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THE ASYMPTOTIC DISTRIBUTION OF THE STRENGTH OF A SERIES-PARALLEL SYSTEM WITH EQUAL LOAD-SHARING[1]

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A classical result due to Daniels is that the strength of a bundle of parallel fibres is asymptotically normally distributed. Extensions of this result are obtained and applied to a series-parallel model consisting of a long chain of bundles arranged in series. This model is of importance in studying the reliability of fibrous materials. Improved approximations are also obtained which reduce the error associated with Daniels' approximation both for the single bundle and for the series-parallel system.

1. Background.

1.1 *Introduction.* We consider a series-parallel model in which individual *elements* are connected in parallel to form *components* which are connected in series to form a *system*. A non-negative tensile load is applied to the system. Elements fail in random fashion and the load on failed elements is redistributed over unfailed elements within the same component. The system is said to fail if all the elements of any single component fail under their original or redistributed loads; otherwise the system survives. The general problem is to determine the distribution of the strength of the system (i.e., the largest load the system is capable of supporting without failing) given the distributions of the strengths of the individual elements and the *load-sharing rule* which determines the pattern of redistribution of load within each component. Particular interest focuses on asymptotic theory as both the number of components and the number of elements per component tend to infinity. Throughout we adopt the convention that the system load is measured in terms of load per element, i.e., $1/n$ times the total load on the system if n is the number of elements in each component. A similar convention is adopted with regard to system strength.

The simplest model within this class consists of a single component of n elements in parallel with the applied load shared equally among surviving elements. Then if the strengths of the individual elements are denoted by $X_1, X_2, \cdots, X_n$ and their ordered values by $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$, the strength of the system is given by

$$Q_n^* = \max\{X_{(k)}\cdot(n-k+1)/n : 1 \le k \le n\}. \tag{1.1}$$

This model is appropriate for a bundle of parallel fibres stretched between two clamps. The distribution of Q_n^* was investigated by Daniels (1945) under the assumption that the fibre strengths $X_1, X_2, \cdots, X_n$ are independent random variables with known common distribution function. Exact and asymptotic ($n \to \infty$) results were obtained. In particular, Daniels obtained positive constants μ^* and σ^* such that $n^{1/2}(Q_n^* - \mu^*)$ converges in distribution to a normal random variable with mean zero and standard deviation σ^*.

The generalization which is the principal subject of this paper consists of K independent

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components in series, each component behaving like a bundle of fibres satisfying Daniels' assumptions. Thus, the strength of the system is the minimum of K independent copies of Q_n^*. We obtain limit theorems as $K \to \infty$, $n \to \infty$ simultaneously. This involves us in the consideration of probabilities of large deviations in Daniels' model. A secondary problem concerns the rate of convergence to asymptotic normality in Daniels' original model. This is shown to be slow, but improved approximations are available which lead to significant reductions in error.

The series-parallel model considered here was first proposed by Gücer and Gurland (1962) as a model for a composite material consisting of strong, stiff fibres embedded in a ductile matrix whose effect is to bond the fibres together. When a fibre breaks, it is of course unable to support any stress at the point of the break, but nevertheless it is able to support almost the original load a short distance away. Define the "ineffective length" δ to be (roughly) the total length of the region surrounding a break within which the fibre is unable to support load. Then the system may be viewed as a long chain of short sections, each of length δ, which behave independently as individual bundles of fibres. This is the series-parallel model. The same model is used in the study of loose fibres twisted together; here there is no matrix, but interfibre friction has the same effect. The model may even be of interest in studying the molecular structure of polymers (Becht *et al.* (1971), DeVries and Williams (1973), DeVries *et al.* (1975)). Polymer fibres are composed of long, thin fibrils which consist of chains of molecules packed into alternately crystalline and amorphous regions. The weak amorphous regions may be viewed as bundles of molecular "fibres" connected in series by the strong crystalline regions which act as crack arrestors. However, little is known about the distribution of stress over the molecules within each amorphous region; there is some evidence that it is highly nonuniform and this would invalidate our model as a literal description of the system. On the other hand, polymers with such molecular structure have been found to be much stronger than other fibrous materials of comparable size and weight. The study of series-parallel systems gives some insight into why this should be so, and may be important in the future development of strong polymers.

In this paper the assumption of equal load-sharing (within each component) is made; this assumption, however, is certainly not true for all materials. For many kinds of composite materials the load is concentrated on those elements near to broken elements. This phenomenon of *local load-sharing* has been the subject of much attention in recent years; see Smith (1980) for a review and for recent results. We also assume the fibres are homogeneous and all of the same length. A generalization of Daniels' model to allow such features as mixtures of fibre types and random slack in fibres was discussed by Phoenix and Taylor (1973). It is an open problem to extend the results of the present paper to the Phoenix-Taylor model. Finally, no allowance is made for the time-dependence of failure phenomena. An extensive discussion of fatigue models is contained in two recent papers of Phoenix (1978, 1979), based on the pioneering work of B. D. Coleman. Borges (1978) studied the series-parallel model for a particular special case in which fibre failure times are exponentially distributed and the bundle failure time may be represented as a sum of independent random variables. Borges' thesis contains an extensive discussion of the relation between large deviations theory and extreme value theory.

1.2 *Preliminary results.* We assume the fibre strengths $X_1, \cdots, X_n$ are independent random variables with continuous distribution function F satisfying $F(0) = 0$ and $\int_0^\infty x^2\, dF(x) < \infty$. Define the empirical distribution function

$$\hat{F}_n(x) = n^{-1}\#\{i: 1 \le i \le n, X_i \le x\}.$$

(Here "#" denotes cardinality.) Then (1.1) may be rewritten as

$$Q_n^* = \sup\{x(1 - \hat{F}_n(x)) : x \ge 0\}. \tag{1.2}$$

Writing $q(t)$ for $F^{-1}(t)$ and $U_n(t)$ for $\hat{F}_n(F^{-1}(t))$ in $0 \le t < 1$, this is equivalent to

$$Q_n^* = \sup\{q(t)\cdot(1 - U_n(t)) : 0 \le t < 1\} \tag{1.3}$$

where $\{U_n(t), 0 \le t < 1\}$ is equivalent in distribution to the empirical distribution function associated with n independent random variables uniformly distributed on [0, 1]. Define $Q_n(t) = q(t)\cdot(1 - U_n(t))$, $\mu(t) = q(t)\cdot(1-t)$, $\sigma^2(t) = q^2(t)\cdot t(1-t)$. Then $Q_n(t)$ has mean $\mu(t)$ and variance $n^{-1}\sigma^2(t)$. Suppose the maximum of $\mu(t)$ is achieved at a unique point t^*, and let $\mu^* = \mu(t^*)$, $\sigma^* = \sigma(t^*)$. Then $n^{1/2}(Q_n(t^*) - \mu^*)$ converges in distribution to a normal random variable with mean zero and standard deviation σ^*. The same result will be true of $n^{1/2}(Q_n^* - \mu^*)$ provided $n^{1/2}(Q_n(t^*) - Q_n^*) \to_p 0$. It will be shown that $n^{1/2}(Q_n(t^*) - Q_n^*)$ is $O_p(n^{-1/6})$, and moreover that this result is independent of the values of $Q_n(t^*)$ provided $n^{1/2}(Q_n(t^*) - \mu^*)$ is $o(n^{1/6})$. These results form the basis of the whole paper. They imply, for instance, that the rate of convergence to normality is no faster than $O(n^{1/6})$. It is confirmed in Section 3 that this is the exact rate of convergence; however, a correction based upon the asymptotic distribution of $n^{2/3}(Q_n^* - Q_n(t^*))$ leads to an improved approximation.

The problem considered here is a special case of the more general problem of determining the probability that a random walk or empirical distribution function crosses a curved boundary. Daniels (1945, 1973, 1974) has approached these problems by a renewal argument, applying asymptotic analysis to the resulting integral equation. Our approach is to embed the process in the appropriate Brownian process and use a rescaling to study the local fluctuations around t^*. The method has been previously used by Barbour (1975). The key features of our approach are the use of the Komlós-Major-Tusnády embedding theorem (restated as Lemma 2.2) and some boundary-crossing inequalities for Brownian motion (mostly derived from Lemma 2.5) to obtain explicit error estimates.

Before proceeding with the asymptotic analysis, we mention Daniels' recursive formula for $F_n(x) = P\{Q_n^* \le x\}$,

$$F_n(x) = \textstyle\sum_{k=1}^n (-1)^{k+1}\binom{n}{k} F(x)^k F_{n-k}(nx/(n-k)). \tag{1.4}$$

In the author's experience (using APL), this formula is useful for n up to about 40. Beyond that, numerical instability problems become overwhelming. For this range of n, a direct comparison is possible between the exact probability of failure and the normal approximation, and it will be seen that the error involved in the approximation is considerable.

1.3 *Notations and conventions.* Throughout the paper $K_1, K_2, K_3, \cdots$, denote positive constants. We use the notations $x_+ = \max(x, 0)$, $x_- = \max(-x, 0)$, $I\{E\}$ the indicator function of the set E and $\Phi(x)$ the standard normal distribution function $\int_{-\infty}^x (2\pi)^{-1/2}\exp(-t^2/2)\,dt$. We also use $\psi(x)$ to denote the distribution function of the random variable $\sup\{B(t) - t^2, t \ge 0\}$, where $B(t)$ is standard Brownian Motion with $B(0) = 0$. We use $(1 + x^{-2})^{-1}x^{-1}(2\pi)^{-1/2}\exp(-x^2/2) \le 1 - \Phi(x) \le x^{-1}(2\pi)^{-1/2}\exp(-x^2/2)$ (Feller (1970), page 175) and $1 - \Phi(x) \le (\exp(-x^2/2))/2$, valid when $x > 0$.

2. Mathematical development of the model.

2.1 *Assumptions and preliminary definitions.* Let $\{U_n(t), 0 \le t \le 1\}$ be the empirical distribution function associated with a sample of size n from the uniform distribution on [0, 1]. Let $\{q(t), 0 \le t < 1\}$ be a given function. Let

$$Q_n(t) = q(t)(1 - U_n(t)), \qquad 0 \le t < 1$$

$$\mu(t) = q(t)(1-t), \qquad 0 \le t < 1$$

$$Q_n^* = \sup\{Q_n(t) : 0 < t < 1\}.$$

Assume:
(I) $q(t)$ is increasing in $[0, 1]$, $q(0) = 0$, $\int_0^1 q^2(t)\,dt < \infty$.
(II) $\mu(t)$ has a unique maximum μ^* attained at a point t^*. This maximum is unique in the sense that, for some $t_0 < t^*$ and some $t_1 > t^*$, the function $\mu(t)$ is strictly increasing in $[t_0, t^*]$, strictly decreasing in $[t^*, t_1]$, and

$$\sup\{\mu(t) : t \notin [t_0, t_1]\} < \mu^*.$$

(III) $\mu(t)$ is three times continuously differentiable in a neighbourhood of $t = t^*$, and $\mu''(t^*) < 0$.

REMARKS. 1. We have already observed that (I) is satisfied when $q = F^{-1}$ where F is a continuous distribution function with $F(0) = 0$ and $\int_0^\infty x^2\,dF(x) < \infty$. Note that (I) implies $\mu(t) \to 0$ as $t \to 0$ or 1.

2. The assumption that $\mu(t)$ has a unique maximum is necessary for asymptotic normality. In the case that the maximum is not unique, Phoenix and Taylor (1973) show that the limit law is that of the maximum of a Gaussian process over the set on which $\mu(t)$ attains its maximum. We do not consider this case.

3. Assumption (III) can be generalized to:
(III′) μ is $(r + 1)$ times continuously differentiable in a neighbourhood of t^*, $\mu^{(k)} = 0$ for $k < r$ and $\mu^{(r)} < 0$. Here r is an arbitrary positive even integer.

If (III′) holds with $r = 2$, we have (III). Throughout this work the assumption is that (III) holds; however, the same method works for the more general case $r > 2$ in (III′), and we indicate some of the results which hold in this case.

We now make some definitions. Let $K_1 = -\mu''(t^*)/2 > 0$ and let $t_0 < t^* < t_1$ be such that

(i) $$\mu \in C^3[t_0, t_1],$$

(ii) $$K_1(t^* - t)^2/2 < \mu(t^*) - \mu(t) < 2K_1(t^* - t)^2 \quad \text{for} \quad t \in [t_0, t_1],$$

(iii) $$q'(t^*)/2 < q'(t) < 2q'(t^*) \quad \text{for} \quad t \in [t_0, t_1],$$

(iv) $$\mu(t_1) = \sup\{\mu(t) : t \geq t_1\} \quad \text{and} \quad \mu(t_0) = \sup\{\mu(t) : t \leq t_0\}.$$

With regard to (iii), note that it follows trivially from the assumptions that $q \in C^1[t_0, t_1]$ with $q'(t^*) > 0$.

Let $\Delta_{n1} = \sup\{Q_n(t) - Q_n(t^*) : t^* \leq t < 1\}$ and let $\Delta_{n2} = \sup\{Q_n(t) - Q_n(t^*) : 0 < t \leq t^*)$. Let $\Delta_n = \max(\Delta_{n1}, \Delta_{n2}) = Q_n^* - Q_n(t^*)$. We are going to study the conditional distribution of Δ_n given $k = nQ_n(t^*)$. More specifically, in this section we obtain limit theorems as $n \to \infty$ and $k \to \infty$ subject to certain restrictions on k. These results are then used in later sections to obtain the desired limit theorems about the distribution of Q_n^*.

Note that Δ_{n1} and Δ_{n2} are conditionally independent given k. It is therefore sufficient to study separately the conditional distributions of Δ_{n1} and Δ_{n2}.

2.2 *Distribution of* Δ_{n1}. For each $m \geq 1$ let $U_m(t)$ be the empirical distribution function associated with m independent uniforms and define $Z_m(t) = m^{1/2}(t - U_m(t))$. Let $Z_0(t)$ be the Brownian Bridge on $[0, 1]$, i.e. a zero-mean continuous path Gaussian process with $EZ(s)Z(t) = s(1 - t)$ for $0 \leq s \leq t \leq 1$. We quote as Lemmas 2.1 and 2.2 two well-known theorems on these processes.

LEMMA 2.1. *Let* ϕ *be a non-negative function nondecreasing on* $[0, \theta]$ *where* $0 < \theta < 1$. *There exists a constant* $C_\theta > 0$ *such that for any* $m \geq 0$,

$$P\{|Z_m(t)| > \phi(t) \quad \text{for some} \quad t \in [0, \theta]\} \leq C_\theta \int_0^\theta \phi^{-2}(t)\,dt.$$

PROOF. This is Lemma 2.2 of Pyke and Shorack (1968). □

Remark. It is an immediate consequence that if ϕ is non-negative and nonincreasing on $[\theta, 1]$, then

$$P\{|Z_m(t)| > \phi(t) \quad \text{for some} \quad t \in [\theta, 1]\} \le C_\theta^1 \int_\theta^1 \phi^{-2}(t)\, dt$$

with $C_\theta^1 = C_{1-\theta}$. This is true because the processes $Z_m(t)$ and $-Z_m(1-t)$ are identical in law.

Lemma 2.2. *For each $m \ge 1$ there exists a version of the processes Z_0, Z_m such that*

$$\text{(2.1)} \quad P\{m^{1/2} |Z_m(t) - Z_0(t)| > x, \quad \textit{some} \quad t \in [0, 1]\} \le K_2 \exp\{-K_3(x - K_4 \log m)\}$$

whenever $x > K_4 \log m$. Here K_2, K_3, K_4 are absolute positive constants.

Proof. This is Theorem 3 of Komlós, Major and Tusnády (1975). □

For each $n \ge 1$ and $1 \le k \le n-1$ let processes Z_0, Z_{n-k} be defined on a probability space $(\Omega_{nk}, \mathscr{F}_{nk}, P_{nk})$ so as to satisfy (2.1) with $m = n - k$. Let $V_{n-k}(t) = t - (n-k)^{-1/2} Z_{n-k}(t)$ for $t \in [0, 1]$ and define $U_n'(t) = k/n + ((n-k)/n) V_{n-k}((t - t^*)/(1 - t^*))$, for $t \in [t^*, 1]$. Now V_{n-k} is itself an empirical distribution function and it is readily seen that U_n' is identical in law to the original process U_n conditioned on $U_n(t^*) = k/n$, for $t \ge t^*$. Having observed this we drop the prime on U_n' and think of P_{nk} as the conditional measure of $\{U_n(t) : t \in [t^*, 1]\}$ given $U_n(t^*) = k/n$. After making substitutions it is seen that

$$Q_n(t) = \mu(t)(1 - k/n)/(1 - t^*) + (n-k)^{1/2} n^{-1} q(t) Z_{n-k}((t - t^*)/(1 - t^*)).$$

Write $e_{nk} = 1 - k/n$. Then we have

$$\text{(2.2)} \quad \begin{aligned} \Delta_{n1} = \sup\{&-(\mu(t^*) - \mu(t)) e_{nk}/(1 - t^*) \\ &+ e_{nk}^{1/2} n^{-1/2} q(t) Z_{n-k}((t - t^*)/(1 - t^*)) : t \in [t^*, 1)\}. \end{aligned}$$

Define

$$\text{(2.3)} \quad \begin{aligned} \Delta_{n1}^* = \sup\{&-(\mu(t^*) - \mu(t)) e_{nk}/(1 - t^*) \\ &+ e_{nk}^{1/2} n^{-1/2} q(t) Z_0((t - t^*)/(1 - t^*)) : t \in [t^*, 1)\}. \end{aligned}$$

Thus Δ_{n1}^* is Δ_{n1} with Z_0 replacing Z_{n-k}. It follows from Lemma 2 that $|\Delta_{n1}^* - \Delta_{n1}|$ is not too large; however, before making this notion precise we need to show that the suprema in (2.2) and (2.3) are, with high probability, attained in $t \le t_1$.

