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SPECIAL INVITED PAPER

SOJOURNS AND EXTREMES OF STATIONARY PROCESSES[1]

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Let $X(t)$, $-\infty < t < \infty$, be a real stationary stochastic process with continuous sample functions. For $t > 0$, put $L_t(u)$ = Lebesgue measure of $\{s: 0 \le s \le t, X(s) > u\}$ and $M(t) = \max(X(s): 0 \le s \le t)$. For several years the author has studied the limiting properties of these random variables in the case where $X(t)$ is a Gaussian process and under two kinds of limiting operations: i) t fixed and $u \to \infty$; ii) $t \to \infty$ and $u = u(t) \to \infty$ as a function of t. The purpose of this paper is to show how the methods developed in the Gaussian case can be extended to the general, not necessarily Gaussian case. This is illustrated by applications of some of the results to specific examples of non-Gaussian processes, and classes of processes containing a Gaussian subclass.

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References

1. Introduction. Let $X(t)$, $-\infty < t < \infty$, be a real separable measurable stationary stochastic process. For $t > 0$ and $u > 0$, we define the "sojourn of X above u on the interval $[0, t]$" as the integral

$$L_t(u) = \int_0^t I_{[X(s)>u]}\,ds, \tag{1.1}$$

where I is the indicator function. The purpose of this work is to present conditions under which several kinds of limit theorems hold for $L_t(u)$ and for another random variable closely related to it, namely,

$$\sup_{0\le s\le t} X(s).$$

The conditions are stated in terms of the finite-dimensional distributions of the process. The four main results are summarized below:

I) There exists an absolutely continuous nonincreasing function $\Gamma(x)$ with a nonincreasing Radon-Nikodym derivative $\Gamma'(x)$, $x > 0$, and a function $v = v(u)$ with $\lim_{u\to\infty} v(u) = \infty$ such that for fixed $t > 0$,

$$\lim_{u\to\infty} \frac{P(vL_t(u) > x)}{E(vL_t(u))} = -\Gamma'(x), \qquad x > 0, \tag{1.2}$$

on the continuity set of Γ'.

II) The function $\Gamma'(x)$ is extended to the value $x = 0$, and the constant $\Gamma'(0)$ appears in an estimate of the tail of the distribution of the supremum:

$$\lim_{u\to\infty} \frac{P(\sup_{[0,t]} X(s) > u)}{E(vL_t(u))} = -\Gamma'(0). \tag{1.3}$$

III) Let u be defined as a function $u(t)$ which increases with t in such a way that the denominator in (1.2) remains a constant, equal to 1. Then, for $s > 0$,

$$E\exp(-svL_t(u)) \to \exp\left\{\int_0^\infty (1 - e^{-sx})\,d\Gamma'(x)\right\} \tag{1.4}$$

for $t \to \infty$; thus, $vL_t(u)$ has a limiting infinitely divisible distribution with a spectral function given by (1.2).

IV) Let $u = u(t)$ tend to ∞ as in III) above. There exists a function $w = w(u)$ such that

$$\lim_{t\to\infty} P\{w(\sup_{[0,t]} X(s) - u)\} \le x = \exp(\Gamma'(0)e^{-x}) \tag{1.5}$$

for every x.

Each of these four results had been previously established for general classes of stationary Gaussian processes. The main point of this work is to formulate more general sets of conditions on a stationary process for the validity of these theorems. This study is in the area known as extreme value theory for continuous parameter processes. The two primary sources for the methods and concepts in it are

i) extreme value theory for sequences of independent identically distributed random variables and its extension to stationary sequences with some dependence structures;

ii) sample function properties, such as continuity and level crossing characteristics, for stationary, continuous parameter processes.

There is a large literature on i); we refer the reader to the recent survey in the book of Galambos [14], Chapter 3. Most of the published research in the case of dependence has been for the stationary Gaussian sequence, where the hypotheses are conveniently expressed in terms of the covariance sequence of the process. The early work in the general, not necessarily Gaussian case, often involved mixing conditions on the distributions of the process that were hard to apply, and could not be verified in the particular Gaussian case. However, in recent years the mixing conditions have been sufficiently simplified so that they also cover the Gaussian case. The most recent report of such results is that of Leadbetter, Lindgren and Rootzen [24]. In the area ii) above, the most interesting and detailed results have been obtained in the Gaussian case. The convenient sources are the book by Cramer and Leadbetter [10] and the survey paper of Dudley [11]. But there is also a persistent interest in the general, not necessarily Gaussian case.

