \bigcap (\tau_o \cap U_{(o,j)})$$

where the intersection is over all j such that $|F_{\tau_o \cap U(o,j)}| > a/3$. Then $\tau_1, \tau_2 \in \Gamma(\mathscr{L})$. Now if $\omega \in \Lambda_u$, then $|F_{[o,t]}| > a$ for some minimal $t = (i, j)$, and for such a minimal t,

$$(\tau_o \cap U_{(i,o)}) \cup (\tau_o \cap U_{(o,j)}) = \tau_o, \qquad (\tau_o \cap U_{(i,o)}) \cap (\tau_o \cap U_{(o,j)}) = [0, t],$$

so

$$\omega \in \left\{|F_{\tau_o}| > \frac{a}{3}\right\} \cup \left\{|F_{\tau_1}| > \frac{a}{3}\right\} \cup \left\{|F_{\tau_2}| > \frac{a}{3}\right\} = \left\{|F_{\tau_1}| > \frac{a}{3}\right\} \cup \left\{|F_{\tau_2}| > \frac{a}{3}\right\}.$$

Now $P\{|F_{\tau_1}| > a/3\} \leq (3/a)E|F_{\tau_1}|$, and similarly for τ_2, so we have

$$P(\Lambda_u) \leq \frac{6}{a} \sup_{\tau \in \Gamma(\mathscr{L})} E|F_\tau|. \tag{9}$$

Now let u increase; in the limit we get the required maximal inequality. □

7. Corollary. *Let $I = \mathbb{N} \times \mathbb{N}$. If (F_S) is an additive amart and $X_t = F_{[o,t]}$ is L^1-bounded, then X_t converges* a.s.

Proof. Combine Theorems 1, 3, and 5. □

3. Vector-valued amarts. Many of the above results can be extended to the case of processes with values in a Banach space E, by making appropriate changes in the proofs. We begin with a brief discussion of the case of general directed set.

Let J be a directed set, let $(\mathscr{F}_t)_{t \in J}$ be a stochastic basis, and let E be a Banach space. An adapted family $(X_t)_{t \in J}$ of Bochner-integrable E-valued random variables is an *amart* iff the net $(E(X_\tau))_{\tau \in \Gamma(J)}$ converges for the norm topology of E. The adapted family $(X_t)_{t \in J}$ is a *uniform amart* iff

$$\lim_t \sup_{\sigma \geq t} E\,\| E[X_\sigma | \mathscr{F}_t] - X_t \| = 0.$$

The basic references are [5] [6] [1] [2]. In the case $J = \mathbb{N}$, it is known that an L^1-bounded uniform amart with values in a Banach space with the Radon-Nikodym property converges (strongly) a.s. [2]; and that an amart such that $\sup_\tau E\,\|X_\tau\| < \infty$ with values in a space E, where E and E^* both have the Radon-Nikodym property, converges weakly a.s. [5, Corollary 5.3].

Let E be a Banach space, let J be a directed set, let X_t be an E-valued Bochner-measurable random variable for each $t \in J$, and let X_∞ be an E-valued random variable. We will say X_t converges to X_∞ in $L^1(E)$, or in Bochner norm, iff

$$\lim_t E\,\|X_t - X_\infty\| = 0.$$

We will say X_t converges to X_∞ in $P^1(E)$, or in Pettis norm, iff for every $\varepsilon > 0$ there exists $t_o \in J$ such that for all $t \geq t_o$ and all norm-one linear functionals $f \in E^*$,

$$E|f(X_t - X_\infty)| < \varepsilon.$$

We will say X_t converges to X_∞ in $L^o(E)$, or in probability, iff for every $\varepsilon > 0$

$$\lim_t P\{\|X_t - X_\infty\| > \varepsilon\} = 0.$$

We will say X_t converges to X_∞ in $P^o(E)$ iff for every $\varepsilon > 0$, $\eta > 0$ there exists $t_o \in J$ such that for all $t \geq t_o$ and all norm-one linear functionals $f \in E^*$,

$$P\{|f(X_t - X_\infty)| > \varepsilon\} < \eta.$$

All four of these topologies are metrizable linear space topologies. The spaces $L^1(E)$ of Bochner-integrable and $L^o(E)$ of Bochner-measurable random variables are complete in their respective topologies.

The following is well-known to the experts, but I was unable to find it in the literature.