Lemma 2.3. *There exists $K_5 > 0$ such that*

$$\text{(2.4)} \quad \begin{aligned} P_{nk}\{&(\mu(t^*) - \mu(t)) e_{nk}/(1 - t^*) \\ &< e_{nk}^{1/2} n^{-1/2} q(t) Z_{n-k}((t - t^*)/(1 - t^*)), \quad \textit{some} \quad t \ge t_1\} < K_5/(n e_{nk}), \end{aligned}$$

and moreover the same result holds with Z_0 replacing Z_{n-k}.

Proof. Since $\mu(t) \le \mu(t_1)$ on $t \ge t_1$ it is sufficient to show

$$\text{(2.5)} \quad \begin{aligned} P_{nk}\{&(\mu(t^*) - \mu(t_1)) e_{nk}/(1 - t^*) \\ &< e_{nk}^{1/2} n^{-1/2} q(t) Z_{n-k}((t - t^*)/(1 - t^*)), \quad \text{some} \quad t \ge t_1\} < K_5/(n e_{nk}). \end{aligned}$$

Let $q_1(t) = q(t^* + t(1 - t^*))$ for $t \in [0, 1)$. Thus $q(t) = q_1((t - t^*)/(1 - t^*))$, $t \in [t^*, 1)$. Note $\int_0^1 q_1^2(t)\, dt < \infty$. Let $t_2 = (t_1 - t^*)/(1 - t^*)$. Then

$$P_{nk}\{Z_{n-k}((t-t^*)/(1-t^*)) > (\mu(t^*)-\mu(t_1))e_{nk}^{1/2}n^{1/2}q^{-1}(t)(1-t^*)^{-1}, \quad \text{some} \quad t \geq t_1\}$$
$$= P_{nk}\{Z_{n-k}(t) > (\mu(t^*)-\mu(t_1))e_{nk}^{1/2}n^{1/2}q_1^{-1}(t)(1-t^*)^{-1}, \quad \text{some} \quad t \geq t_2\}$$
$$\leq C_{t_2}^1(1-t^*)^2(\mu(t^*)-\mu(t_1))^{-2}e_{nk}^{-1}n^{-1}\int_{t_2}^1 q_1^2(t)\,dt$$

from which (2.5) follows with $K_5 = C_{t_2}^1(1-t^*)^2(\mu(t^*)-\mu(t_1))^{-2}\int_{t_2}^1 q_1^2(t)\,dt$. (2.4) follows. We have used the remark following Lemma 2.1 here. The argument holds verbatim for Z_0. □

We now make precise the notion that $|\Delta_{n1}^* - \Delta_{n1}|$ is small.

LEMMA 2.4. *If $z > K_4 q(t_1)n^{-1}\log n$ then*

$$P_{nk}\{|\Delta_{n1}^* - \Delta_{n1}| > z\} < K_2\exp\{-K_3(zn/q(t_1) - K_4\log n)\} + 2K_5/(ne_{nk}).$$

PROOF. In view of (2.2), (2.3) and Lemma 2.3 it suffices to show

$$P_{nk}\{e_{nk}^{1/2}n^{-1/2}q(t)|Z_{n-k}((t-t^*)/(1-t^*)) - Z_0((t-t^*)/(1-t^*))| > z$$
$$\text{for some} \quad t \in [t^*, t_1]\} < K_2\exp\{-K_3(zn/q(t_1) - K_4\log n)\}.$$

But this follows at once from (2.1) using the fact that $q(t)$ is increasing on $[t^*, t_1]$. □

Now the supremum defined by (2.3) is necessarily attained near to t^* when n is large. But $\mu(t^*) - \mu(t)$ behaves locally like a quadratic in $t^* - t$ and Z_0 behaves locally like a Brownian motion. These ideas are exploited in studying the asymptotic behaviour of Δ_{n1}^* as $n \to \infty$.

For $x \geq 0$ define $B(x) = (1+x)Z_0(x/(1+x))$. $B(x)$ is a Brownian motion with $B(0) = 0$, and $Z_0(t) = (1-t)B(t/(1-t))$ for $t \in [0, 1]$. Writing $x = (t-t^*)/(1-t)$ in (2.3), we get

$$\Delta_{n1}^* = \sup\{-[\mu(t^*) - \mu((t^*+x)/(1+x))]e_{nk}/(1-t^*) \tag{2.6}$$
$$+ e_{nk}^{1/2}n^{-1/2}(1-t^*)^{-1}\mu((t^*+x)/(1+x))B(x) : x \geq 0\}.$$

Let $x_1 = (t_1 - t^*)/(1-t_1)$. It follows from the Taylor expansion of μ about t^* that there exists a constant $K_6 > 0$ such that the inequalities

$$(1-\varepsilon)(1-t^*)^2K_1x^2 \leq \mu(t^*) - \mu((t^*+x)/(1+x)) \leq (1+\varepsilon)(1-t^*)^2K_1x^2,$$
$$(1-\varepsilon)\mu(t^*) \leq \mu((t^*+x)/(1+x)) \leq (1+\varepsilon)\mu(t^*)$$

are satisfied on $0 \leq x \leq K_6\varepsilon$ whenever $0 < \varepsilon < x_1/K_6$. For such ε define

$$^{\varepsilon}\Delta_{n1} = \sup\{-(1-\varepsilon)(1-t^*)K_1x^2e_{nk} + e_{nk}^{1/2}n^{-1/2}\mu(t^*)(1-t^*)^{-1}(1+\varepsilon)B(x) : x \geq 0\}, \tag{2.7}$$

$$_{\varepsilon}\Delta_{n1} = \sup\{-(1+\varepsilon)(1-t^*)K_1x^2e_{nk} + e_{nk}^{1/2}n^{-1/2}\mu(t^*)(1-t^*)^{-1}(1-\varepsilon)B(x) : x \geq 0\}. \tag{2.8}$$

Now $_{\varepsilon}\Delta_{n1} \leq \Delta_{n1}^* \leq {}^{\varepsilon}\Delta_{n1}$ on the set, which we shall call $E_1(\varepsilon)$, on which the suprema defining Δ_{n1}^* and $_{\varepsilon}\Delta_{n1}$ in (2.6) and (2.8) are achieved for $x \in [0, K_6\varepsilon]$. The next step is to find a lower bound on $P_{nk}\{E_1(\varepsilon)\}$.

First we give an elementary result about Brownian motion.

LEMMA 2.5. *Let $\{B(x), x \geq 0\}$ be the Brownian motion with $B(0) = 0$.*
(i) *Let $a > 0$, $b > 0$. Then*

$$P\{B(x) \geq a + bx \quad \text{for some} \quad x > 0\} = e^{-2ab}.$$

(ii) *Let $x_0 > 0$, $b > 0$ and let a be any real number. Then*

$$P\{B(x) \geq a + bx \quad \text{for some} \quad x \geq x_0\} = 1 - \Phi((a+bx_0)x_0^{-1/2}) + e^{-2ab}\Phi((a-bx_0)x_0^{-1/2}).$$

PROOF. (i) is Corollary 5.1, page 361, of Karlin and Taylor (1975).

(ii) Using (i) and the Markov property,

$$P\{B(x_0) < a + bx_0, B(x) \geq a + bx \quad \text{for some} \quad x > x_0\}$$
$$= \int_{-\infty}^{a+bx_0} (2\pi x_0)^{-1/2} \exp(-u^2/2x_0)\exp(-2b(a + bx_0 - u))\, du$$
$$= e^{-2ab}\Phi((a - bx_0)x_0^{-1/2})$$

after completing the square in the exponent and rearranging terms. The result follows by adding on $P\{B(x_0) \geq a + bx_0\}$. □

LEMMA 2.6. *Let $\{B(x), x \geq 0)$ be Brownian motion with $B(0) = 0$. Let $A > 0$, $x > x_0$. Then*

(i) $P\{B(x) \geq Ax, \text{ some } x > x_0\} \leq \exp\{-A^2x_0/2\}$

(ii) $P\{B(x) \geq Ax^2, \text{ some } x > x_0\} \leq \exp\{-A^2x_0^3/2\}$.

PROOF. First we note that if, in the statement of Lemma 2.5(ii), it is assumed that $a + bx_0 > 0$ and $a - bx_0 < 0$, then the inequality $1 - \Phi(x) \leq (\exp(-x^2/2))/2$ for $x \geq 0$ can be applied to the conclusion to obtain

$$P\{B(x) \geq a + bx, \qquad \text{some} \quad x \geq x_0\} \leq \exp\{-(a + bx_0)^2/2x_0\}$$

Part (i) follows immediately by writing $a = 0$, $b = A$.

Part (ii) follows by writing $a = -Ax_0^2$, $b = 2Ax_0$ and observing that, with a and b so defined, the inequality $a + bx \leq Ax^2$ holds for all x. □

LEMMA 2.7. *There exists a positive constant K_7 such that, whenever $0 > \varepsilon > x_1/K_6$,*

$$P_{nk}\{E_1(\varepsilon)\} > 1 - K_5n^{-1}e_{nk}^{-1} - 2\exp(-K_7\varepsilon^3ne_{nk}).$$

PROOF. $E_1(\varepsilon)^c \subset A_1 \cup A_2 \cup A_3$ where

$$A_1 = \{e_{nk}^{1/2}n^{-1/2}q(t)Z_0((t - t^*)/(1 - t^*)) > -\mu(t^*)(\mu(t))e_{nk}/(1 - t^*),$$
$$\text{some } t \geq t_1\},$$
$$A_2 = \{e_{nk}^{1/2}n^{-1/2}(1 - t^*)^{-1}\mu(t^*)B(x) > K_1(1 - t^*)x^2e_{nk}/(2(1 + x_1)^2),$$
$$\text{some } x > K_6\varepsilon\},$$
$$A_3 = \{e_{nk}^{1/2}n^{-1/2}\mu(t^*)(1 - t^*)^{-1}(1 - \varepsilon)B(x) > (1 + \varepsilon)(1 - t^*)K_1x^2e_{nk},$$
$$\text{some } x > K_6\varepsilon\}.$$

Events A_1, A_2, A_3 include, respectively, the events that the supremum in (2.6) is achieved for $x \geq x_1$, the supremum in (2.6) is achieved for $K_6\varepsilon < x < x_1$, and the supremum in (2.8) is achieved for $K_6\varepsilon < x$. The representation for A_1 makes use of the equivalent form (2.3); that for A_2 uses the inequalities

$$\mu(t^*) - \mu((t^* + x)/(1 + x)) \geq K_1((t^* + x)/(1 + x) - t^*)^2/2$$
$$\geq K_1(1 - t^*)^2x^2/(2(1 + x_1)^2),$$
$$\mu((t^* + x)/(1 + x)) \leq \mu(t^*),$$

valid in $x \leq x_1$.

By Lemma 2.3, $P_{nk}\{A_1\} \leq K_5/(ne_{nk})$.

By Lemma 2.6, $P_{nk}\{A_2\}$ and $P_{nk}\{A_3\}$ are both bounded above by $\exp\{-[K_1(1 - t^*)^2$

$n^{1/2}e_{nk}^{1/2}/(2\mu(t^*)(1 + x_1)^2)]^2(K_6\varepsilon)^3/2\}$. The result follows with $K_7 = [K_1(1 - t^*)^2/(2\mu(t^*)(1 + x_1)^2)]^2K_6^3/2$. □

We now simplify the study of ${}_\varepsilon\Delta_{n1}$ and ${}^\varepsilon\Delta_{n1}$ by means of a time rescaling. Let $x = Ly$, where $L > 0$ will be specified later. Since $B(x) =_d L^{1/2}B(y)$, we have

$$(2.9)\qquad {}^\varepsilon\Delta_{n1} =_d \sup\{-(1-\varepsilon)(1-t^*)K_1L^2y^2e_{nk} + e_{nk}^{1/2}n^{-1/2}\mu(t^*)(1-t^*)^{-1}(1+\varepsilon)L^{1/2}B(y): y \geq 0\}.$$

Choose L so that the coefficients of the two terms are equal, i.e., $L = [e_{nk}^{1/2}\ n^{-1/2}\mu(t^*)\cdot(1-t^*)^{-1}(1+\varepsilon)]^{2/3}[(1-\varepsilon)(1-t^*)K_1e_{nk}]^{-2/3}$.

Then

$${}^\varepsilon\Delta_{n1} =_d [(1-\varepsilon)(1-t^*)K_1e_{nk}]^{-1/3}[e_{nk}^{1/2}n^{-1/2}\ \mu(t^*)(1-t^*)^{-1}(1+\varepsilon)]^{4/3} \cdot\sup\{B(y) - y^2 : y \geq 0\}.$$

Suppose we are given $\varepsilon_1 > 0$. Then choose ε such that

$$(1-\varepsilon)^{-1/3}(1+\varepsilon)^{4/3} < (1-\varepsilon_1)^{-1},$$
$$(1+\varepsilon)^{-1/3}(1-\varepsilon)^{4/3} > (1+\varepsilon_1)^{-1}.$$

This may be achieved by $\varepsilon = K_8\varepsilon_1$, some constant $K_8 > 0$, provided ε_1 is sufficiently small. It is assumed throughout that $0 < K_6\varepsilon = K_6K_8\varepsilon_1 < x_1$.

Let $h(e) = (e/(1-t^*))^{1/3}$ for $e \in [0, 1]$ and let

$$K_9 = (1-t^*)^{1/3}[(1-t^*)K_1]^{-1/3}[\mu(t^*)/(1-t^*)]^{4/3} = K_1^{-1/3}(q(t^*))^{4/3}.$$

Then

$$(2.10)\qquad {}^\varepsilon\Delta_{n1} < (1-\varepsilon_1)^{-1}h(e_{nk})K_9n^{-2/3}\sup\{B(y) - y^2 : y \geq 0\}.$$

Similarly, it can be shown

$$(2.11)\qquad {}_\varepsilon\Delta_{n1} > (1+\varepsilon_1)^{-1}h(e_{nk})K_9n^{-2/3}\sup\{B(y) - y^2 : y \geq 0\}.$$

Define

$$(2.12)\qquad \psi(z) = P\{B(y) - y^2 \leq z \quad \text{for} \quad y \geq 0\}.$$

We now give the main result about the distribution of Δ_{n1}.

LEMMA 2.8.
(i) *Let* $0 < \varepsilon_1 < x_1/(K_6K_8)$, $0 < \varepsilon_2 < 1$. *Then*

$$\psi((1-\varepsilon_1)(1-\varepsilon_2)n^{2/3}K_9^{-1}h(e_{nk})^{-1}z) - 3K_5n^{-1}e_{nk}^{-1} - 2\exp\{-K_7K_8^3\varepsilon_1^3ne_{nk}\} - K_2\exp\{-K_3(n\varepsilon_2zq(t_1)^{-1} - K_4\log n)_+\}$$
$$\leq P_{nk}\{\Delta_{n1} \leq z\}$$
$$\leq \psi((1+\varepsilon_1)(1+\varepsilon_2)_n{}^{2/3}K_9^{-1}h(e_{nk})^{-1}z) + 3K_5n^{-1}e_{nk}^{-1} + 2\exp\{-K_7K_8^3\varepsilon_1^3ne_{nk}\} + K_2\exp\{-K_3(n\varepsilon_2zq(t_1)^{-1} - K_4\log n)_+\}$$

(ii) *If* $z > 0$ *and if* $\{k(n), n \geq 1\}$ *is a sequence of integers satisfying* $1 \leq k(n) \leq n - 1$, $n^{-1}k(n) \to t^*$, *then*

$$\lim_{n\to\infty}P_{nk(n)}\{\Delta_{n1} \leq K_9n^{-2/3}z\} = \psi(z).$$

PROOF.

(i) $P_{nk}\{\Delta_{n1} \le z\} \ge P_{nk}\{{}^{\varepsilon}\Delta_{n1} \le z(1-\varepsilon_2)\} - P_{nk}\{{}^{\varepsilon}\Delta_{n1} < \Delta^*_{n1}\}$

$$-P_{nk}\{|\Delta_{n1} - \Delta^*_{n1}| > z\varepsilon_2\}, \quad \text{where} \quad \varepsilon = K_8\varepsilon_1.$$

The first inequality then follows from (2.10), (2.12) and Lemmas 2.4 and 2.7. The second inequality is similar.

(ii) With arbitrary ε_1 and ε_2, apply (i) to $z_n = K_9 n^{-2/3} z$. Since $e_{nk(n)} \to 1 - t^*$, it follows that $h(e_{nk(n)}) \to 1$. Then

$$\limsup_{n\to\infty} P_{nk(n)}\{\Delta_{n1} \le z_n\} \le \psi((1+\varepsilon_1)(1+\varepsilon_2)z),$$

and

$$\liminf_{n\to\infty} P_{nk(n)}\{\Delta_{n1} \le z_n\} \ge \psi((1-\varepsilon_1)(1-\varepsilon_2)z).$$

Since ψ is obviously a continuous distribution function, the result follows by letting $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$. □

2.3 *Distribution of* Δ_{n2}. The behaviour of Δ_{n2} is similar to that of Δ_{n1}, but the differences are significant enough to necessitate a separate treatment.