However, despite the activity in extreme value theory for stationary sequences and for stationary, continuous parameter Gaussian processes, there have been few investigations in the non-Gaussian case for continuous parameter processes. Two of the early papers in this area are those of Newell [33] and Berman [1] in the case of stationary diffusion processes. The only other results in the non-Gaussian case are, to the best of my knowledge, the recent work of Lindgren [27] and Sharpe [39] on the Chi squared process, and the report of Leadbetter [23] on a general class of processes. Further results of Leadbetter, Lindgren and Rootzen appear in [26]. The challenge of formulating useful theorems in the general case is in that there are so few processes whose finite-dimensional distributions can be explicitly represented as in the Gaussian case.

While most authors in the area of extreme value theory for stationary Gaussian processes have used the tool of level crossings to investigate the maximum of the sample function, the present author has always been more interested in the sojourn of the sample function as a tool for the maximum. In the Gaussian case, one's preference for level crossings or sojourns may be a matter of taste; however, in the present approach to the study of the extremes of a stationary, not necessarily Gaussian process, the concept of the sojourn has led to many results which are of interest themselves and for which I have found no conveniently stated versions for level crossings.

The methods of the present work are abstractions and extensions of those that the author has used in earlier work in the Gaussian case. They are based most directly on those in the papers [7] and [8]. The hypotheses in the various theorems are stated as conditions on the k-dimensional distributions of the process for fixed $k \ge 1$. One of the convenient features of our results is that many of the conditions depend on the k-dimensional distributions only for $k = 1, 2, 3$. We indicate, on the basis of our previous work, that all of these conditions are satisfied by the stationary Gaussian processes which were previously studied. We also show that there are various classes of non-Gaussian processes which satisfy the conditions which are sufficient for the result (1.2). It is our intention to continue this study in the future, and show that the results (1.3)–(1.5) may also be applied to these non-Gaussian processes. The outline of such a plan is given in the concluding Section 20.

The extreme value limit distribution in the Gaussian case is, with sufficient asymptotic independence, of the double exponential distribution type $\exp(-e^{-x})$. This type also arises in the more general case considered in this paper. A natural question that may be asked is whether the results can be extended to limiting distributions of the other two extreme value types. We indicate in Section 20 how this may be done.

One of the very attractive features of studying the stationary Gaussian process is that the process is completely described by its covariance function, and so all conditions on the process may be phrased entirely in terms of that function. If $r(t)$ is the covariance function, the local conditions on the process are described by conditions on $r(t)$, for t near 0; and the ergodic or mixing conditions are described by conditions on $r(t)$ for $t \to \infty$. In the formulation of similar results for general, not necessarily Gaussian processes, the local and mixing conditions cannot be stated in terms of a single function. Instead, the local conditions are stated in terms of the joint distributions of the process at a finite set of points in a small interval, and the mixing conditions are similarly stated for sets of points which are mutually distant on the time axis. The results (1.2) and (1.3) are local results, so that they depend only on the local conditions for the finite dimensional distributions. On the other hand, (1.4) and (1.5) depend on the local results and also on the mixing conditions, so that their hypotheses are more comprehensive.

Let us indicate the relations among the hypotheses of the four main results. If $\mathrm{Hyp}(A) \subset \mathrm{Hyp}(B)$ means that the hypothesis of a proposition A is part of the hypothesis of proposition B, then our results are related as follows:

$$\mathrm{Hyp(I)} \subset \mathrm{Hyp(II)} \qquad \mathrm{Hyp(I)} \subset \mathrm{Hyp(III)}$$
$$\mathrm{Hyp(II)} \subset \mathrm{Hyp(IV)} \qquad \mathrm{Hyp(III)} \subset \mathrm{Hyp(IV)}.$$

We present a brief summary of each of the sections that follow:

2. The convergence (1.2) represents a type of limiting operation on a family of distributions which we first used in [8], and which has been used without a descriptive term in all of our work up to this study. We will now call it "tail convergence in distribution." It differs from ordinary convergence in distribution because the denominator in (1.2) converges to 0 in the applications in this work. Lemma 2.1 provides a simple sufficient condition for tail convergence.