8. THEOREM. *Let J be a directed set, let $(\mathscr{F}_t)_{t\in J}$ be a stochastic basis, and let E be a Banach space with the Radon-Nikodym property. Suppose $(X_t)_{t\in J}$ is an L^1-bounded E-valued amart. Then X_t converges in $P^o(E)$.*

PROOF. By Astbury's Riesz decomposition [1, Theorem 2.1], X_t can be written $X_t = Y_t + Z_t$, where Y_t is a martingale and Z_t converges to 0 in $P^1(E)$. For any increasing sequence $t_1 < t_2 < t_3 < \cdots$, the process $(Y_{t_n})_{n=1}^\infty$ is an L^1-bounded martingale, so by Chatterji's theorem [3] it converges a.e. and hence it converges in $L^o(E)$. But $L^o(E)$ is a complete metric space, so it follows by the Cauchy criterion that the process $(Y_t)_{t\in J}$ converges in $L^o(E)$. Therefore, in particular, (Y_t) converges in $P^o(E)$. Also (Z_t) converges to 0 in $P^o(E)$, so (X_t) converges in $P^o(E)$. □

I conclude by stating the vector-valued versions of the theorems proved above. The proofs are omitted, since they involve nothing new: they use Theorem 8, part (b) of Lemma 2, and the vector-valued techniques used in the past ([5], [6], [1], [2], [4], [8]).

9. THEOREM. *Let E be a Banach space. Let $I = N \times N$, and let $(F_S)_{S\in\mathscr{L}(I)}$ be an additive process with values in E. Suppose*

$$\sup_{\tau\in\Gamma} E\|F_\tau\| < \infty,$$

and write $X_t = F_{[o,t]}$.

(a) $P\{\sup_{t\in I}\|X_t\|t > a\} \leq \frac{6}{a}\sup_{\tau\in\Gamma(\mathscr{L})} E\|F_\tau\|$.

(b) *If (F_S) is an amart, then for all $f \in E^*$, the net $f(X_t)$ converges a.s.*

(c) *If, in addition, E has the Radon-Nikoydm property, then there is a random variable X_∞ so that $f(X_t) \to f(X_\infty)$ a.s. for all $f \in E^*$.*

(d) *If, in addition, E^* has the Radon-Nikodym property, then X_t converges weakly a.s.*

(e) *If (F_S) is a uniform amart and E has the Radon-Nikodym property, then X_t converges strongly a.s.*

REFERENCES

[1] ASTBURY, K. (1978). Amarts indexed by directed sets. *Ann. Probability* **6** 267–278.
[2] BELLOW, A. (1978). Uniform amarts. *Z. Wahrsch. verw. Gebiete* **41** 177–191.
[3] CHATTERJI, S. D. (1978). Martingale convergence and the Radon-Nikodym theorem. *Math. Scand.* **22** 21–41.
[4] EDGAR, G. A. (1980). Asplund operators and a.e. convergence. *J. Multivariate Anal.* **10** 460–466.
[5] EDGAR, G. A. and SUCHESTON, L. (1976). Amarts: A class of asymptotic martingales. *J. Multivariate Anal.* **6** 193–221.
[6] EDGAR, G. A. and SUCHESTON, L. (1976). The Riesz decomposition for vector-valued amarts. *Z. Wahrsch. verw. Gebiete* **36** 85–92.
[7] FOUQUE, J.-P. and MILLET, A. (1980). Régularité à gauche des martingales fortes à plusieurs indices. *Comptes Rendus Acad. Sci. Paris* **290** A773–776.
[8] GHOUSSOUB, N. and SUCHESTON, L. (1968). A refinement of the Riesz decomposition for amarts and semiamarts. *J. Multivariate Anal.* **8** 146–150.

[9] MILLET, A. and SUCHESTON, L. (1979). Characterizations of Vitali conditions with overlap in terms of convergence of classes of amarts. *Canadian J. Math* **31** 1033–1046.

[10] MILLET, A. and SUCHESTON, L. (1980). On convergence of L_1-bounded Martingales indexed by directed sets. *Probab. Math. Statist.* **1** 151–169.

[11] MILLET, A. and SUCHESTON, L. (1980). Convergence et régularité des martingales a indices multiples. *Comptes Rendus Acad. Sci. Paris* **291** A147–150.

[12] MILLET, A. and SUCHESTON, L.(1981). On regularity of multiparameter amarts and martingales. *Z. Wahrsch. verw. Gebiete* **56** 21–25.

[13] WALSH, J. B. (1979). Convergence and regularity of multiparameter strong martingales. *Z. Wahrsch. verw. Gebiete* **46** 177–192.

DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
231 WEST 18TH AVE.
COLUMBUS, OH 43210