Let Z_0 be a Brownian Bridge and Z_k an empirical process, constructed on the space $(\Omega_{nk}, \mathscr{F}_{nk}, P_{nk})$ to satisfy (2.1) with $m = k$. For definiteness, these processes are assumed independent of the ones constructed in Section 2.2; however, the point will not cause confusion as the two sets of processes are never used simultaneously. Let $V_k(t) = t - k^{-1/2}Z_k(t)$ for $t \in [0, 1]$, and $U'_n(t) = (k/n)V_k(t/t^*)$ for $t \in [0, t^*]$. Then the process $U'_n(t)$, $t \in [0, t^*]$, is identical in law to the original process $U_n(t)$, $t \in [0, t^*]$, conditioned on $U_n(t^*) = k/n$. We therefore drop the prime on U'_n and think of P_{nk} as the conditional measure of $\{U_n(t), t \in [0, t^*]\}$ given $U_n(t^*) = k/n$.

Writing $d_{nk} = k/n = 1 - e_{nk}$, we have

$$\Delta_{n2} = \sup\{-(\mu(t^*) - \mu(t))\, d_{nk}/t^* + (q(t^*) - q(t))(d_{nk}/t^* - 1) + d_{nk}^{1/2} n^{-1/2} q(t) Z_k(t/t^*) : t \in [0, t^*]\}. \tag{2.13}$$

Comparing this with (2.2), it is seen that there is an extra term (the middle one) in the argument.

Proceeding along the lines of Section 2.1, define

$$\Delta^*_{n2} = \sup\{-(\mu(t^*) - \mu(t)\, d_{nk}/t^* + (q(t^*) - q(t))(d_{nk}/t^* - 1) + d_{nk}^{1/2} n^{-1/2} q(t) Z_0(t/t^*) : t \in [0, t^*)\}. \tag{2.14}$$

LEMMA 2.9. *If* $z > K_4 q(t^*) n^{-1} \log n$ *then*

$$P_{nk}\{|\Delta^*_{n2} - \Delta_{n2}| > z\} < K_2 \exp\{-K_3(zn/q(t^*) - K_4 \log n)\}.$$

PROOF. This follows at once from (2.1) and the monotonicity of q. □

For $x \ge 0$ define $B(x) = (1+x)Z_0(1/(1+x))$. Then $B(x)$ is Brownian Motion with $B(0) = 0$. Equation (2.14) becomes

$$\Delta^*_{n2} = \sup\{-[\mu(t^*) - \mu(t^*/(1+x))]\, d_{nk}/t^* + [q(t^*) - q(t^*/(1+x))] \cdot (d_{nk}/t^* - 1) + d_{nk}^{1/2} n^{-1/2} (1+x)^{-1} q(t^*/(1+x)) B(x) : x \ge 0\}. \tag{2.15}$$

Let $x_0 = (t^* - t_0)/t_0$. It follows from the Taylor expansions of μ and q that there exists a constant $K_{10} > 0$ such that the inequalities

$$(1-\varepsilon)K_1(t^*)^2x^2 \le \mu(t^*) - \mu(t^*/(1+x)) \le (1+\varepsilon)K_1(t^*)^2x^2,$$
$$(1-\varepsilon)q'(t^*)t^*x \le q(t^*) - q(t^*/(1+x)) \le (1+\varepsilon)q'(t^*)t^*x,$$
$$(1-\varepsilon)q(t^*) \le q(t^*/(1+x))/(1+x) \le (1+\varepsilon)q(t^*)$$

are satisfied on $0 \le x \le K_{10}\,\varepsilon$ whenever $0 < \varepsilon < x_0/K_{10}$. For such ε define

$$(2.16)\quad {}^{\varepsilon}\Delta_{n2} = \sup\{-(1-\varepsilon)t^*K_1x^2\,d_{nk} + (1+\varepsilon)q'(t^*)(d_{nk}/t^*-1)_+t^*x + d_{nk}^{1/2}n^{-1/2}q(t^*)(1+\varepsilon)B(x) : x \ge 0\},$$

$$(2.17)\quad {}_{\varepsilon}\Delta_{n2} = \sup\{-(1+\varepsilon)t^*K_1x^2\,d_{nk} - (1+\varepsilon)q'(t^*)(d_{nk}/t^*-1)_-t^*x + d_{nk}^{1/2}n^{-1/2}q(t^*)(1-\varepsilon)B(x) : x \ge 0\}.$$

Now ${}_{\varepsilon}\Delta_{n2} \le \Delta^*_{n2} \le {}^{\varepsilon}\Delta_{n2}$ on the set $E_2(\varepsilon)$ on which the suprema defining Δ^*_{n2} and ${}_{\varepsilon}\Delta_{n2}$ in (2.15) and (2.17) are achieved for $x \in [0, K_{10}\varepsilon]$.

LEMMA 2.10. *There exist positive constants K_{11}, K_{12} such that if $0 < \varepsilon < x_0/K_{10}$ and $d_{nk}/t^* < 1 + K_{11}\varepsilon$ then*

$$P_{nk}\{E_2(\varepsilon)\} > 1 - 3\exp(-K_{12}\varepsilon^3 n\,d_{nk}).$$

PROOF. $E_2(\varepsilon)^c \subset A_1 \cup A_2 \cup A_3$ where

$$A_1 = \{d_{nk}^{1/2}n^{-1/2}(1+x)^{-1}q(t^*/(1+x))B(x) > [\mu(t^*) - \mu(t^*/(1+x))]\,d_{nk}/t^* - [q(t^*) - q(t^*/(1+x))](d_{nk}/t^*-1),\quad \text{some}\quad x \ge x_0\},$$
$$A_2 = \{d_{nk}^{1/2}n^{-1/2}q(t^*)B(x) > K_1t^*\,d_{nk}x^2/(2(1+x_0)^2) - 2q'(t^*)t^*x(d_{nk}/t^*-1),\quad \text{some}\quad x > K_{10}\varepsilon\},$$
$$A_3 = \{d_{nk}^{1/2}n^{-1/2}q(t^*)B(x) > K_1t^*\,d_{nk}x^2,\quad \text{some}\quad x > K_{10}\varepsilon\}.$$

Events A_1, A_2, A_3 include, respectively, the events that the supremum in (2.15) is achieved for $x \ge x_0$, the supremum in (2.15) is achieved for $K_{10}\,\varepsilon < x < x_0$, and the supremum in (2.17) is achieved for $x > K_{10}\,\varepsilon$. The representation for A_2 uses the inequalities

$$q(t^*/(1+x))/(1+x) \le q(t^*),\ \mu(t^*) - \mu(t^*/(1+x)) \ge K_1(t^* - t^*/(1+x))^2/2 \ge K_1t^{*2}x^2/(2(1+x_0)^2),$$
$$q(t^*) - q(t^*/(1+x)) \le (t^* - t^*/(1+x))\sup\{q'(t): t_0 < t < t^*\} \le 2t^*xq'(t^*),$$

valid in $x < x_0$.

Assume d_{nk} satisfies the inequalities

$$[\mu(t^*) - \mu(t^*/(1+x_0))]\,d_{nk}/t^* > 2q(t^*)(d_{nk}/t^*-1),$$
$$K_1t^*\,d_{nk}K_{10}\varepsilon/(2(1+x_0)^2) > 4q'(t^*)t^*(d_{nk}/t^*-1).$$

The condition that these inequalities be satisfied may be written as $d_{nk}/t^* < 1 + K_{11}\varepsilon$, some $K_{11} > 0$, for $\varepsilon \in [0, x_0/K_{10}]$.

Then $A_i \subset A_i'$ for $i = 1, 2$, where

$$A_1' = \{d_{nk}^{1/2}n^{-1/2}(1+x)^{-1}q(t^*/(1+x))B(x) > [\mu(t^*) - \mu(t^*/(1+x))]\,d_{nk}/(2t^*),\quad \text{some}\quad x \ge x_0\},$$
$$A_2' = \{d_{nk}^{1/2}n^{-1/2}q(t^*)B(x) > K_1t^*\,d_{nk}x^2/(4(1+x_0)^2),\quad \text{some}\quad x > K_{10}\varepsilon\}.$$

By Lemma 2.6 (ii),

$$P_{nk}\{A_3\} \le P_{nk}\{A_2'\} \le \exp\{-[d_{nk}^{1/2}n^{1/2}K_1t^*/(4(1+x_0)^2q(t^*))]^2(K_{10}\varepsilon)^3/2\}.$$

By Lemma 2.6(i),

$$\begin{aligned}P_{nk}\{A_1'\} &\le P\{B(x) > [\mu(t^*) - \mu(t^*/(1+x_0))]\, d_{nk}^{1/2}n^{1/2}x/(2t^*q(t^*)), \quad \text{some} \quad x \ge x_0\}\\ &\le \exp\{-[\mu(t^*) - \mu(t^*/(1+x_0))]\, d_{nk}^{1/2}n^{1/2}/(2t^*q(t^*)]^2x_0/2\}.\end{aligned}$$

Now choose K_{12} so that both of these quantities are less than $\exp(-K_{12}\varepsilon^3n\,d_{nk})$ whenever $0 < \varepsilon < x_0/K_{10}$. □

Now let $x = Ly$ where

$$L = [d_{nk}^{1/2}n^{-1/2}q(t^*)(1+\varepsilon)]^{2/3}[(1-\varepsilon)t^*K_1\,d_{nk}]^{-2/3}.$$

Then

$$\begin{aligned}{}^{\varepsilon}\Delta_{n2} =_d {}& [(1-\varepsilon)t^*K_1\,d_{nk}]^{-1/3}[d_{nk}^{1/2}n^{-1/2}q(t^*)(1+\varepsilon)]^{4/3}\\ &\cdot\sup\{B(y) - y^2 + (1+\varepsilon)q'(t^*)(d_{nk}/t^* - 1)_+t^*[d_{nk}^{1/2}n^{-1/2}q(t^*)(1+\varepsilon)]^{-2/3}\\ &\cdot[(1-\varepsilon)t^*K_1\,d_{nk}]^{-1/3}y : y \ge 0\}.\end{aligned} \tag{2.18}$$

Suppose we are given $\varepsilon_1 > 0$. Then choose ε such that

$$(1+\varepsilon)^{4/3}(1-\varepsilon)^{-1/3} < (1-\varepsilon_1)^{-1},$$
$$(1-\varepsilon)^{4/3}(1+\varepsilon)^{-1/3} > (1+\varepsilon_1)^{-1},$$
$$(1+\varepsilon)\cdot(1+\varepsilon)^{-2/3}(1-\varepsilon)^{-1/3} < 1+\varepsilon_1,$$
$$(1+\varepsilon)\cdot(1-\varepsilon)^{-2/3}(1+\varepsilon)^{-1/3} < 1+\varepsilon_1.$$

This may be achieved by $\varepsilon = K_{13}\varepsilon_1$ when $0 < \varepsilon_1 < x_0/(K_{10}K_{13})$. Let $h_1(d) = (d/t^*)^{1/3}$, $h_2(d) = (d/t^*)^{-2/3}$ for $d \in (0, 1]$, and let

$$\begin{aligned}K_{14} &= (t^*)^{-2/3}q'(t^*)t^*[q(t^*)]^{-2/3}[K_1t^*]^{-1/3}\\ &= q'(t^*)[q(t^*)]^{-2/3}K_1^{-1/3}.\end{aligned}$$

Also note that

$$(t^*)^{1/3}[q(t^*)]^{4/3}[K_1t^*]^{-1/3} = K_9.$$

Then

$$\begin{aligned}{}^{\varepsilon}\Delta_{n2} < {}& (1-\varepsilon_1)^{-1}h_1(d_{nk})K_9n^{-2/3}\\ &\cdot\sup\{B(y) - y^2 + (1+\varepsilon_1)(d_{nk}/t^* - 1)_+n^{1/3}h_2(d_{nk})K_{14}y : y \ge 0\},\end{aligned} \tag{2.19}$$

and similarly

$$\begin{aligned}{}_{\varepsilon}\Delta_{n2} > {}& (1+\varepsilon_1)^{-1}h_1(d_{nk})K_9n^{-2/3}\\ &\cdot\sup\{B(y) - y^2 - (1+\varepsilon_1)(d_{nk}/t^* - 1)_-n^{1/3}h_2(d_{nk})K_{14}y : y \ge 0\}.\end{aligned} \tag{2.20}$$

The next lemma estimates the effect of the linear term in the supremum.

LEMMA 2.11. *There exists a constant* $K_{15} > 0$ *such that for* $0 < \varepsilon < ½$, $|a| < K_{15}\varepsilon$ *the inequality*

$$\psi(-\varepsilon + z(1-\varepsilon)) \le P\{B(y) - y^2 + ay \le z \quad \text{for all} \quad y \ge 0\} \le \psi(\varepsilon + z(1+\varepsilon))$$

holds for all $z > 0$.

PROOF. We consider only the case $a > 0$; a similar argument works for $a < 0$.

It is trivial that $P\{B(y) - y^2 + ay \le z \text{ for all } y \ge 0\} \le P\{B(y) - y^2 \le z \text{ for all } y \ge 0\} = \psi(z) \le \psi(\varepsilon + z(1+\varepsilon))$ when $a > 0$, so we only have to prove the left-hand inequality.

Now $y^2 - ay + z \ge (1-a)y^2 - ca + z$ for all $y \ge 0$, where $c = 1/4$. Then

$$\begin{aligned} P\{B(y) - y^2 + ay \le z \quad &\text{for all} \quad y \ge 0\} \\ &\ge P\{B(y) - (1-a)y^2 \le z - ca \quad \text{for all} \quad y \ge 0\} \\ &= P\{(1-a)^{-1/3}B(t) - (1-a)^{-1/3}t^2 \le z - ca \quad \text{for all} \quad t \ge 0\} \\ &\quad (\text{putting} \quad y = (1-a)^{-2/3}t) \\ &= \psi((z - ca)(1-a)^{1/3}) \\ &\ge \psi(z(1-a)^{1/3} - ca). \end{aligned}$$

We therefore have to choose a such that $ca \le \varepsilon$ and $(1-a)^{1/3} \ge 1 - \varepsilon$. This is satisfied by $0 < a < K_{15}\varepsilon$ for suitable K_{15}. □

We are now able to prove the analogue of Lemma 2.8 for Δ_{n2}.

LEMMA 2.12.

(i) Let $0 < \varepsilon_1 < x_0/(K_{10}K_{13})$, $0 < \varepsilon_2 < 1$, $0 < \varepsilon_3 < 1/2$. Suppose

(a) $d_{nk}/t^* < 1 + K_{11}K_{13}\varepsilon_1$,

(b) $(1+\varepsilon_1)\,|\,d_{nk}/t^* - 1\,|\,n^{1/3}h_2(d_{nk})K_{14} < K_{15}\varepsilon_3$. Then

$$\begin{aligned} \psi((1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3)_N{}^{2/3}K_9^{-1}\, h_1(d_{nk})^{-1}z - \varepsilon_3) - 3\exp(-K_{12}K_{13}^3\,\varepsilon_1^3 n\, d_{nk}) \\ - K_2\exp\{-K_3(n\varepsilon_2 zq(t^*)^{-1} - K_4\log n)_+\} \end{aligned}$$

$$\begin{aligned} &\le P_{nk}\{\Delta_{n2} \le z\} \\ &\le \psi((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)n^{2/3}K_9^{-1}h_1(d_{nk})^{-1}z + \varepsilon_3) + 3\exp(-K_{12}K_{13}^3\varepsilon_1^3 n\, d_{nk}) \\ &\quad + K_2\exp\{-K_3(n\varepsilon_2 zq(t^*)^{-1} - K_4\log n)_+\}. \end{aligned}$$

(ii) If $z > 0$ and if $\{k(n), n \ge 1\}$ is a sequence of integers satisfying $1 \le k(n) \le n-1$, $n^{1/3}\,|\,k(n)/n - t^*\,| \to 0$, then

$$\lim_{n\to\infty} P_{nk(n)}\{\Delta_{n2} \le K_9 n^{-2/3}z\} = \psi(z).$$

PROOF. (i) Conditions (a) and (b) guarantee, respectively, that Lemma 2.10 (with $\varepsilon = K_{13}\varepsilon_1$) and Lemma 2.11 (with $\varepsilon = \varepsilon_3$, $a = \pm(1+\varepsilon_1)\,|\,d_{nk}/t^* - 1\,|\,n^{1/3}h_2(d_{nk})K_{14}$) are applicable. The result then follows by Lemmas 2.9, 2.10 and 2.11, and Equations (2.19) and (2.20). (ii) Fix ε_1, ε_2, ε_3 and apply (i) to $z_n = K_9 n^{-2/3}z$. The condition $n^{1/3}\,|\,k(n)/n - t^*\,| \to 0$ guarantees that conditions (a) and (b) are satisfied for all n sufficiently large. The result then follows by applying (i) and then letting $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$. □

2.4 *Asymptotic distribution of* Δ_n.

THEOREM 2.1. *Let* $z > 0$ *and let* $\{k(n), n \ge 1\}$ *be a sequence of positive integers such that* $1 \le k(n) \le n-1$, $n^{1/3}\,|\,k(n)/n - t^*\,| \to 0$. *Then* $\lim_{n\to\infty} P_{nk(n)}\{\Delta_n \le n^{-2/3}K_9 z\} = \psi^2(z)$.

PROOF. This follows from Lemma 2.8 and 2.12, noting that $\Delta_n = \max(\Delta_{n1}, \Delta_{n2})$ and that Δ_{n1} and Δ_{n2} are conditionally independent given $U_n(t^*)$. □

Recall $\mu^* = \mu(t^*)$ and define $\sigma^* = q(t^*)[t^*(1-t^*)]^{1/2}$.

THEOREM 2.2 *Let z_1 be real and $z_2 > 0$. Then* $\lim_{n\to\infty} P\{n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z_1, n^{2/3} K_9^{-1} \Delta_n \le z_2\} = \Phi(z_1)\psi^2(z_2)$.