3. The Sojourn Limit Theorem (Theorem 3.1) is the starting point of the entire work; all the other results depend on it. It states that the sojourn (1.1), after multiplication by a suitable normalizing function $v = v(u)$, is tail convergent in distribution for $u \to \infty$, and the limit is explicitly given. The hypothesis of the theorem is that the process is stationary and has the following two characteristics:

i) When the process is conditioned to be above a high level u at some fixed point (for example, at $t = 0$), then it "forgets" exactly how high above u it is, and, upon appropriate normalization, converges in finite dimensional distributions to a limiting process $Z(t)$. This is a natural condition to impose; indeed, we would expect the convergence of the functional (1.1) to be closely related to the conditional convergence of the underlying process X.

ii) After going above a high level u, the sample function tends to fall quickly to some point below u. The condition is implied by a very simple asymptotic condition on the tails of the bivariate distributions. This characteristic implies that the significant contribution to a high level sojourn on a fixed, finite interval is made in the neighborhood of at most one point of the interval.

The Sojourn Limit Theorem was first proved in the Gaussian case in [2] for mean square differentiable processes. The most general result in the Gaussian case is in [7]. The proof in the latter is the basis of the proof in the current more general case.

4. As an offshoot of the theory of extreme values in sequences of independent random variables, it is natural for our theory to include the same hypothesis on the marginal distribution required by the former theory. Let $F(x)$ represent the common marginal distribution of the process. We show that the hypothesis of the Sojourn Limit Theorem actually implies that F is in the domain of attraction of the extreme value type $\exp(-e^{-x})$.

Furthermore, it is shown that the normalizing function $w = w(u)$ appearing in the hypothesis of that theorem is actually the familiar "extremal intensity function," $w(u) = F'(u)/(1 - F(u))$ (see Gumbel [18], page 20).

5. The conditioning event in the statement of the Sojourn Limit Theorem is $X(0) > u$. In some applications, including the Gaussian case, the calculation of the conditional distributions given $X(0) > u$ is done by conditioning at a fixed point x above u, and then integrating over $x > u$. Two theorems in Section 5 provide conditions under which the conditioning at a fixed point $u + y/w$ leads to the same limit as conditioning by $X(0) > u$.

6. The conditions of the Sojourn Limit Theorem can be simplified in the case where the process $X(t)$ is assumed to have a stochastic derivative. Aside from the hypothesis of stationarity, the only conditions on the process are those which are expressible in terms of the joint distribution of $X(t)$, $X(0)$ and $X'(0)$.

7–10. In Sections 7–10 we apply the Sojourn Limit Theorem to specific stationary processes. We begin with the special case in which the Sojourn Limit Theorem was first proved, namely, the stationary Gaussian process. Our second example is the random cosine wave $Y \cos(t - Z)$, $-\infty < t < \infty$, where Y and Z are independent random variables, Y is nonnegative, and Z is uniformly distributed on $[0, 2\pi]$. Y has a general distribution so that the process is not necessarily Gaussian. (If Y has the Rayleigh distribution then the process is Gaussian.) Next we consider a class of stationary Markov processes having a transition density. Our last example is the Chi squared process $\|\mathbf{X}(t)\|^2$, where $X_1(t), \cdots, X_k(t)$ are independent copies of a stationary Gaussian process, and $\|\mathbf{X}(t)\|^2 = \sum_{i=1}^{k} X_i^2(t)$.

11. One of the immediate consequences of the Sojourn Limit Theorem is an inequality giving a lower bound for the asymptotic tail of the distribution of $\sup_{[0,t]} X(s)$. We show that (1.3) above holds with "lim" and "=" replaced by "liminf" and "≥", respectively.

12. In this section we establish the formula (1.3) for a large class of stochastically differentiable processes. It is shown that the right hand member of (1.3) may be identified with a constant directly computable from the limiting process $Z(t)$.