PROOF. We shall prove the equivalent statement that for $-\infty < z_0 < z_1 < \infty$, $z_2 > 0$, we have $\lim_{n\to\infty} P\{\sigma^* z_0 \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z_1, n^{2/3} K_9^{-1}\Delta_n \le z_2\} = [\Phi(z_1) - \Phi(z_0)]\psi^2(z_2)$. Now $\sigma^{*-1}(Q_n(t^*) - \mu^*) = (t^*(1-t^*))^{-1/2}(t^* - U_n(t^*))$. For each n, let $I_n = \{k\colon z_0 \le n^{1/2}(t^*(1-t^*))^{-1/2}(t^* - k/n) \le z_1\}$. Then

$$\begin{aligned} &P\{\sigma^* z_0 \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z_1, n^{2/3} K_9^{-1}\Delta_n \le z_2\} \\ &= \textstyle\sum_{k\in I_n} P\{U_n(t^*) = k/n\} P_{nk}\{\Delta_n \le n^{-2/3} K_9 z_2\} \\ &\le (\textstyle\sum_{k\in I_n} P\{U_n(t^*) = k/n\}) \sup_{k\in I_n} P_{nk}\{\Delta_n \le n^{-2/3} K_9 z_2\} \\ &\to (\Phi(z_1) - \Phi(z_0))\psi^2(z_2) \end{aligned}$$

the first factor converging to $\Phi(z_1) - \Phi(z_0)$ by the Central Limit Theorem for $U_n(t^*)$ and the second factor converging to $\psi^2(z_2)$ by Theorem 2.1. The condition of Theorem 2.1 is satisfied because when $k \in I_n$,

$$n^{1/3}|k/n - t^*| = n^{-1/6} n^{1/2}|k/n - t^*| \le n^{-1/6}(t^*(1-t^*))^{1/2}\max(|z_0|, |z_1|),$$

and this last quantity tends to zero.

Hence

$$\limsup_{n\to\infty} P\{\sigma^* z_0 \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z_1, n^{2/3} K_9^{-1}\Delta_n \le z_2\} \le (\Phi(z_1) - \Phi(z_0))\psi^2(z_2).$$

A similar argument holds for the limit inferior, and the two together give the result. □

Theorem 2.2 is a striking result because it shows that $Q_n(t^*)$ and Δ_n are asymptotically independent and it gives their limiting distributions. Daniels' original result is a consequence:

COROLLARY 2.1. *For all real z,* $\lim_{n\to\infty} P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} = \Phi(z)$.

PROOF. Write $n^{1/2}(Q_n^* - \mu^*) = n^{1/2}(Q_n(t^*) - \mu^*) + n^{1/2}\Delta_n$. It will suffice to show $n^{1/2}\Delta_n \to_p 0$. But this follows at once after setting $z_1 = \infty$ in Theorem 2.2. □

REMARK. In the case of that Assumption (III) is false but (III′) holds with $r > 2$, Theorem 2.2 becomes

$$\lim_{n\to\infty} P\{n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z_1, n^{r/(2r-1)} K_9^{-1}\Delta_n \le z_2\} = \Phi(z_1)\psi_r^2(z_2)$$

where

$$\psi_r(z) = P\{B(y) - y^r \le z \quad \text{for all} \quad y \ge 0\}, \qquad K_9 = [-\mu^{(r)}(t^*)/r!]^{-1/(2r-1)}[q(t^*)]^{2r/(2r-1)}.$$

3. Bounds for the absolute error.

3.1 *The absolute error in the normal approximation.* The purpose of this section is to show that an analogue of the Berry-Esseen Theorem holds for the normal approximation to the bundle strength distribution. The rate of convergence, however, is only $O(n^{-1/6})$. This compares with a rate of convergence of $O(n^{-1/2})$ in the Berry-Esseen Theorem as applied to the approximation of the distribution function of a sum of independent random variables by a normal distribution function.

First we give some preliminary lemmas.

LEMMA 3.1. *Let X and Y be independent random variables with distribution functions*

Φ, ψ respectively. There exists a constant $K_{16} > 0$ such that for all $\theta > 0$ and all real z

(3.1) $$|P\{X + \theta Y \le z\} - \Phi(z)| < K_{16}\theta.$$

PROOF. Clearly $P\{X + \theta Y \le z\}$ is bounded above by $\Phi(z)$; we require a lower bound.

Let $\{B(t), t \ge 0\}$ be Brownian motion with $B(0) = 0$. Then $\theta Y =_d \sup\{\theta B(t) - \theta t^2 : t \ge 0\} =_d \sup\{B(t) - \theta^{-3}t^2 : t \ge 0\}$ so $X + \theta Y =_d \sup\{B(t) - \theta^{-3}(t-1)^2 : t \ge 1\}$.

For fixed z let a be real and b be positive such that

(3.2) $$a + bt \le z + \theta^{-3}(t-1)^2 \quad \text{for all} \quad t.$$

Then

$$\begin{aligned} P\{X + \theta Y > z\} &= P\{B(t) > z + \theta^{-3}(t-1)^2, \quad \text{some} \quad t \ge 1\} \\ &\le P\{B(t) > a + bt, \quad \text{some} \quad t \ge 1\} \\ &= 1 - \Phi(a+b) + e^{-2ab}\Phi(a-b) \end{aligned}$$

by Lemma 2.5. If, in addition, $b > a$, we can write

$$P\{X + \theta Y > z\} \le 1 - \Phi(a+b) + (2\pi)^{-1/2}(b-a)^{-1}\exp\{-(b+a)^2/2\}.$$

Now (3.2) is satisfied if $z - a - b \ge 0$ and $4(z - a - b) \ge b^2\theta^3$. We may set $z - a - b = L_1\theta$ and $b = L_2\theta^{-1}$ where L_1 is an arbitrary positive constant (independent of z) and $L_2 = 2L_1^{1/2}$. Thus $a = z - L_1\theta - L_2\theta^{-1}$. We consider two cases.

Case (i). $z \le 3L_2\theta^{-1}/2 + L_1\theta$.
Then $2a \le b$ and $(b-a)^{-1} \le 2/b$. Then

$$\begin{aligned} P\{X + \theta Y > z\} &\le 1 - \Phi(z - L_1\theta) + (2\pi)^{-1/2}2L_2^{-1}\theta \exp\{-(z - L_1\theta)^2/2\} \\ &\le 1 - \Phi(z) + (2\pi)^{-1/2}L_1\theta + (2\pi)^{-1/2}2L_2^{-1}\theta \end{aligned}$$

so

$$P\{X + \theta Y \le z\} \ge \Phi(z) - (2\pi)^{-1/2}(L_1 + 2L_2^{-1})\theta.$$

Case (ii). $z > 3L_2\theta^{-1}/2 + L_1\theta$.
In this case we set $b = L_2\theta^{-1}$ as before, $a = (½)L_2\theta^{-1} < z - L_1\theta - L_2\theta^{-1}$, so (3.2) is still satisfied. Then

$$P\{X + \theta Y > z\} \le 1 - \Phi(3L_2\theta^{-1}/2) + (2\pi)^{-/12}2L_2^{-1}\theta$$

so

$$\begin{aligned} P\{X + \theta Y > z\} - (1 - \Phi(z)) &\le \Phi(z) - \Phi(3L_2\theta^{-1}/2) + (2\pi)^{-1/2}2L_2^{-1}\theta \\ &\le 1 - \Phi(3L_2\theta^{-1}/2) + (2\pi)^{-1/2}2L_2^{-1}\theta \end{aligned}$$

which is $O(\theta)$ as $\theta \to 0$. Choosing $K_{16} > 0$ such that $(2\pi)^{-1/2}(L_1 + 2L_2^{-1}) < K_{16}$ and $1 - \Phi(3L_2\theta^{-1}/2) + (2\pi)^{-1/2}2L_2^{-1}\theta < K_{16}\theta$ for all $\theta > 0$, (3.1) is proved. □

Note that (3.1) can also be written in the form

(3.3) $$\left| \int_{-\infty}^{z} \psi((z-u)/\theta)\, d\,\Phi(u) - \Phi(z) \right| < K_{16}\theta.$$

COROLLARY 3.1. *Suppose in Lemma* 3.1 *the random variable Y has distribution function ψ^2 (instead of ψ). Then the same conclusion holds with $2K_{16}$ in place of K_{16}.*

PROOF. Use (3.3) and the inequality $\psi^2 \ge 2\psi - 1$. □

LEMMA 3.2. *If* $K_{17} = 3^{-3/2} \cdot 8$ (≈ 1.539) *then* $1 - \psi(z) \le \exp\{-K_{17}z^{3/2}\}$ *for all* $z > 0$.

PROOF. If $t^2 \ge -a + bt$ for all t, where $a > 0$ and $b > 0$, then

$$P\{B(t) \ge t^2 + z, \quad \text{some} \quad t \ge o\} \le P\{B(t) \ge z - a + bt, \quad \text{some} \ t \ge 0\}$$
$$= \exp\{-2b(z-a)_+\}$$

by Lemma 2.5. Choose a and b to maximize $b(z-a)$ under the constraint $b^2 \le 4$a. □

Let δ be a fixed positive constant, say $\delta = 1$.

LEMMA 3.3 *There exists a constant* $K_{18} > 0$ *such that*

$$P\{n^{1/2}\Delta_n > \delta\sigma^*\} < K_{18}n^{-1}, \quad \textit{for all} \quad n.$$

PROOF. Fix $\varepsilon_1, \varepsilon_2, \varepsilon_3$ positive. Let a be a positive number to be specified below.

$$P\{n^{1/2}\Delta_{n2} > \delta\sigma^*\} = \textstyle\sum_{k=0}^{n} P\{U_n(t^*) = k/n\}P_{nk}\{n^{1/2}\Delta_{n2} > \delta\sigma^*\}$$
$$\le P\{U_n(t^*) - t^* > an^{-1/3}\} + P\{U_n(t^*) - t^* < -an^{-1/3}\}$$
$$+ \sup\{P_{nk}\{n^{1/2}\Delta_{n2} > \delta\sigma^*\} : |k/n - t^*| \le an^{-1/3}\}.$$

The first two terms are $o(n^{-1})$ as $n \to \infty$; and if we choose a sufficiently small, so that the conditions of Lemma 2.12(i) are satisfied, then the third term is $o(n^{-1})$ also.

Hence $P\{n^{1/2}\Delta_{n2} > \delta\sigma^*\}$ is $o(n^{-1})$. A similar argument applies to $P\{n^{1/2}\Delta_{n1} > \delta\sigma^*\}$ except that, referring to Lemma 2.8(i), this is $O(n^{-1})$. The result is then proved. □

We quote for future reference:

BERRY-ESSEEN THEOREM (Feller (1971); Section XVI.5, page 542). *Let* $X_1, X_2, \ldots, X_n$ *be independent identically distributed random variables with* $EX_1 = 0$, $EX_1^2 = \sigma^2 > 0$, $E|X_1|^3 = \rho < \infty$. *Then*

$$|P\{X_1 + \cdots + X_n \le z\sigma\sqrt{n}\} - \Phi(z)| < 3\rho\sigma^{-3}n^{-1/2}$$

for all z *and* n.

An application of this theorem yields

$$|P\{n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^*z\} - \Phi(z)| < K_{19}n^{-1/2} \tag{3.4}$$

where $K_{19} = 3(t^{*2} + (1-t^*)^2)[t^*(1-t^*)]^{-1/2}$.

THEOREM 3.1. *There exists a constant* $K_{20} > 0$ *such that for all* z *and* n

$$|P\{n^{1/2}(Q_n^* - \mu^*) < \sigma^*z\} - \Phi(z)| < K_{20}n^{-1/6}. \tag{3.5}$$

PROOF. First we deal with the trivial case when $|z| > \log n$. If $z > \log n$ then

$$P\{n^{1/2}(Q_n^* - \mu^*) > \sigma^*z\} \le P\{n^{1/2}(Q_n(t^*) - \mu^*) > \sigma^*(\log n - \delta)\} + P\{n^{1/2}\Delta_n > \delta\sigma^*\}$$
$$\le [1 - \Phi(\log n - \delta)] + K_{19}n^{-1/2} + K_{18}n^{-1}$$

using Equation (3.4) and Lemma 3.3. This quantity is $O(n^{-1/2})$ as $n \to \infty$. A similar argument applies when $z < -\log n$. Also $\Phi(-z) = 1 - \Phi(z) = o(n^{-1/2})$ when $z > \log n$. Hence it suffices to prove (3.5) in the case $|z| \le \log n$.

For each n and k define $Y_n(k) = n^{1/2}(q(t^*)(1 - k/n) - \mu^*)/\sigma^*$, i.e., the value of $n^{1/2}(Q_n(t^*) - \mu^*)/\sigma^*$ when $U_n(t^*) = k/n$.

Define $I_n(z) = \{k : z - \delta \le Y_n(k) \le z\}$.

For $|z| \le \log n$, $k \in I_n(z)$ we have $n^{1/2}|d_{nk}/t^* - 1| = O(\log n)$ so for suitable $A_1 > 0$ we can define $\varepsilon_{n1} = \varepsilon_{n3} = A_1 n^{-1/6} \log n$ such that the inequalities $d_{nk}/t^* < 1 + K_{11} K_{13} \varepsilon_{n1}$, $(1 + \varepsilon_{n1}) | d_{nk}/t^* - 1 | n^{1/3} h_2(d_{nk}) K_{14} < K_{15} \varepsilon_{n3}$ are satisfied. Also define $\varepsilon_{n2} = n^{-1/6}$. Note that for $|z| \le \log n$, $k \in I_n(z)$, $|1 - h_1(d_{nk})|$ and $|1 - h(e_{nk})|$ are both $O(n^{-1/2} \log n)$. It then follows that there exists n_0 such that, whenever $n \ge n_0$, $|z| \le \log n$, $k \in I_n(z)$, the inequalities

$$1 - 3A_1 n^{-1/6} \log n < (1 - \varepsilon_{n1})(1 - \varepsilon_{n2})(1 - \varepsilon_{n3}) h_1(d_{nk})^{-1}$$
$$< (1 + \varepsilon_{n1})(1 + \varepsilon_{n2})(1 + \varepsilon_{n3}) h_1(d_{nk})^{-1} < 1 + 3A_1 n^{-1/6} \log n,$$
$$1 - 3A_1 n^{-1/6} \log n < (1 - \varepsilon_{n1})(1 - \varepsilon_{n2}) h(e_{nk})^{-1} < (1 + \varepsilon_{n1})(1 + \varepsilon_{n2}) h(e_{nk})^{-1}$$
$$< 1 + 3A_1 n^{-1/6} \log n$$

are satisfied.

Applying Lemmas 2.12(i) and 2.8(i), it follows under the same restrictions on n, z and k that for any $x > 0$

$$P_{nk}\{\Delta_n \le x\} \ge \psi^2((1 - 3A_1 n^{-1/6} \log n) n^{2/3} K_9^{-1} x - A_1 n^{-1/6} \log n) \tag{3.6}$$
$$- A_2 n^{-1} - 2K_2 \exp\{-K_3(n^{5/6} x q(t_1)^{-1} - K_4 \log n)_+\},$$

for some $A_2 > 0$.

For $n \ge n_0$, $|z| \le \log n$ we have

$$P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\}$$
$$= P\{\sigma^*(z - \delta) \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z,\ n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} \tag{3.7}$$
$$+ P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(z - \delta),\ n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\}.$$

But by (3.4), we have

$$P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(z - \delta)\} \ge \Phi(z - \delta) - K_{19} n^{-1/2} \tag{3.8}$$

and by Lemma 3.3 we have

$$P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(z - \delta),\ n^{1/2}(Q_n^* - \mu^*) > \sigma^* z\} \tag{3.9}$$
$$\le P\{n^{1/2} \Delta_n > \delta \sigma^*\} \le K_{18} n^{-1}.$$

By (3.8) and (3.9), it follows that

$$P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(z - \delta),\ n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} \tag{3.10}$$
$$\ge \Phi(z - \delta) - K_{19} n^{-1/2} - K_{18} n^{-1}.$$

Also

$$P\{\sigma^*(z - \delta) \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z,\ n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\}$$
$$= \textstyle\sum_{k \in I_n(z)} P\{U_n(t^*) = k/n\} P_{nk}\{n^{1/2} \Delta_n \le \sigma^*(z - Y_n(k))\}$$
$$\ge \textstyle\sum_{k \in I_n(z)} P\{U_n(t^*) = k/n\}$$
$$\cdot [\psi^2((1 - 3A_1 n^{-1/6} \log n) n^{1/6} \sigma^* K_9^{-1}(z - Y_n(k)) - A_1 n^{-1/6} \log n]$$
$$- A_2 n^{-1} - 2K_2 \exp\{-K_3(n^{1/3} \sigma^*(z - Y_n(k)) q(t_1)^{-1} - K_4 \log n)_+\},$$

using (3.6). Write this sum as $S_{n1} - S_{n2} - S_{n3}$. Then

$$S_{n2} = \textstyle\sum_{k \in I_n(z)} P\{U_n(t^*) = k/n\} A_2 n^{-1} < A_2 n^{-1}. \tag{3.11}$$

Next,

$$S_{n3} = \textstyle\sum_{k \in I_n(z)} P\{U_n(t^*) = k/n\} 2K_2 \exp\{-K_3(n^{1/3} \sigma^*(z - Y_n(k)) q(t_1)^{-1} - K_4 \log n)_+\}.$$

Break the sum into two parts: part 1 is that part for which

$$K_3(n^{1/3}\sigma^*(z - Y_n(k))q(t_1)^{-1} - K_4 \log n) > \log n$$

or equivalently $z - Y_n(k) > n^{-1/3}q(t_1)\sigma^{*-1}(K_3^{-1} + K_4)\log n$, and part 2 is the remainder. Part 1 is less than $2K_2n^{-1}$, and part 2 is less than $2K_2P\{z - n^{-1/3}q(t_1)\sigma^{*-1}(K_3^{-1} + K_4)\log n \le Y_n(nU_n(t^*)) \le z\}$ which is less than $2K_2[(2\pi)^{-1/2}n^{-1/3}q(t_1)\sigma^{*-1}(K_3^{-1} + K_4) \log n + 2K_{19}n^{-1/2}]$, using (3.4) and the boundedness of the normal density. This last expression is $O(n^{-1/3} \log n)$. Hence there exists $A_3 > 0$ such that

$$S_{n3} \le A_3n^{-1/3} \log n. \tag{3.12}$$

Write S_{n1} in the form

$$S_{n1} = \textstyle\sum_{k\in I_n(z)} P\{U_n(t^*) = k/n\}\psi^2(\alpha_n(z - Y_n(k)) - \beta_n)$$

where $\alpha_n = (1 - 3A_1n^{-1/6} \log n)n^{1/6}\sigma^*K_9^{-1}$, $\beta_n = A_1n^{-1/6} \log n$. Then

$$\begin{aligned}
S_{n1} &= \int_0^\delta P\{z - \delta \le Y_n(nU_n(t^*)) \le z - u\}\, d\psi^2(\alpha_n u - \beta_n)\\
&\ge \int_0^\delta (\Phi(z - u) - \Phi(z - \delta) - 2K_{19}n^{-1/2})\, d\psi^2(\alpha_n u - \beta_n)\\
&\ge \int_0^\infty (\Phi(z - u) - \Phi(z - \delta) - 2K_{19}n^{-1/2})\, d\psi^2(\alpha_n u - \beta_n)\\
&= \int_0^\infty \Phi(z - u)\, d\psi^2(\alpha_n u - \beta_n) - \Phi(z - \delta) - 2K_{19}n^{-1/2}\\
&= \int_0^\infty \Phi(z - u - \alpha_n^{-1}\beta_n)\, d\psi^2(\alpha_n u) - \Phi(z - \delta) - 2K_{19}n^{-1/2}\\
&\ge \int_0^\infty \Phi(z - u)\, d\psi^2(\alpha_n u) - (2\pi)^{-1/2}\alpha_n^{-1}\beta_n - \Phi(z - \delta) - 2K_{19}n^{-1/2}.
\end{aligned}$$

Now $\alpha_n^{-1}\beta_n = (1 - 3A_1n^{-1/6} \log n)^{-1}\sigma^{*-1}K_9A_1n^{-1/3} \log n$ which is $O(n^{-1/3} \log n)$. Hence there exists $A_4 > 0$ such that, whenever $n \ge n_0$ and $|z| \le \log n$,

$$S_{n1} \ge \int_0^\infty \Phi(z - u)\, d\psi^2(\alpha_n u) - \Phi(z - \delta) - A_4n^{-1/3} \log n. \tag{3.13}$$

Equations (3.11), (3.12) and (3.13) give

$$\begin{aligned}
&P\{\sigma^*(z - \delta) \le n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^*z,\ n^{1/2}(Q_n^* - \mu^*) \le \sigma^*z\}\\
&\quad\ge \int_0^\infty \Phi(z - u)\, d\psi^2(\alpha_n u) - \Phi(z - \delta) - A_4n^{-1/3} \log n - A_2n^{-1} - A_3n^{-1/3} \log n.
\end{aligned} \tag{3.14}$$

Equations (3.7), (3.10) and (3.14) give

$$\begin{aligned}
P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^*z\} \ge \int_0^\infty \Phi(z - u)\, d\psi^2(\alpha_n u) - A_4n^{-1/3} \log n - A_2n^{-1}\\
- A_3n^{-1/3} \log n - K_{19}n^{-1/2} - K_{18}n^{-1}.
\end{aligned}$$

Hence there exists $A_5 > 0$ such that, whenever $n \geq n_0$ and $|z| \leq \log n$,

$$(3.15) \qquad P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* z\} \geq \int_0^\infty \Phi(z-u)\, d\psi^2(\alpha_n u) - A_5 n^{-1/3} \log n.$$

Recall that $\alpha_n = (1 - 3A_1 n^{-1/6} \log n) n^{1/6} \sigma^* K_9^{-1}$. An analogous argument gives the upper bound, with $\alpha_n = (1 + 3A_1 n^{-1/6} \log n) n^{1/6} \sigma^* K_9^{-1}$, of

$$(3.16) \qquad P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* z\} \leq \int_0^\infty \Phi(z-u)\, d\psi^2(\alpha_n u) + A_5 n^{-1/3} \log n.$$

Equation (3.16) will be useful later on, built for the present purpose we use the simpler upper bound

$$(3.17) \quad P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* z\} \leq P\{n^{1/2}(Q_n(t^*) - \mu^*) \leq \sigma^* z\} \leq \Phi(z) + K_{19} n^{-1/2}.$$

By Corollary 3.1,

$$(3.18) \qquad \left| \int_0^\infty \Phi(z-u)\, d\psi^2(\alpha_n u) - \Phi(z) \right| = O(\alpha_n^{-1}) = O(n^{-1/6}).$$

(3.15), (3.17) and (3.18) then give the result

$$|P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* z\} - \Phi(z)| = O(n^{-1/6})$$

whenever $n \geq n_0$ and $|z| \leq \log n$. But as noted at the beginning, this is sufficient for (3.5) to hold. □

3.2 An improvement upon the normal approximation. The purpose of this section is to define and study the improved approximation suggested by Theorem 2.2.

LEMMA 3.4. *There exists a positive constant K_{21} such that, whenever $0 < |\varepsilon| < \frac{1}{2}$ and $0 < \theta|\varepsilon| < e^{-1}$,*

$$\left| \int_{-\infty}^z \psi^2((z-u)(1-\varepsilon)\theta^{-1})\, d\Phi(u) - \int_{-\infty}^z \psi^2((z-u)\theta^{-1})\, d\Phi(u) \right| < K_{21} \theta |\varepsilon| (-\log \theta |\varepsilon|)^{2/3}$$

for all real z.

PROOF. We prove only for $\varepsilon > 0$; the proof for $\varepsilon < 0$ runs along the same lines. Given θ and ε, let a be a positive number to be specified below.

$$\begin{aligned}
\int_{-\infty}^z \psi^2((z-u)(1-\varepsilon)\theta^{-1})\, d\Phi(u) &= \int_0^\infty \Phi(z-u)\, d\psi^2(u(1-\varepsilon)\theta^{-1}) \\
&> \int_0^a \Phi(z - u/(1-\varepsilon))\, d\psi^2(u\theta^{-1}) \\
&\geq \int_0^a \Phi(z-u)\, d\psi^2(u\theta^{-1}) - \sup\{\Phi(z-u) - \Phi(z - u/(1-\varepsilon)) : 0 \leq u \leq a\} \\
&\geq \int_0^\infty \Phi(z-u)\, d\psi^2(u\theta^{-1}) - [1 - \psi^2(a\theta^{-1})] - (2\pi)^{-1/2} a\varepsilon/(1-\varepsilon) \\
&\geq \int_0^\infty \Phi(z-u)\, d\psi^2(u\theta^{-1}) - 2\exp(-K_{17}(a\theta^{-1})^{3/2}) - 2(2\pi)^{-1/2} a\varepsilon
\end{aligned}$$

where we have used Lemma 3.2 together with the obvious inequality $1 - \psi^2 \le 2(1 - \psi)$. Choose a such that $K_{17}(a\theta^{-1})^{3/2} = -\log \theta\varepsilon$, or $a = \theta(-K_{17}^{-1} \log \theta\varepsilon)^{2/3}$; then

$$\begin{aligned} 2 \exp(-K_{17}(a\theta^{-1})^{3/2}) + 2(2\pi)^{-1/2}a\varepsilon &= 2\theta\varepsilon + 2(2\pi)^{-1/2}\theta\varepsilon(-K_{17}^{-1} \log \theta\varepsilon)^{2/3} \\ &< \theta\varepsilon(-\log \theta\varepsilon)^{2/3}(2 + 2(2\pi)^{-1/2}K_{17}^{-2/3}) \\ &= \theta\varepsilon(-\log \theta\varepsilon)^{2/3}K_{21} \end{aligned}$$

with $K_{21} = (2 + 2(2\pi)^{-1/2}K_{17}^{-2/3}) = 2 + 3(8\pi)^{-1/2}$.

Then

$$\int_{\infty-}^{z} \psi^2((z-u)(1-\varepsilon)\theta^{-1})\, d\Phi(u) \ge \int_{-\infty}^{z} \psi^2((z-u)\theta^{-1})\, d\Phi(u) - K_{21}\theta\varepsilon(-\log \theta\varepsilon)^{2/3}.$$

Since also

$$\int_{-\infty}^{z} \psi^2((z-u)(1-\varepsilon)\theta^{-1})\, d\Phi(u) \le \int_{-\infty}^{z} \psi^2((z-u)\theta^{-1})\, d\Phi(u),$$

the result follows. □

The improved approximation is the convolution of the asymptotic distributions of $Q_n(t^*)$ and Δ_n. Recall that $K_9 = [-\mu''(t^*)/2]^{-1/3}[q(t^*)]^{4/3}$. Define:

$$\Phi_n(z) = \int_{-\infty}^{z} \psi^2(n^{1/6}\sigma^* K_9^{-1}(z-u))\, d\Phi(u). \tag{3.19}$$

It is proposed to use $\Phi_n(z)$ rather than $\Phi(z)$ as an approximation to $P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\}$.

Theorem 3.2. *There exists a constant $K_{22} > 0$ such that for all z and $n > 1$,*

$$|P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} - \Phi_n(z)| < K_{22}n^{-1/3}(\log n)^2. \tag{3.20}$$

Proof. By Equation (3.15), for $n \ge n_0$ and $|z| < \log n$,

$$P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} \ge \int_0^{\infty} \Phi(z-u)\, d\psi^2(\alpha_n u) - A_5 n^{-1/3} \log n$$

where $\alpha_n = (1 - \varepsilon)\theta^{-1}$ with $\varepsilon = 3A_1 n^{-1/6} \log n$, $\theta = n^{-1/6}\sigma^{*-1}K_9$. Applying Lemma 3.4, provided n_0 is large enough we have

$$\begin{aligned} &\left| \int_0^{\infty} \Phi(z-u)\, d\psi^2(\alpha_n u) - \int_0^{\infty} \Phi(z-u)\, d\psi^2(n^{1/6}\sigma^* K_9^{-1} u) \right| \\ &\qquad < K_{21} 3A_1 n^{-1/3}\sigma^{*-1}K_9 \log n[-\log(3A_1 n^{-1/3}\sigma^{*-1}K_9 \log n)]^{2/3} \\ &\qquad = o(n^{-1/3}(\log n)^2) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Then a constant K_{22} exists such that

$$P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} \ge \Phi_n(z) - K_{22}n^{-1/3}(\log n)^2$$

for all $n \ge n_0$ and all $|z| \le \log n$; but the extension to all z is immediate, for the same reasons as in Theorem 3.1, and then K_{22} may be chosen so that the result holds for all $n > 1$.

A similar argument, using Equation (3.16) and the second part of Lemma 3.4 (with $\varepsilon <$

0) gives

$$P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} \le \Phi_n(z) + K_{22} n^{-1/3}(\log n)^2.$$

This proves Equation (3.20). □

REMARKS. When assumption (III′) holds with $r \ge 2$, the definition of Φ_n becomes

$$\Phi_n(z) = \int_{-\infty}^{z} \psi_r^2(n^{1/(4r-2)}\sigma^* K_9^{-1}(z-u))\, d\Phi(u)$$

with $\psi_r(z) = P\{B(y) - y^r \le z \text{ for all } y \ge 0\}$,

$$K_9 = [-\mu_{(r)}(t^*)/r!]^{-1/(2r-1)}[q(t^*)]^{2r/(2r-1)}.$$

The results analogous to Theorems 3.1 and 3.2 are then

$$|P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} - \Phi(z)| < K_{20} n^{-1/(4r-2)},$$

$$|P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\} - \Phi_n(z)| < K_{22} n^{-1/(2r-1)}(\log n)^2.$$

Although this improved approximation has nice properties, it is not particularly easy to compute. Daniels (1974) and Barbour (1975), discussing a closely related problem arising in an epidemics model, proposed a simpler form of improved approximation. To introduce this, observe that Φ_n is the distribution function of

$$Z + n^{-1/6} K_9 \sigma^{*-1} W$$

where Z and W are independent random variables, Z being standard normal and W having distribution function ψ^2. The idea of Daniels and Barbour was to replace W by its expected value. Thus if $K_9 EW = b^*$, we replace $\Phi_n(z)$ by $\Phi(z - n^{-1/6} b^* \sigma^{*-1})$. This may be justified by the following lemma.

LEMMA 3.5.

$$\sup_z |\Phi_n(z + n^{-1/6} b^* \sigma^{*-1}) - \Phi(z)| = O(n^{-1/3}).$$

PROOF. By first conditioning on W, it may be seen that

$$\Phi_n(z) = E\{\Phi(z - n^{-1/6} K_9 \sigma^{*-1} W)\}.$$

Expanding in a Taylor series,

$$\begin{aligned}\Phi_n(z + n^{-1/6} b^* \sigma^{*-1}) &= E\{\Phi(z - n^{-1/6} K_9 \sigma^{*-1}(W - EW))\} \\ &= E\{\Phi(z) - n^{-1/6} K_9 \sigma^{*-1}(W - EW)\Phi'(z) \\ &\quad + \tfrac{1}{2}[n^{-1/6} K_9 \sigma^{*-1}(W - EW)]^2 \Phi''(\hat{z})\}\end{aligned}$$

where $\hat{z}$ depends on z and W. Hence

$$|\Phi_n(z + n^{-1/6} b^* \sigma^{*-1}) - \Phi(z)| = \tfrac{1}{2} n^{-1/3} K_9^2 \sigma^{*-2} |E\{(W - EW)^2 \Phi''(\hat{z})\}|.$$

But Φ'' is bounded, while Lemma 3.2 shows that W has finite variance. Hence the right hand side is bounded by a constant of order $n^{-1/3}$, independent of z. □

Lemma 3.5 together with Therorem 3.2 yields at once the following result:

THEOREM 3.3.

$$\sup_z |P\{n^{1/2}(Q_n^* - \mu^* - n^{-2/3} b^*) \le \sigma^* z\} - \Phi(z)| = O(n^{-1/3}(\log n)^2).$$

In its new form, the improved approximation to the distribution of Q_n^* is normal, with mean $\mu^* + n^{-2/3} b^*$ and standard deviation $n^{-1/2}\sigma^*$. Compared with Daniels' original approximation, the only change is to shift the mean by $n^{-2/3}b^*$. The only remaining problem is the calcuation of b^*. This, however, depends only on EW, which is a universal constant.

We may write

$$W = \sup\{B_i(t) - t^2 : i = 1, 2, t \geq 0\}$$

where B_1, B_2 are independent Brownian Motions with $B_i(0) = 0$. Barbour (1975) defined a universal constant λ by

$$\lambda = E\{\sup[B_i(t) - t^2/2 : i = 1, 2, t \geq 0]\}$$

and stated that $\lambda \approx 1$. More precise calcuations have shown $\lambda = 0.996$ to three figures (personal communication from Professor Daniels).

It is easily seen by a rescaling argument that

$$EW = 2^{-1/3}\lambda$$

which combined with the expression for K_9 yields

$$b^* = [-\mu''(t^*)]^{-1/3}[q(t^*)]^{4/3}\lambda.$$

SUMMARY. Daniels' (1945) approximation to the distribution of T_n is normal with mean μ^* and standard deviation $n^{-1/2}\sigma^*$. The error of approximation is uniformly $O(n^{-1/6})$. The improved approximation is normal with mean $\mu^* + n^{-2/3}b^*$ and the same standard deviation. The error is at most $O(n^{-1/3}(\log n)^2)$. The derivation of the approximation (see also Barbour (1975)) suggests that the error is in fact $O(n^{-1/3})$, but we do not have a rigorous proof of this.

4. A numerical example: Weibull distribution. In applications, the distribution

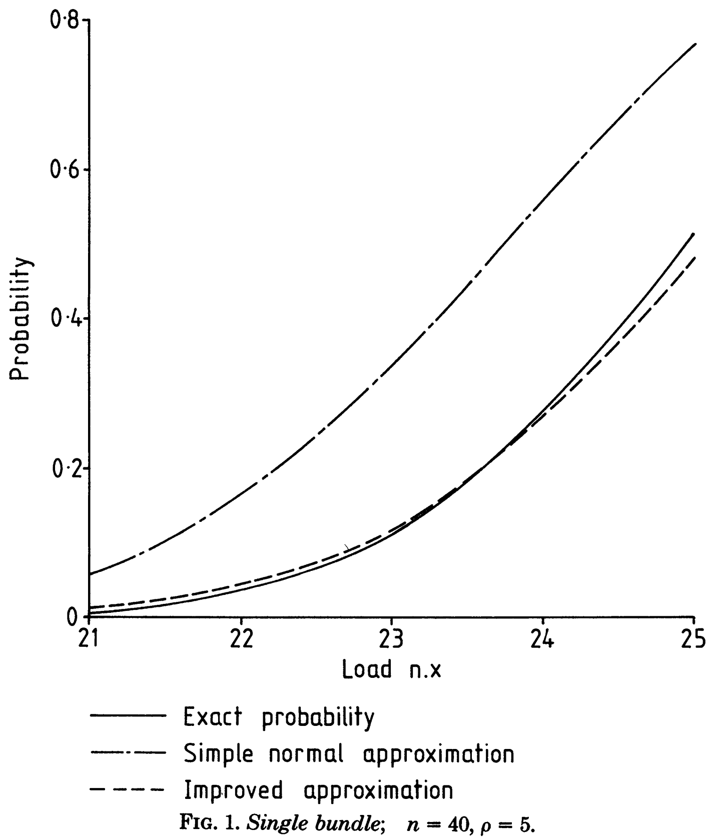

FIG. 1. *Single bundle*; $n = 40$, $\rho = 5$.

function F of fibre strengths is often taken as the Weibull distribution

$$F(x) = 1 - \exp(-cx^{\rho}), \qquad x \geq 0,$$

where c and ρ are positive parameters. This is sometimes justified by reference to the weakest-link property of materials; Coleman (1958) reviews the Weibull distribution in this context. By change of scale we may assume $c = 1$. Then $q = F^{-1}$ is given by

$$q(t) = [-\log(1 - t)]^{1/\rho}.$$

Straightforward calculations show that

$$\mu^* = \rho^{-1/\rho} e^{-1/\rho},$$

$$\sigma^* = [e^{-1/\rho}(1 - e^{-1/\rho})]^{1/2} \cdot \rho^{-1/\rho},$$

$$b^* = \rho^{-1/\rho - 1/3} e^{-1/3\rho}$$

(taking $\lambda = 1$ for practical purposes).