13. Here we begin our effort to establish (1.3) without the assumption of stochastic differentiability. In the Gaussian case we were able to get estimates of the tail of the distribution of the maximum by appropriate modifications of the well known Fernique Inequality [12]. However, this result is not applicable to general stationary processes. Nevertheless, it has been known for some time that the proof of the Fernique Inequality depends on the Gaussian property only at certain points. We prove a nonparametric version of the Fernique Inequality by placing suitable conditions on the distributions of the increments of the process. Then we generalize the calculations in [7] to obtain a preliminary inequality giving an upper bound for the tail of the distribution of the maximum. The hypotheses that are added in this section involve only the two- and three-dimensional distributions of the process.

14. The verification of (1.3) for processes satisfying the previous conditions is completed. The estimates for the distribution of the maximum from Section 13 are used in the proof to show that the convergence of the finite dimensional distributions in the hypothesis of the Sojourn Limit Theorem is accompanied by weak convergence over the corresponding Banach space of continuous functions. The main result, Theorem 14.1, is called the Maximum Limit Theorem.

15. It is shown that the conditions of the Maximum Limit Theorem are satisfied by the stationary Gaussian process satisfying the hypothesis of Section 7. We also discuss the relation of $-\Gamma'(0)$ in (1.3) to the constant H_α of Pickands [34] in the Gaussian case.

16. Sections 16–18 are devoted to the establishment of (1.4). The basic tool is our compound Poisson limit theorem in [8]. As we noted in that paper, this theorem is entirely independent of the context of Gaussian processes, and we had noted our expectation to use it for more general processes. This result is recorded in this section as Theorem 16A. Finally, we define the relation between t and u under which (1.4), and also (1.5), hold.

17. The conditions of Theorem 16A are of three kinds: strictly local, local mixing, and global mixing. The first involves the finite dimensional distributions of the process in the neighborhood of a fixed point. The second involves the joint distribution of a pair of

variables of the process at two points which are "far," but not "too far," from each other. The third involves the joint distributions of sets of variables of arbitrary but fixed size which are mutually far from each other. In Section 17 we state conditions on the finite dimensional distributions of the process which are sufficient for the strictly local and local mixing conditions of the convergence theorem.

18. In Section 18 we formulate the global mixing conditions. A new concept in the theory of mixing is introduced: that of a net of distributions over a directed set. It is shown that the topology of the space of the distributions may be characterized by sequential convergence, so that the standard convergence theorems may be applied.

19. In this section, we combine the results II and III above to establish IV. Both differentiable and nondifferentiable processes are covered; however, the results for the former may be obtained more simply by the method indicated at the end of Section 12.

20. In this last section, we briefly indicate some possible extensions of the main results. In particular, we indicate how larger classes of processes might be shown to fall in the domain of the techniques introduced in this work.

We close this introduction with a comment about the place of this work in the current and recent literature on the maximum of stationary Gaussian processes. This subject is featured by the contrast between the simplicity of the stated results and the complexity of the calculations needed to prove them. Indeed, the hypothesis of each theorem is a statement about the covariance function of the process, and the proof is typically a calculation involving the explicit finite dimensional distributions. The purpose of this work is to focus attention on the latter and demonstrate that they are not restricted to the context of Gaussian processes. We have supplied several examples of non-Gaussian processes which satisfy the hypothesis of the first main result (1.2), and we intend, in future work, to extend these to the other results. These plans are sketched in Section 20. In this way we hope to demonstrate that our previous methods in the Gaussian case are "robust" in the more general case. This has been the aim of similar work of Leadbetter [23] and his joint work with Lindgren and Rootzen [25]; and of Marcus [30].

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2. Tail convergence in distribution.