A numerical study was performed, using (1.4) for an exact calculation, with $n = 40$ and $\rho = 5$; the latter value is typical of glass or polymer fibres. Figure 1 shows the exact probability, the simple normal approximation of Daniels (1945), and the improved approximation. It is seen that the improved approximation really is a considerable practical improvement. Very similar results have been obtained for other values of n and ρ.

5. Large deviations. Let $\{x_n, n \geq 1\}$ be a sequence of positive numbers such that $x_n \to \infty$, $n^{-1/6} x_n \to 0$. This sequence is assumed to be fixed throughout this section.

It follows from classical theory that

$$\lim_{n \to \infty} P\{n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} / \Phi(-x_n) = 1. \tag{5.1}$$

The purpose of this section is to show that (5.1) continues to hold when $Q_n(t^*)$ is replaced by Q_n^*.

Equation (5.1) is a consequence of the Large Deviations theory of sums of independent, identically distributed random variables. $Q_n(t^*)$ is a constant multiple of $nU_n(t^*)$, which has the binomial distribution, and (5.1) follows from Theorem 1, Section XVI.7, page 549, of Feller (1971).

Define

$$F_n^*(x) = P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* x\},$$

$$F_n'(x) = P\{n^{1/2}(Q_n(t^*) - \mu^*) \leq \sigma^* x\}.$$

Then Equation (5.1) may be written as $F_n'(-x_n)/\Phi(-x_n) \to 1$.

THEOREM 5.1. *$F_n^*(-x_n)/\Phi(-x_n) \to 1$ as $n \to \infty$.*

PROOF. In view of (5.1), it suffices to show $F_n^*(-x_n)/F_n'(-x_n) \to 1$. But $Q_n^* \geq Q_n(t^*)$, so $F_n^*(x)/F_n'(x)$ is the same as the conditional probability $P\{n^{1/2}(Q_n^* - \mu^*) \leq \sigma^* x \mid n^{1/2}(Q_n(t^*) - \mu^*) \leq \sigma^* x\}$. It is therefore sufficient to show

$$P\{n^{1/2}(Q_n^* - \mu^*) > -\sigma^* x_n \mid n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} \to 0. \tag{5.2}$$

Let $\{y_n, n \geq 1\}$ be a sequence of positive numbers such that $x_n y_n \to 0$, $n^{1/6} y_n \to \infty$; for instance, take $y_n = (n^{1/6} x_n)^{-1/2}$. Now

$$\begin{aligned}
&P\{n^{1/2}(Q_n^* - \mu^*) > -\sigma^* x_n \mid n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} \\
&\quad \leq P\{n^{1/2}(Q_n(t^*) - \mu^*) > -\sigma^*(x_n + y_n) \mid n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} \\
&\qquad + P\{n^{1/2}(Q_n^* - Q_n(t^*)) > \sigma^* y_n \mid -2\sigma^* x_n < n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} \\
&\qquad + P\{n^{1/2}(Q_n(t^*) - \mu^*) \leq -2\sigma^* x_n \mid n^{1/2}(Q_n(t^*) - \mu^*) \leq -\sigma^* x_n\} \\
&\quad = P_{n1}' + P_{n2}' + P_{n3}', \quad \text{say.}
\end{aligned}$$

Now

$$P'_{n1} = 1 - F'_n(-x_n - y_n)/F'_n(-x_n)$$
$$= 1 - \frac{F'_n(-x_n - y_n)}{\Phi(-x_n - y_n)} \cdot \frac{\Phi(-x_n)}{F'_n(-x_n)} \cdot \frac{\Phi(-x_n - y_n)}{\Phi(-x_n)}.$$

But by 5.1, $F'_n(-x_n - y_n)/\Phi(-x_n - y_n) \to 1$, $\Phi(-x_n)/F'_n(-x_n) \to 1$. Also $\Phi(-x_n - y_n)/\Phi(-x_n) \sim (x_n/(x_n + y_n))\exp\{-(x_n + y_n)^2/2 + x_n^2/2\}$ which $\to 1$ by virtue of $x_n y_n \to 0$. Thus $P'_{n1} \to 0$. Next,

$$P'_{n3} = F'_n(-2x_n)/F'_n(-x_n)$$
$$= \frac{F'_n(-2x_n)}{\Phi(-2x_n)} \cdot \frac{\Phi(-x_n)}{F'_n(-x_n)} \cdot \frac{\Phi(-2x_n)}{\Phi(-x_n)}.$$

Again by (5.1), $F'_n(-2x_n)/\Phi(-2x_n) \to 1$ and $\Phi(-x_n)/F'_n(-x_n) \to 1$. Also $\Phi(-2x_n)/\Phi(-x_n) \to 0$. Thus $P'_{n3} \to 0$.

Define $I_n = \{k: -2\sigma^* x_n < n^{1/2}(q(t^*)(1 - k/n) - \mu^*) \le -\sigma^* x_n\}$. Then $P'_{n2} \le \sup\{P_{nk}(n^{1/2}\Delta_n > \sigma^* y_n): k \in I_n\}$.

But since $n^{-1/6}x_n \to 0$, it follows that $\sup\{n^{1/3}|k/n - t^*|: k \in I_n\} \to 0$. Hence by Theorem 2.1, for any $z > 0$,

$$\lim_{n\to\infty}\sup\{P_{nk}(\Delta_n > n^{-2/3}K_9 z): k \in I_n\} = 1 - \psi^2(z).$$

But $P_{nk}(\Delta_n > n^{-2/3}K_9 z) = P_{nk}(n^{1/2}\Delta_n > n^{-1/6}K_9 z) > P_{nk}(n^{1/2}\Delta_n > \sigma^* y_n)$, for n sufficiently large, for any arbitrarily large z.This is because $n^{1/6}y_n \to \infty$. Hence $\lim_{n\to\infty}\sup\{P_{nk}(n^{1/2}\Delta_n > \sigma^* y_n): k \in I_n\} = 0$, from which it follows that $P'_{n2} \to 0$.

So $P'_{n1} + P'_{n2} + P'_{n3} \to 0$, from which (5.2) and hence the theorem follow immediately. □

REMARKS. 1. In case that Assumption (III′) holds with $r > 2$, Theorem 5.1 holds under the condition that $x_n n^{-1/(4r-2)} \to 0$.

2. A natural question is whether the analogous result holds for the upper tail; i.e., whether $(1 - F_n^*(x_n))/(1 - \Phi(x_n)) \to 1$. The answer is no. Stronger assumptions than have been made are required on the behaviour of $q(t)$ as $t \to 1$.

6. A limit thoerem for the series-parallel model. Assume we are given a sequence $\{K(n), n \ge 1\}$ of positive integers tending to infinity. For each n, consider the system consisting of $K(n)$ independent bundles, each of n fibres, in series. The strength of the system is the minimum of $K(n)$ independent copies of Q_n^*. If $G_n(x)$ is the probability that the system fails under load x, then G_n is related to F_n through the formula

$$G_n(x) = 1 - (1 - F_n(x))^{K(n)}. \tag{6.1}$$

We would like to find constants $a_n > 0$ and b_n real and a limit distribution function H such that, for all real x,

$$G_n(a_n x + b_n) \to H(x) \quad \text{as} \quad n \to \infty. \tag{6.2}$$

Of course the limit behaviour of G_n as $n \to \infty$ depends on the sequence $\{K(n)\}$. Here we consider the case where $K(n)$ increases "not too fast"; the precise condition is $\log K(n) = o(n^{1/3})$. The opposite extreme, where $K(n) \to \infty$ "very fast", is studied by Harlow, Smith and Taylor (1978). More precisely, they study the case $K \to \infty$, n fixed, and show under quite general conditions that a limit theorem holds with the Weibull distribution as the limit H. The theorem here leads to the limit law $H(x) = 1 - \exp(-\exp(x))$. Thus the manner in which the constants K and n vary determines the form of the limit law as well as the renormalization constants.

THEOREM 6.1. *Assume G_n is defined by* (6.1) *and that* $K(n) \to \infty$, $n^{-1/3}\log K(n) \to 0$

as $n \to \infty$. Define

$$a_n = \sigma^*(2n \log K(n))^{-1/2},$$

$$b_n = \mu^* + \sigma^*(\log(4\pi) + \log\log K(n) - 4\log K(n))\cdot(8n\log K(n))^{-1/2}.$$

Then for all real x,

$$G_n(a_n x + b_n) \to 1 - \exp(-\exp(x)) \quad as \quad n \to \infty. \tag{6.3}$$

PROOF. Given (6.1), the result (6.3) is equivalent to

$$K(n)F_n(a_n x + b_n) \to e^x \quad \text{as} \quad n \to \infty, \quad \text{for all real} \quad x. \tag{6.4}$$

Let $\alpha_n = (2\log K(n))^{-1/2}$, $\beta_n = (\log(4\pi) + \log\log K(n) - 4\log K(n))\cdot(8\log K(n))^{-1/2}$. Here $\beta_n \to -\infty$, $\alpha_n/\beta_n \to 0$, $-\beta_n = O((\log K(n))^{1/2})$. It follows from the approximation $\Phi(-x) = (2\pi)^{-1/2}x^{-1}\exp(-x^2/2)(1 + o(1))$ as $x \to \infty$ that

$$K(n)\Phi(\alpha_n x + \beta_n) \to e^x \quad \text{as} \quad n \to \infty, \quad \text{for all real} \quad x. \tag{6.5}$$

Fix an x. Then $-(\alpha_n x + \beta_n)$ is $O((\log K(n))^{1/2})$ which is $o(n^{1/6})$. Then by Theorem 5.1,

$$F_n^*(\alpha_n x + \beta_n)/\Phi(\alpha_n x + \beta_n) \to 1. \tag{6.6}$$

Hence from (6.5) and (6.6),

$$K(n)F_n^*(\alpha_n x + \beta_n) \to e^x. \tag{6.7}$$

But $F_n^*(x) = F_n(\mu^* + n^{-1/2}\sigma^* x)$. It follows from the definition of a_n and b_n that $F_n^*(\alpha_n x + \beta_n) = F_n(a_n x + b_n)$. Hence (6.7) is equivalent to (6.4). This is what we had to prove. □

REMARKS. 1. In case that Assumption (III′) holds with $r > 2$, Theorem 6.1 holds under the stronger condition $K(n) \to \infty$, $n^{-1/(2r-1)}\log K(n) \to 0$.

2. For a more detailed discussion of the connection between Large Deviations theory and extreme value theory, see the thesis of Borges (1978), which discusses a similar fibre-bundle model.

7. An improved lower tail approximation. Although we have now obtained a limit theorem for the series-parallel model, we have not discussed the possibility of using an improved approximation to obtain strengthened versions of Theorems 5.1 and 6.1. A natural approach is to use the improved approximation of Section 3 in order to reduce the error, but the objective here is to extend the zone within which the relative error tends to zero. For this it is necessary to use a "lower tail improvement," designed specifically to reduce the relative error in the lower tail of the distribution.

The results presented are motivated by the work of Borges (1978) on the relation between Large Deviations theory and extreme values for row sums of triangular arrays. Let $\{W_{nj}, n \geq 1, 1 \leq j \leq n\}$ be a triangular array of random variables with $EW_{nj} = 0$, $\sum_{j=1}^n EW_{nj}^2 = 1$ for each n. Let W_n be the probability distribution function of the row sum $\sum_{j=1}^n W_{nj}$. According to a theorem of Book (1976), if the array $\{W_{nj}\}$ satisfies certain conditions then

$$1 - W_n(x_n) = (1 - \Phi(x_n))\exp\{n^{-1/2}x_n^3\lambda_n(n^{-1/2}x_n) + O(n^{-1/2}x_n)\}. \tag{7.1}$$

$$W_n(-x_n) = \Phi(-x_n)\exp\{-n^{-1/2}x_n^3\lambda_n(-n^{-1/2}x_n) + O(n^{-1/2}x_n)\} \tag{7.2}$$

whenever $x_n \to \infty$, $x_n = o(n^{1/2})$. Here λ_n is a power series whose coefficients are determined by the cumulants of $\sum_{j=1}^n W_{nj}$.

The expressions $(1 - \Phi(x_n))\exp\{n^{-1/2}x_n^3\lambda_n(n^{-1/2}x_n)\}$ and $\Phi(-x_n)\exp\{-n^{-1/2}x_n^3\lambda_n(-n^{-1/2}x_n)\}$ may be regarded as upper and lower tail improvements on the normal approximations to $1 - W_n(x_n)$ and $W_n(-x_n)$ respectively. Borges shows that (7.2) may be used to define constants a_n, b_n such that

$$1 - (1 - W_n(a_n x + b_n))^{K(n)} \to 1 - \exp(-\exp(x)) \tag{7.3}$$

whenever $\log K(n) = o(n)$; an analogous relation exists between (7.1) and the maximum df $W_n^{K(n)}$.

For our problem it is not possible to get such complete and elegant results but we do obtain a lower tail improvement which has performed well in empirical studies.

Throughout this section $\{x_n, n \geq 1\}$ is taken to be a fixed sequence of positive numbers such that $x_n \to \infty$, $x_n = o(n^{1/6}(\log n)^{1/3})$. For real a define

$$\psi(z; a) = P\{B(y) - y^2 + ay \leq z \quad \text{for all} \quad y \geq 0\}$$

when $\{B(y), y \geq 0\}$ is a Brownian motion with $B(0) = 0$. Also let $a_n = n^{-1/6}x_n((1 - t^*)/t^*)^{1/2}K_{14}$ (recall $K_{14} = q'(t^*)q(t^*)^{-2/3}K_1^{-1/3}$ and $K_1 = -\mu''(t^*)/2$) and define

$$\Omega_n(z) = \psi(z; a_n)\psi(z).$$

Here $\psi(z)$ has the same meaning as before; it is the same as $\psi(z; 0)$.

The following theorem will be proved:

THEOREM 7.1.

$$F_n^*(-x_n) \sim \exp(-\lambda_1 n^{-1/2}x_n^3) \int_{-\infty}^{-x_n} \Omega_n(n^{1/6}\sigma^* K_9^{-1}(-x_n - u))\, d\Phi(u) \text{ as } n \to \infty.$$

where

$$\lambda_1 = (1 - 2t^*)[36t^*(1 - t^*)]^{-1/2}.$$

(The symbol "$\sim$" here and subsequently means that the ratio of the two sides tends to 1.)

Let $\{\delta_n, n \geq 1\}$ be an arbitrary sequence of positive numbers such that $\delta_n \to 0$, $\delta_n n^{1/4} \cdot x_n^{-1} \to \infty$. As in Section 3, define

$$Y_n(k) = n^{1/2}(q(t^*)(1 - k/n) - u^*)/\sigma^* = n^{1/2}(t^* - d_{nk})[t^*(1 - t^*)]^{-1/2},$$

and let

$$\begin{aligned} I_n &= \{k: - x_n - \delta_n \leq Y_n(k) \leq - x_n\} \\ &= \{k: x_n((1 - t^*)t^*)^{1/2} \leq n^{1/2}(d_{nk}/t^* - 1) \leq (x_n + \delta_n)((1 - t^*)t^*)^{1/2}\}. \end{aligned}$$

We aim first to prove a sharpened version of Lemma 2.12 for $P_{nk}\{\Delta_{n2} \leq z\}$, for $k \in I_n$. This will appear as Lemma 7.3. Recall from (2.15),

$$\begin{aligned} \Delta_{n2}^* = \sup\{&-[\mu(t^*) - \mu(t^*/(1 + x))]\, d_{nk}/t^* + [q(t^*) - q(t^*/(1 + x))] \\ &\cdot (d_{nk}/t^* - 1) + d_{nk}^{1/2}n^{-1/2}(1 + x)^{-1}q(t^*/(1 + x))B(x) : x \geq 0\} \end{aligned} \tag{7.4}$$

where $\{B(x), x \geq 0\}$ is a Brownian motion with $B(0) = 0$. For $\varepsilon > 0$ define

$$\begin{aligned} {}^{\varepsilon}\Delta_{n3} = \sup\{&-(1 - \varepsilon)t^*K_1x^2\, d_{nk} + (1 + \varepsilon)q'(t^*)(d_{nk}/t^* - 1)t^*x \\ &+ d_{nk}^{1/2}n^{-1/2}q(t^*)(1 + \varepsilon)B(x) : x \geq 0\}, \end{aligned} \tag{7.5}$$

$$\begin{aligned} {}_{\varepsilon}\Delta_{n3} = \sup\{&-(1 + \varepsilon)t^*K_1x^2\, d_{nk} + (1 - \varepsilon)q'(t^*)(d_{nk}/t^* - 1)t^*x \\ &+ d_{nk}^{1/2}n^{-1/2}q(t^*)(1 - \varepsilon)B(x) : x \geq 0\}. \end{aligned} \tag{7.6}$$

Then provided $d_{nk} > t^*$ (which it is on I_n), the inequality ${}_{\varepsilon}\Delta_{n3} \leq \Delta_{n2}^* \leq {}^{\varepsilon}\Delta_{n3}$ is satisfied on the set $E_3(\varepsilon)$ on which the suprema in (7.4) and (7.6) are achieved in $0 \leq x \leq K_{10}\varepsilon$ with K_{10} as before.