Let (Y_n) be a sequence of nonnegative random variables with finite expectations. We say that (Y_n) is *tail convergent in distribution* if there is a monotone nonincreasing function $G(x)$, $x > 0$, such that

$$\lim_{n\to\infty} \frac{P(Y_n > x)}{EY_n} = G(x) \tag{2.1}$$

on the continuity set of G. Observe that the function

$$F_n(x) = (EY_n)^{-1} \int_0^x P(Y_n > y)\, dy, \qquad x > 0, \tag{2.2}$$

is a probability distribution function because

$$EY_n = \int_0^\infty P(Y_n > y)\, dy.$$

LEMMA 2.1. *If F_n has a weak limit F on the nonnegative axis, then F is absolutely continuous, and* (2.1) *holds with $G(x) = F'(x)$ almost everywhere.*

PROOF. The proof is exactly the same as that for the particular case proved in [7], Theorem 6.1. We use the facts that i) (F_n) is uniformly absolutely continuous, so that F is absolutely continuous; ii) as a monotone function, $P(Y_n > y)/EY_n$ has, for $n \to \infty$, a weak subsequential limit; and iii) the latter limit must coincide with G and F' almost everywhere.

The case of particular interest to us will be when $EY_n \to 0$, so that the numerator in (2.1) also tends to 0 for each x.

3. The Sojourn Limit Theorem. Let $X(t)$, $-\infty < t < \infty$, be a separable, measurable real stochastic process. For $u > 0$ and $t > 0$, define

$$L_t(u) = \int_0^t I_{[X(s)>u]}\, ds, \tag{3.1}$$

where $I_{[\ldots]}$ is the indicator function. $L_t(u)$ is called the sojourn time of the process above the level u in the interval $[0, t]$. We recall an identity which we have used in previous work [6], [7]:

$$\int_x^\infty P(L_t(u) > y)\, dy = \int_0^t P(L_s(u) > x, X(s) > u)\, ds.$$

Define the process $X^*(s)$ as $X^*(s) = X(t - s)$, $0 \le s \le t$, with t fixed. The corresponding sojourn time $L_s^*(u)$, $0 \le s \le t$, defined by (3.1) with X^* in the place of X, is $L_s^* = L_t - L_{t-s}$. The identity above becomes

$$\int_x^\infty P(L_t(u) > y)\, dy = \int_0^t P(L_t(u) - L_{t-s}(u) > x, X(t-s) > u)\, ds.$$

Under the hypothesis of stationary, this becomes

$$\int_x^\infty P(L_t(u) > y)\, dy = \int_0^t P(L_s(u) > x, X(0) > u)\, ds. \tag{3.2}$$

THEOREM 3.1. (*Sojourn Limit Theorem*). *Let $X(t)$, $-\infty < t < \infty$, be separable, measurable, and stationary. Suppose that the following two conditions hold*:

Assumption 3.I. *There exists a measurable process $Z(t)$, $-\infty < t < \infty$, with continuous finite dimensional distributions, and functions $v = v(u)$ and $w = w(u)$, $u > 0$, with*

$$\lim_{u\to\infty} v(u) = \infty, \tag{3.3}$$

such that the finite dimensional distributions of the process

$$w(u)(X(t/v) - u), \qquad -\infty < t < \infty, \tag{3.4}$$

conditioned by

$$X(0) > u, \tag{3.5}$$

converge to those of $Z(t)$.

Assumption 3.II. *The function v satisfies*

$$\lim_{d\to\infty} \limsup_{u\to\infty} v \int_{d/v}^t P(X(s) > u \mid X(0) > u)\, ds = 0, \tag{3.6}$$

for $0 < t \le T$, for some $T > 0$. Define

$$\Gamma(x) = P\left(\int_0^\infty I_{[Z(s)>0]}\, ds > x\right), \qquad x > 0. \tag{3.7}$$

Our conclusion is:

$$\lim_{u\to\infty} \int_x^\infty \frac{P(vL_t(u) > y)}{E(vL_t(u))}\, dy = \Gamma(x) \tag{3.8}$$

at all continuity points $x > 0$ of Γ, for $0 < t \le T$.

PROOF. By stationarity and Fubini's theorem, we have

$$EL_t(u) = \int_0^t P(X(s) > u)\, ds = tP(X(0) > u). \tag{3.9}$$

By a change of variable of integration, the numerator on the left hand side of (3.8) becomes

$$v \int_{x/v}^{\infty} P(L_t(u) > y)\, dy,$$

which, by the identity (3.2), is equal to

$$v \int_0^t P(vL_s(u) > x, X(0) > u)\, ds.$$

From this and (3.9) it follows that the ratio on the left hand side of (3.8) is equal to

$$(1/t) \int_0^t P(vL_s(u) > x \,|\, X(0) > u)\, ds. \tag{3.10}$$

Let $d > 0$ be arbitrary but fixed. In the evaluation of the limit of (3.10) for $u \to \infty$, it suffices, according to (3.3), to replace (3.10) by

$$(1/t) \int_{d/v}^t P(vL_s(u) > x \,|\, X(0) > u)\, ds. \tag{3.11}$$

For convenience, we now suppress the argument u and write $L_t = L_t(u)$; then $L_s = (L_s - L_{d/v}) + L_{d/v}$. Since $L_{d/v}$ is nonnegative, (3.11) is bounded below by

$$(1/t) \int_{d/v}^t P(v(L_s - L_{d/v}) > x \,|\, X(0) > u)\, ds. \tag{3.12}$$

For arbitrary $\epsilon > 0$, (3.11) is bounded above by the sum of two terms,

$$(1/t) \int_{d/v}^t P(vL_{d/v} > x(1-\epsilon) \,|\, X(0) > u)\, ds, \tag{3.13}$$

and

$$(1/t) \int_{d/v}^t P(v\cdot[L_s - L_{d/v}] > x\epsilon \,|\, X(0) > u)\, ds. \tag{3.14}$$

(Indeed, $a + b > x$ implies $a > x(1-\epsilon)$ or $b > x\epsilon$.) Finally, by a change of variable of integration and the introduction of the normalizing function w, we have, suppressing the argument u of w,

$$vL_{d/v} = \int_0^d I_{[w\cdot(X(s/v)-u)>0]}\, ds. \tag{3.15}$$

We find the limiting conditional distribution of (3.15), given $X(0) > u$, for $u \to \infty$. Fubini's theorem implies

$$E\left\{\left[\int_0^d I_{[w\cdot(X(s/v)-u)>0]}\, ds\right]^k \,\middle|\, X(0) > u\right\}$$
$$= \int_0^d \cdots \int_0^d P(w\cdot(X(s_j/v) - u) > 0, j = 1, \cdots, k \,|\, X(0) > u)\, ds_1 \cdots ds_k,$$

for $k \geq 1$. By Assumption 3.I, the integral above converges for $u \to \infty$ to

$$\int_0^d \cdots \int_0^d P(Z(s_j) > 0, j = 1, \cdots, k)\, ds_1, \cdots, ds_k,$$

which, by Fubini's theorem, is equal to

$$E\left(\int_0^d I_{[Z(s)>0]}\, ds\right)^k .$$

Since the random variable $\int_0^d I\, ds$ is bounded by d, the moment convergence theorem may be applied, and so we conclude that the random variable (3.15) converges in conditional distribution to the corresponding integral for the process $Z(t)$. It follows that (3.12) converges to

$$P\left(\int_0^d I_{[Z(s)>0]}\, ds > x\right) \tag{3.16}$$

and that (3.13) converges to

$$P\left(\int_0^d I_{[Z(s)>0]}\, ds > x(1-\epsilon)\right) \tag{3.17}$$

at all points of continuity.

Next we estimate (3.14). By Markov's inequality, it is at most equal to

$$(v/tx\epsilon) \int_{d/v}^t \int_{d/v}^s P(X(r) > u \mid X(0) > u)\, dr\, ds,$$

which is at most equal to

$$(v/x\epsilon) \int_{d/v}^t P(X(s) > u \mid X(0) > u)\, ds. \tag{3.18}$$

From the lower bound (3.12) and its limit (3.16), it now follows that, for $d > 0$,

$$P\left(\int_0^d I_{[Z(s)>0]}\, ds > x\right) \le \text{liminf}_{u\to\infty}\, (1/t) \int_0^t P(vL_s(u) > x \mid X(0) > u)\, ds. \tag{3.19}$$

Similarly, from the upper bound of the sum of (3.13) and (3.14) and their limits (3.17) and (3.18), respectively, we see that

$$\begin{aligned} &\text{limsup}_{u\to\infty}\, (1/t) \int_0^t P(vL_s(u) > x \mid X(0) > u)\, ds \\ &\le P\left(\int_0^d I_{[Z(s)>0]}\, ds > x(1-\epsilon)\right) \\ &\quad + (1/x\epsilon)\, \text{limsup}_{u\to\infty}\, v \int_{d/v}^t P(X(s) > u \mid X(0) > u)\, ds. \end{aligned} \tag{3.20}$$

Let $d \to \infty$ in (3.19) and (3.20), and apply (3.6); finally, let $\epsilon \to 0$ to conclude (3.8).