LEMMA 7.1. *There exist positive constants K_{24}, K_{25} such that, if $0 < \varepsilon < x_0/K_{10}$ and $1 < d_{nk}/t^* < 1 + K_{24}\varepsilon$ then*

$$P_{nk}\{E_3(\varepsilon)\} \geq 1 - 3\exp(-K_{25}\varepsilon^3 n d_{nk}).$$

PROOF. This is proved along the same lines as Lemma 2.10. □

Given $\varepsilon_1 > 0$, choose $\varepsilon > 0$ such that

$$(1+\varepsilon)^{4/3}(1-\varepsilon)^{-1/3} < (1-\varepsilon_1)^{-1},$$
$$(1-\varepsilon)^{4/3}(1+\varepsilon)^{-1/3} > (1+\varepsilon_1)^{-1},$$
$$(1+\varepsilon)\cdot(1+\varepsilon)^{-2/3}(1-\varepsilon)^{-1/3} < 1+\varepsilon_1,$$
$$(1-\varepsilon)\cdot(1-\varepsilon)^{-2/3}(1+\varepsilon)^{-1/3} > 1-\varepsilon_1.$$

These inequalities may be satisfied with $\varepsilon = K_{26}\varepsilon_1$ in the range $0 < \varepsilon_1 < x_0/(K_{10}K_{26})$, for a suitable positive constant K_{26}.

Using a time rescaling as before, we obtain when $d_{nk} > t^*$

$$(7.7)\qquad {}^{\varepsilon}\Delta_{n3} < (1-\varepsilon_1)^{-1}h_1(d_{nk})K_9 n^{-2/3} \cdot\sup\{B(y) - y^2 + (1+\varepsilon_1)(d_{nk}/t^* - 1)n^{1/3}h_2(d_{nk})K_{14}y : y \geq 0\},$$

$$(7.8)\qquad {}_{\varepsilon}\Delta_{n3} > (1+\varepsilon_1)^{-1}h_1(d_{nk})K_9 n^{-2/3} \cdot\sup\{B(y) - y^2 + (1-\varepsilon_1)(d_{nk}/t^* - 1)n^{1/3}h_2(d_{nk})K_{14}y : y \geq 0\}.$$

We now require a lemma concerning the continuity of the function $\psi(z; a)$.

LEMMA 7.2. *If* $a > 0$, $z > 0$ *and* $0 < \varepsilon < 1$, *then*

$$(7.9)\qquad \psi(z(1-\varepsilon) - 2a^2\varepsilon; a) \leq \psi(z; a(1+\varepsilon)) \leq \psi(z; a),$$

$$(7.10)\qquad \psi(z; a) \leq \psi(z; a(1-\varepsilon)) \leq \psi(z(1+\varepsilon) + 2a^2\varepsilon; a).$$

PROOF. We shall only prove (7.10); (7.9) is very similar. Note that $\psi(z; a)$ is nondecreasing in z and nonincreasing in a. The left-hand inequality in (7.10) is trivial and we therefore concentrate on the right-hand inequality.

Since $y^2 - a(1-\varepsilon)y \leq (1+\varepsilon)y^2 - a(1+\varepsilon)y + a^2\varepsilon$ for all y, we have

$$\begin{aligned}\psi(z; a(1-\varepsilon)) &= P\{B(y) \leq y^2 - a(1-\varepsilon)y + z \quad \text{for all} \quad y \geq 0\}\\ &\leq P\{B(y) \leq (1+\varepsilon)y^2 - a(1+\varepsilon)y + z + a^2\varepsilon \quad \text{for all} \quad y \geq 0\}\\ &= P\{(1+\varepsilon)^{-1/3}B(t) \leq (1+\varepsilon)^{-1/3}t^2 - a(1+\varepsilon)^{1/3}t + z + a^2\varepsilon\\ &\qquad \text{for all} \quad y = t(1+\varepsilon)^{-2/3} \geq 0\}\\ &= \psi((z + a^2\varepsilon)(1+\varepsilon)^{1/3}; a(1+\varepsilon)^{2/3})\\ &\leq \psi(z(1+\varepsilon) + 2a^2\varepsilon; a). \quad \square\end{aligned}$$

Return to (7.7). For $k \in I_n$, $(d_{nk}/t^* - 1) \leq n^{-1/2}(x_n + \delta_n)((1-t^*)/t^*)^{1/2}$. Also $h_2(d_{nk}) - 1 = O(d_{nk}/t^* - 1) = O(n^{-1/2}x_n) = o(\delta_n x_n^{-1})$ as $n \to \infty$. Define $\varepsilon_1 \equiv \varepsilon_{n1} = n^{-1/4}$. So $\varepsilon_{n1} = o(\delta_n x_n^{-1})$ as $n \to \infty$. Then

$$(1+\varepsilon_{n1})(d_{nk}/t^* - 1)h_2(d_{nk}) < n^{-1/2}x_n(1 + 2\delta_n x_n^{-1})((1-t^*)/t^*)^{1/2}$$

for $k \in I_n$, for all n sufficiently large. Hence (7.7) becomes

$$(7.11)\qquad {}^{\varepsilon}\Delta_{n3} < (1-\varepsilon_{n1})^{-1}h_1(d_{nk})K_9 n^{-2/3}\sup\{B(y) - y^2 + (1 + 2\delta_n x_n^{-1})a_n y : y \geq 0\}$$

with $a_n = n^{-1/6}x_n((1-t^*)/t^*)^{1/2}K_{14}$.

Similarly, (7.8) becomes

$$(7.12)\qquad {}_{\varepsilon}\Delta_{n3} > (1+\varepsilon_{n1})^{-1}h_1(d_{nk})K_9 n^{-2/3}\sup\{B(y) - y^2 + (1-2\delta_n x_n^{-1})a_n y : y \geq 0\}.$$

Using (7.11) and (7.12) together with Lemma 7.2, the inequalities

$$(7.13)\qquad P_{nk}\{{}^{\varepsilon}\Delta_{n3} \leq z\} \geq \psi((1-n^{-1/4})n^{2/3}K_9^{-1}h_1(d_{nk})^{-1}z(1-2\delta_n x_n^{-1}) - 2a_n^2(2\delta_n x_n^{-1}); a_n),$$

$$(7.14)\qquad P_{nk}\{{}_{\varepsilon}\Delta_{n3} \leq z\} \leq \psi((1+n^{-1/4})n^{2/3}K_9^{-1}h_1(d_{nk})^{-1}z(1+2\delta_n x_n^{-1}) + 2a_n^2(2\delta_n x_n^{-1}); a_n),$$

hold for all $k \in I_n$, for all n sufficiently large.

LEMMA 7.3. *Let $\varepsilon_{n2} > 0$ for each n. Then the inequalities*

$$\begin{aligned}
&\psi((1-n^{-1/4})(1-\varepsilon_{n2})(1-2\delta_n x_n^{-1})n^{2/3}K_9^{-1}h_1(d_{nk})^{-1}z - 2a_n^2(2\delta_n x_n^{-1}); a_n)\\
&\quad - 3\exp(-K_{25}K_{26}^3 n^{1/4}d_{nk}) - K_2\exp\{-K_3(n\varepsilon_{n2}zq(t^*)^{-1} - K_4\log n)_+\}\\
&\qquad \leq P_{nk}\{\Delta_{n2} \leq z\}\\
&\qquad \leq \psi((1+n^{-1/4})(1+\varepsilon_{n2})(1+2\delta_n x_n^{-1})n^{2/3}K_9^{-1}h_1(d_{nk})^{-1}z + 2a_n^2(2\delta_n x_n^{-1}); a_n)\\
&\qquad + 3\exp(-K_{25}K_{26}^3 n^{1/4}d_{nk}) + K_2\exp\{-K_3(n\varepsilon_{n2}zq(t^*)^{-1} - K_4\log n)_+\}
\end{aligned}$$

hold for all $k \in I_n$, for all n sufficiently large.

PROOF. For fixed n let $\varepsilon_{n1} = n^{-1/4}$, $\varepsilon = K_{26}\varepsilon_{n1}$. For $k \in \mathrm{I}_n$,

$$1 < d_{nk}/t^* \leq 1 + n^{-1/2}(x_n + \delta_n)((1-t^*)/t^*)^{1/2} = 1 + o(\varepsilon_{n1}) \quad \text{as} \quad n \to \infty.$$

Hence the condition for Lemma 7.1 is satisfied for $k \in I_n$ for all n sufficiently large. The result then follows from the representations (7.13), (7.14) together with Lemma 2.9 and Lemma 7.1. □

Lemma 7.3 will be a key tool in the proof of Theorem 7.1. Before proceeding, we prove one additional technical lemma.

LEMMA 7.4. *If $4z \geq (a+2)^2$ then $\psi(z; a) > 1/2$, for $a > 0$.*

PROOF. The condition on z guarantees $t + 1/2 \leq t^2 - at + z$ for all t, for the latter inequality is satisfied if $4z \geq (a+1)^2 + 2$, and $(a+2)^2 > (a+1)^2 + 2$ for $a > 0$. But then

$$\begin{aligned}
\psi(z; a) &= P\{B(t) \leq t^2 - at + z \quad \text{for all} \quad t \geq 0\}\\
&\geq P\{B(t) \leq t + 1/2 \quad \text{for all} \quad t \geq 0\}\\
&= 1 - e^{-1} \quad \text{(by Lemma 2.5)}\\
&> 1/2. \ \square
\end{aligned}$$

The proof of Theorem 7.1 uses the local large deviations theory for binomial random variables. Fix p such that $0 < p < 1$ and let $q = 1 - p$. For $n \geq 1$, $0 \leq k \leq n$ define

$$\pi_n(k) = \binom{n}{k}p^k q^{n-k},$$

$$\Pi_n(k) = \textstyle\sum_{k' \leq k} \pi_n(k').$$

Thus π_n, Π_n are respectively the density function and the distribution function of a binomial random variable with parameters n and p. Define

$$\begin{aligned}
\lambda_p(t) &= -\textstyle\sum_{k=0}^{\infty} (p^{k+2} - (-q)^{k+2})(k+3)^{-1}(k+2)^{-1}(pq)^{-((k+1)/2)}t^k\\
&= (q-p)/(6p^{1/2}q^{1/2}) - t(p^3+q^3)/(12pq) + \cdots
\end{aligned}$$

which is a power series in t, convergent for $|t|$ sufficiently small.

Given n and k, let $z_{nk} = (k - np)(npq)^{-1/2}$; $\phi(z) = (2\pi)^{-1/2}\exp(-z^2/2)$. Then if $1 \le z_{nk} = o(n^{1/2})$, we have

$$\pi_n(k) = (pqn)^{-1/2}\phi(z_{nk})\exp\{n^{-1/2}z_{nk}^3\lambda_p(n^{-1/2}z_{nk}) + O(n^{-1/2}\,|\,z_{nk}\,|)\}, \tag{7.15}$$

$$1 - \Pi_n(k) = (1 - \Phi(z_{nk}))\exp\{n^{-1/2}z_{nk}^3\lambda_p(n^{-1/2}z_{nk}) + O(n^{-1/2}z_{nk})\}, \tag{7.16}$$

while if $1 < -z_{nk} = o(n^{1/2})$, (7.15) holds as stated and (7.16) becomes

$$\Pi_n(k) = \Phi(z_{nk})\exp\{n^{-1/2}z_{nk}^3\lambda_p(n^{-1/2}z_{nk}) + O(n^{-1/2}\,|\,z_{nk}\,|)\}. \tag{7.17}$$

(7.15) is due to Khincin (1929); see Richter (1957) for discussion and extensive generalizations. The integrated versions (7.16), (7.17) follow from Khincin's results or from the general theorem on page 552 of Feller (1971).

These results will be applied in the case $p = t^*$. We write $\lambda(t)$ in place of $\lambda_{t^*}(t)$.

Proof of Theorem 7.1. The proof mimics that of Theorem 3.1, but using Lemma 7.3 in place of Lemma 2.12 and the large deviations results (7.15)–(7.17) in place of the Berry-Esseen Theorem. Recall that the sequence $\{x_n\}$ is fixed throughout.

Let $\varepsilon_{n2} = n^{-1/4}$. From Lemma 2.8 and Lemma 7.3, for $k \in I_n$ we have

$$\begin{aligned} P_{nk}\{\Delta_n \le z\} \ge\ & \Omega_n((1 - 3\delta_n x_n^{-1})n^{2/3}K_9^{-1}z - 4\delta_n x_n^{-1}a_n^2) \\ & - 2K_2\exp\{-K_3(n^{3/4}zq(t_1)^{-1} - K_4\log n)_+\} - O(n^{-1}). \end{aligned} \tag{7.18}$$

We have used the fact that $\varepsilon_{n2} = n^{-1/4} \ll \delta_n x_n^{-1}$ as $n \to \infty$. Let $\alpha_n = (1 - 3\delta_n x_n^{-1})n^{1/6}K_9^{-1}\sigma^*$ and $\beta_n = 4\delta_n x_n^{-1}a_n^2$. Then

$$\begin{aligned} P_{nk}\{n^{1/2}\Delta_n \le \sigma^* z\} \ge\ & \Omega_n(\alpha_n z - \beta_n) \\ & - 2K_2\exp\{-K_3(n^{1/4}z\sigma^* q(t_1)^{-1} - K_4\log n)_+\} - O(n^{-1}) \end{aligned} \tag{7.19}$$

for $k \in I_n$. Recall $F_n'(z) = P\{n^{1/2}(Q_n(t^*) - \mu^*) \le \sigma^* z\}$, $F_n^*(z) = P\{n^{1/2}(Q_n^* - \mu^*) \le \sigma^* z\}$. By (7.17), $F_n'(-x_n) \sim \Phi(-x_n)\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\}$ as $n \to \infty$.

As in (3.7), the probability $F_n^*(-x_n)$ may be written

$$\begin{aligned} F_n^*(-x_n) = P\{&\sigma^*(-x_n - \delta_n) \le n^{1/2}(Q_n(t^*) - \mu^*) \\ &\le \sigma^*(-x_n),\ n^{1/2}(Q_n^* - \mu^*) \le -\sigma^* x_n\} \\ &+ P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(-x_n - \delta_n),\ n^{1/2}(Q_n^* - \mu^*) \le -\sigma^* x_n\}. \end{aligned} \tag{7.20}$$

But

$$P\{n^{1/2}(Q_n(t^*) - \mu^*) < \sigma^*(-x_n - \delta_n),\ n^{1/2}(Q_n^* - \mu^*) \le -\sigma^* x_n\} \le F_n'(-x_n - \delta_n)$$

and

$$\begin{aligned} &F_n'(-x_n - \delta_n)/F_n'(-x_n) \\ &\quad\sim (\Phi(-x_n - \delta_n)/\Phi(-x_n))\exp\{-n^{-1/2}(x_n + \delta_n)^3\lambda(-n^{-1/2}(x_n + \delta_n)) + n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\} \\ &\quad\sim \exp\{-(x_n + \delta_n)^2/2 + x_n^2/2 - n^{-1/2}(x_n + \delta_n)^3\lambda(-n^{-1/2}(x_n + \delta_n)) + n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\} \\ &\quad\sim \exp(-x_n\delta_n). \end{aligned}$$

Thus

$$\begin{aligned} P\{n^{1/2}(Q_n(t^*) - \mu^*) &< \sigma^*(-x_n - \delta_n),\ n^{1/2}(Q_n^* - \mu^*) \le -\sigma^* x_n\} \\ &\le F_n'(-x_n)\exp\{-x_n\delta_n + o(1)\} \end{aligned} \tag{7.21}$$

as $n \to \infty$. Using (7.19), we have for the first term in (7.20),

$$P\{\sigma^*(-x_n-\delta_n) \le n^{1/2}(Q_n(t^*)-\mu^*) \le \sigma^*(-x_n),\ n^{1/2}(Q_n^*-\mu^*) \le -\sigma^* x_n\}$$
$$= \textstyle\sum_{k\in I_n} P\{U_n(t^*)=k/n\}P_{nk}\{n^{1/2}\Delta_n \le \sigma^*(-x_n - Y_n(k))\}$$
$$\ge \textstyle\sum_{k\in I_n} P\{U_n(t^*)=k/n\}(\Omega_n(a_n(-x_n-Y_n(k)))-\beta_n)$$
$$- 2K_2\exp\{-K_3(n^{1/4}(-x_n-Y_n(k))\sigma^* q(t_1)^{-1} - K_4\log n)_+\} - O(n^{-1}))$$
$$= S_{n1} - S_{n3} - S_{n2}, \quad \text{say.}$$

If, in addition, S_{n4} denotes the quantity in (7.21), we have, from (7.20), $F_n^*(-x_n) \ge S_{n1} - S_{n2} - S_{n3} + S_{n4}$.

The program is now as follows. It will be shown that $(S_{n2}+S_{n3}+S_{n4})/S_{n1} \to 0$; hence $F_n^*(-x_n) \ge S_{n1}(1+o(1))$. It will then be shown that

$$S_{n1} \sim \exp(-\lambda_1 n^{-1/2}x_n^3)\int_{-\infty}^{-x_n} \Omega_n(n^{1/6}\sigma^* K_9^{-1}(-x_n-u))\, d\Phi(u)$$

which proves half the result. The opposite half will follow by a symmetrical argument to get an upper bound on $F_n^*(-x_n)$. We shall not give the details of this; note, however, that it is for this reason that we need to worry about S_{n4}.

Now

$$S_{n2} = O(n^{-1})\textstyle\sum_{k\in I_n} P\{U_n(t^*)=k/n\} = O(n^{-1}F_n'(-x_n)). \tag{7.22}$$

Also,

$$S_{n3} = \textstyle\sum_{k\in I_n} P\{U_n(t^*)=k/n\}2K_2\exp\{-K_3(n^{1/4}(-x_n-Y_n(k))\sigma^* q(t_1)^{-1} - K_4\log n)_+\}.$$

Break the sum into two parts: part 1 for which $K_3(n^{1/4}(-x_n-Y_n(k))\sigma^* q(t_1)^{-1} - K_4\log n) > \log n$, and part 2 the rest. Then part 1 is less than $2K_2 n^{-1}F_i'(-x_n)$, and part 2 is less than $2K_2$ times

$$P\{-x_n - n^{-1/4}q(t_1)\sigma^{*-1}(K_3^{-1}+K_4)\log n \le Y_n(nU_n(t^*)) \le -x_n\}$$
$$\sim n^{-1/4}q(t_1)\sigma^{*-1}(K_3^{-1}+K_4)\log n\,\phi(-x_n)\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\}$$
$$\sim n^{-1/4}q(t_1)\sigma^{*-1}(K_3^{-1}+K_4)\log n\, x_n\Phi(-x_n)\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\}$$
$$\sim n^{-1/4}q(t_1)\sigma^{*-1}(K_3^{-1}+K_4)\log n\, x_n F_n'(-x_n)$$

where we have used first (7.15) and second (7.17). Hence

$$S_{n3} = O(n^{-1/4}x_n \log n F_n'(-x_n)). \tag{7.23}$$

Finally

$$S_{n1} = \textstyle\sum_{k\in I_n} P\{U_n(t^*)=k/n\}\Omega_n(\alpha_n(-x_n-Y_n(k))-\beta_n)$$
$$\sim \left(\int_{-x_n-\delta_n}^{-x_n} \phi(u)\Omega_n(\alpha_n(-x_n-u)-\beta_n)\,du\right)\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\}$$
$$\sim \left(\int_{-x_n-\delta_n}^{-x_n} \phi(u)\Omega_n(\alpha_n(-x_n-u)-\beta_n)\,du\right)F_n'(-x_n)/\Phi(-x_n)$$

again using (7.15) and then (7.17). But

$$\int_{-x_n-\delta_n}^{-x_n} \phi(u)\Omega_n(\alpha_n(-x_n - u) - \beta_n)\, du = \int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n u - \beta_n)\, du$$

$$= \int_0^{\delta_n-\beta_n/\alpha_n} \phi(-x_n - u - \beta_n/\alpha_n)\Omega_n(\alpha_n u)\, du$$

$$\sim \int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n u)\, du$$

provided $x_n\beta_n/\alpha_n \to 0$ and $\delta_n \gg \beta_n/\alpha_n$ as $n \to \infty$. But $\alpha_n = O(n^{1/6})$, $\beta_n = O(\delta_n n^{-1/3}x_n)$ so $x_n\beta_n/\alpha_n = O(\delta_n n^{-1/2}x_n^2)$ and $\beta_n/(\alpha_n\delta_n) = O(n^{-1/2}x_n)$, both of which tend to zero as required. Then we have

$$S_{n1} \sim \left(\int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n u)\, du\right) (F_n'(-x_n)/\Phi(-x_n)). \tag{7.24}$$

Now $\Omega_n(z) = \psi(z; a_n)\psi(z) > \psi(z; a_n)^2$. By Lemma 7.4, a sufficient condition for $\Omega_n(\alpha_n u) > 1/4$ is $u \ge u_n = (a_n + 2)^2/(4\alpha_n)$. Then provided $u_n < \delta_n$

$$S_{n1} > \left(\frac{1}{4}\right)\left(\int_{u_n}^{\delta_n} \phi(-x_n - u)\, du\right)(F_n'(-x_n)/\Phi(-x_n))(1 + o(1))$$

$$\sim \frac{1}{4}[(\Phi(-x_n - u_n) - \Phi(-x_n - \delta_n))/\Phi(-x_n)]F_n'(-x_n).$$

If $x_n = O(n^{1/6})$ then $u_n = o(n^{-1/6})$ and $u_n\delta_n^{-1} \to 0$ if δ_n is chosen so that $\delta_n \to 0$, $x_n\delta_n \to \infty$. Then $[\Phi(-x_n - u_n) - \Phi(-x_n - \delta_n)]/\Phi(-x_n) \sim \exp(-x_n u_n)$ which is bounded below as $n \to \infty$. In this case it is clear that $(S_{n2} + S_{n3} + S_{n4})/S_{n1} \to 0$. If $n^{-1/6}x_n \to \infty$ then $u_n = O(n^{-1/2}x_n^2) = o(\delta_n)$ so that $(\Phi(-x_n - u_n) - \Phi(-x_n - \delta_n))/\Phi(-x_n) \sim \exp(-x_n u_n) - \exp(-x_n\delta_n) \sim \exp(-x_n u_n)$ and $x_n u_n = O(n^{-1/2}x_n^3)$. In view of (7.21), (7.22) and (7.23) it will be sufficient to show $\exp(-x_n\delta_n)$, n^{-1} and $n^{-1/4}x_n \log n$ all tend to zero faster than $\exp(-x_n u_n)$. The first follows from $u_n/\delta_n \to 0$, and the second and third require $x_n u_n = o(\log n)$ which follows from $x_n = o(n^{1/6}(\log n)^{1/3})$. Hence $(S_{n2} + S_{n3} + S_{n4})/S_{n1} \to 0$.

A PARENTHETICAL REMARK. The inequalities following (7.24) are crude and it is therefore not clear that the condition $x_n = o(n^{1/6}(\log n)^{1/3})$ is necessary, though it can be shown that this lack of precision affects only the power of log n. On the other hand, if x_n is very much too big then S_{n1} will be much smaller than S_{n3}, and the approximation can be expected to perform very badly. We return to this point when discussing the empirical results of Section 8.

The only thing left to show now is that

$$S_{n1} \sim \exp(-\lambda_1 n^{-1/2}x_n^3) \int_{-\infty}^{-x_n} \Omega_n(n^{1/6}\sigma^* K_9^{-1}(-x_n - u))\, d\Phi(u) \quad \text{as} \quad n \to \infty. \tag{7.25}$$

Now $\alpha_n = (1 - 3\delta_n x_n^{-1})n^{1/6}K_9^{-1}\sigma^*$; let $\alpha_n' = n^{1/6}K_9^{-1}\sigma^*$. Then

$$\int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n u)\, du = \frac{\alpha_n'}{\alpha_n}\int_0^{\alpha_n\delta_n/\alpha_n'} \phi(-x_n - \alpha_n' u/\alpha_n)\Omega_n(\alpha_n' u)\, du$$

$$\sim \int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n' u)\, du$$

TABLE 1
Comparison of Approximations to $F_n(x)$ for $n = 40$, $\rho = 5$

nx	Q_0	Q_1	Q_2	Q_3	$Q_1 - Q_0$	$Q_2 - Q_0$	$Q_3 - Q_0$
16.0	−21.299	−12.037	−15.847	−23.169	9.262	5.452	−1.869
16.5	−18.937	−10.776	−14.376	−20.407	8.160	4.561	−1.470
17.0	−16.765	−9.592	−12.983	−17.818	7.173	3.782	−1.053
17.5	−14.773	−8.484	−11.667	−15.426	6.290	3.106	−0.653
18.0	−12.950	−7.451	−10.428	−13.245	5.499	2.702	−0.296
18.5	−11.287	−6.493	−9.266	−11.291	4.794	2.021	−0.004
19.0	−9.774	−5.610	−8.179	−9.556	4.164	1.595	0.219
19.5	−8.403	−4.801	−7.168	−8.038	3.603	1.235	0.365
20.0	−7.167	−4.063	−6.232	−6.748	3.103	0.935	0.418
20.5	−6.057	−3.398	−5.370	−5.684	2.659	0.687	0.373
21.0	−5.067	−2.803	−4.581	−4.740	2.264	0.486	0.327
21.5	−4.190	−2.276	−3.865	−3.895	1.914	0.325	0.295
22.0	−3.420	−1.815	−3.219	−3.145	1.604	0.201	0.275
22.5	−2.750	−1.419	−2.644	−2.480	1.331	0.106	0.270
23.0	−2.174	−1.083	−2.136	−1.901	1.090	0.038	0.273
23.5	−1.686	−0.806	−1.695	−1.485	0.881	−0.009	0.201
24.0	−1.277	−0.581	−1.316	−1.143	0.697	−0.039	0.135
24.5	−0.941	−0.404	−0.998	−0.861	0.536	−0.057	0.080
25.0	−0.674	−0.271	−0.736	−0.633	0.403	−0.062	0.040

Q_0, Q_1, Q_2, Q_3 *are the logarithms of the exact probability, the simple normal approximation, the improved approximation and the lower tail improvement, respectively.*

provided $(x_n + \delta_n)(\delta_n - \delta_n\alpha_n/\alpha_n') \to 0$, i.e., provided $x_n\delta_n(3\delta_n x_n^{-1}) \to 0$. This is satisfied since we assumed $\delta_n \to 0$.

In addition $\int_{\delta_n}^{\infty} \phi(-x_n - u)\Omega_n(\alpha_n' u)\,du < \Phi(-x_n - \delta_n)$ which is negligible in comparison with $\int_0^{\delta_n} \phi(-x_n - u)\Omega_n(\alpha_n' u)\,du$ by an argument similar to that used to prove $S_{n4}/S_{n1} \to 0$. Hence

$$S_{n1} \sim \left(\int_0^{\infty} \phi(-x_n - u)\Omega_n(\alpha_n' u)\,du\right)(F_n'(-x_n)/\Phi(-x_n))$$

$$\sim \left(\int_{-\infty}^{-x_n} \Omega_n(n^{1/6}\sigma^* K_9^{-1}(-x_n - u))\,d\Phi(u)\right)\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\}.$$

But $n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n) = \lambda_1 n^{-1/2}x_n^3 - \lambda_2 n^{-1}x_n^4 + \cdots$ and all these terms but the first are negligible. Here $\lambda_1 = (q - p)/(6p^{1/2}q^{1/2})$ with $p = 1 - q = t^*$; thus $\lambda_1 = (1 - 2t^*)[36t^*(1 - t^*)]^{-1/2}$. Therefore $\exp\{-n^{-1/2}x_n^3\lambda(-n^{-1/2}x_n)\} \sim \exp\{-\lambda_1 n^{-1/2}x_n^3\}$. This gives (7.25), and the proof is complete. □

8. Numerical study of lower-tail probabilities. Three approximations to $F_n(x)$ have been proposed. In Table 1, these three approximations are compared for $n = 40$, assuming a Weibull distribution with $\rho = 5$ for the fibre strengths. Since the emphasis is on relative error in the lower tail, the logarithms of the quantities under study are shown. For each x, we have

$$Q_0 = \log F_n(x),$$

$$Q_1 = \log \Phi(n^{1/2}\sigma^{*-1}(x - \mu^*)),$$

$$Q_2 = \log \Phi(n^{1/2}\sigma^{*-1}(x - \mu^* - n^{-2/3}b^*)),$$

$$Q_3 = \log \Phi_n^{\ell}(n^{1/2}\sigma^{*-1}(x - \mu^*)),$$

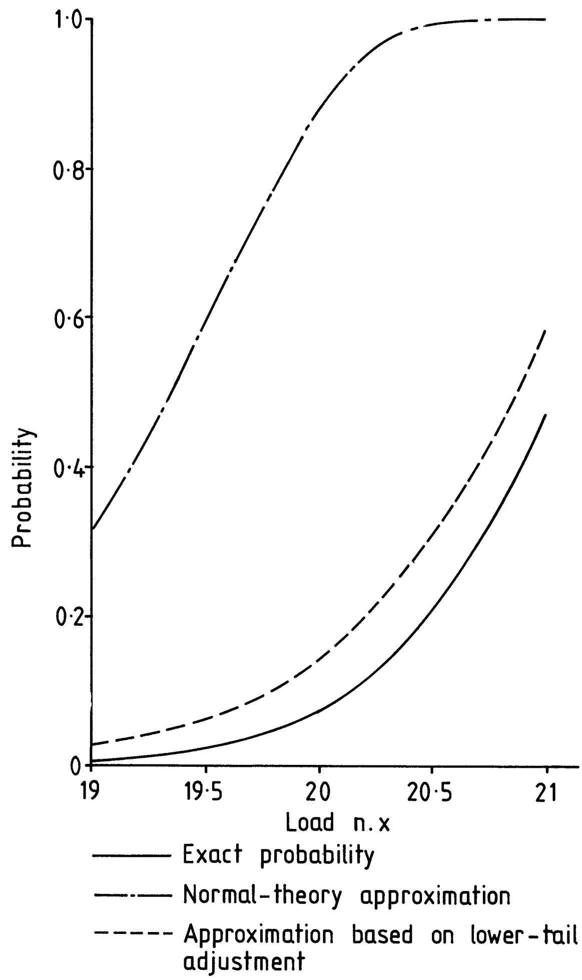

FIG. 2. *Series-parallel system*; $n = 40, K = 100, \rho = 5$.

where

$$\Phi_n^{\ell}(z) = \exp(\lambda_1 n^{-1/2} z^3) \int_{-\infty}^{z} \Omega_n(n^{1/6}\sigma^* K_9^{-1}(z-u))\, d\Phi(u)$$

as in Section 7. The last approximation is specifically designed as a lower-tail improvement.

The differences $Q_1 - Q_0$, $Q_2 - Q_0$, $Q_3 - Q_0$ determine the relative errors of the three approximations. At the bottom of the table (moderate probability), both Q_2 and Q_3 are a substantial improvement on Q_1, but there is not much to choose between them. In the middle of the table (lower-tail probabilities), Q_2 is still a substantial improvement on Q_1, but Q_3 is even better. This provides some confirmation for the theory. However, at the top of the table (extreme lower-tail), Q_3 has overshot: it underestimates the true probability by a large amount. A corresponding table for the case $\rho = 10$ (not reproduced here) showed similar behaviour, but with an even more dramatic overshoot of Q_3 in the extreme lower-tail. Referring now to the Parenthetical Remark of Section 7, it is seen that this was anticipated; when $x_n \gg n^{1/6}(\log n)^{1/3}$ the approximation is much smaller than the principal error terms. Thus Q_3 cannot be used in the extreme lower-tail, but is effective within the "zone of convergence" $x_n = o(n^{1/6}(\log n)^{1/3})$ defined by Theorem 7.1.

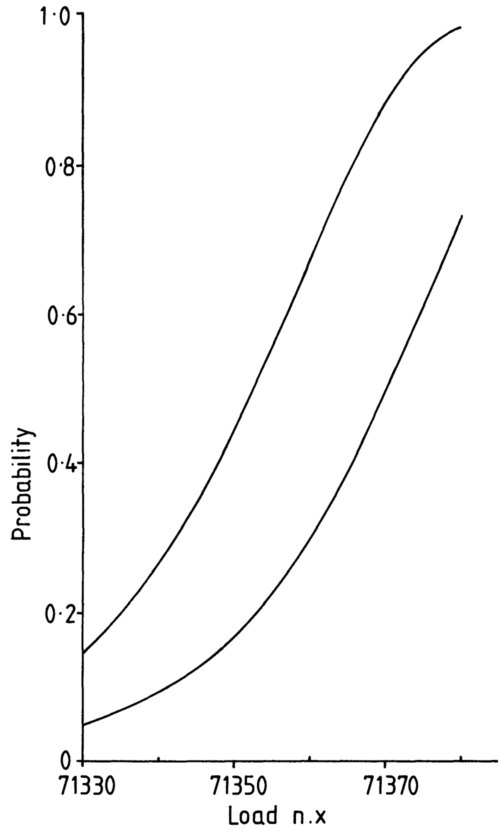

FIG. 3. *Series-parallel system*; $n = K = 10^5$, $\rho = 10$.

We have not yet discussed how to use Q_3 for the series-parallel model. The following procedure is proposed. Given a sequence $\{K(n)\}$, define b_n' for $n \geq 1$ by the formula

(8.1) $$K(n)\Phi_n^{\ell}(n^{1/2}\sigma^{*-1}(b_n' - \mu^*)) = 1.$$

We conjecture that if $K(n) \to \infty$, $\log K(n) = o(n^{1/3}(\log n)^{2/3})$, then

(8.2) $$G_n(a_n x + b_n') \to 1 - \exp(-\exp(x)) \qquad \text{for all} \quad x,$$

where $G_n = 1 - (1 - F_n)^{K(n)}$ and a_n is as in Theorem 6.1. This conjecture is suggested by the main result of Borges (1978).

We thus have two approximations to $G_n(x)$, namely the normal-theory approximation

$$1 - \exp(-\exp((x - b_n)/a_n))$$

studied in detail in Section 6, and the adjusted approximation

$$1 - \exp(-\exp((x - b_n')/a_n)).$$

A comparison is shown in Figure 2.

What does all this mean when n and K are large? In Figure 3, the normal theory and adjusted approximations are compared for the values $n = K = 10^5$, $\rho = 10$, which were the values used by Gücer and Gurland in their study. Here, of course, the exact probability is not computable. There is still quite a gap between the two curves, but the range of x-values covered is small. Thus it appears, on the evidence of this figure, that for large n and K the relative error in determining the operating load is small.

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