By application of Lemma 2.1, we obtain

COROLLARY 3.1. *$\Gamma(x)$ is absolutely continuous, and*

$$\text{lim}_{u\to\infty} \frac{P(vL_t(u) > x)}{E(vL_t(u))} = -\Gamma'(x) \tag{3.21}$$

for almost every $x > 0$, for $0 < t \le T$.

PROOF. Lemma 2.1 and (3.8) imply the tail convergence in distribution of $vL_t(u)$, the absolute continuity of Γ, and the identification of the limit (3.21) as $-\Gamma'$.

We observe that if the sample functions of the process $X(t)$ are assumed to be continuous or even simply bounded, and if the conclusion of the Sojourn Limit Theorem holds with Γ satisfying $\Gamma(0) > 0$, then

$$\lim_{u\to\infty} vP(X(0) > u) = 0. \tag{3.22}$$

Indeed, if $X(\cdot)$ is bounded, then $vL_t(u) = 0$ for all sufficiently large u, so that $P(vL_t(u) > x) \to 0$ for every $x > 0$, which would imply, according to (3.21), that either $\Gamma'(x) = 0$ for almost all x, or $E(vL_t(u)) \to 0$. The former possibility is excluded by the assumption $\Gamma(0) > 0$, and the latter, by (3.9), implies (3.22).

We remark that 3.I and 3.II are natural assumptions for the result stated in the theorem. Indeed, 3.I is a condition on the convergence of the finite dimensional distributions, and 3.II is a tightness condition on the family of distributions of the sojourn. Such a pair of assumptions is typical in establishing the convergence of the distribution of functionals in many contexts.

4. Implications of the hypothesis of the Sojourn Limit Theorem, and the forms of w and v. Let $F(x)$ be the marginal distribution function of $X(t)$; by stationarity, F does not depend on t. Let us recall that there are three types of limiting distributions of maxima in sequences of independent random variables with a common distribution function, and their domains of attraction have been characterized in the classical paper of Gnedenko [17] and the more recent monograph of de Haan [19]. We shall be concerned in this paper with distributions in the domain of attraction of the particular extreme value distribution,

$$\Lambda(x) = \exp(-e^{-x}), \qquad -\infty < x < \infty,$$

the "double exponential" distribution. A distribution function F such that $F(x) < 1$ for all real x belongs to the domain of attraction of Λ if and only if there exists a positive function $w = w(u)$ such that

$$\lim_{u\to\infty} \frac{1 - F(u + x/w)}{1 - F(u)} = e^{-x}, \qquad -\infty < x < \infty. \tag{4.1}$$

The function w is then necessarily of the form

$$w(u) = \frac{1 - F(u)}{\int_u^\infty (1 - F(x))\,dx} \tag{4.2}$$

or is asymptotically equal to it for $u \to \infty$. If $f(x) = F'(x)$ exists and is nonincreasing for all sufficiently large x, then w may be taken as

$$w(u) = \frac{f(u)}{1 - F(u)}; \tag{4.3}$$

see [19], page 88. (These results can be extended to the case where there is a real x_0 such that $F(x_0) = 1$; however, for simplicity, we consider only the case $x_0 = \infty$.)

Our first result is that, under general conditions, Assumption 3.I implies the condition (4.1).

THEOREM 4.1. *Suppose that* 3.I *holds with some function* $w = w(u)$. *If this function satisfies*

$$\lim_{u\to\infty} uw(u) = \infty; \tag{4.4}$$

and if

$$\lim_{u\to\infty, u'\to\infty, u/u'\to 1}(w(u)/w(u')) = 1; \tag{4.5}$$

then F is in the domain of attraction of Λ, *w is asymptotically equal to the function* (4.2), *and* $Z(0)$ *is exponentially distributed.*

PROOF. Define $Q(x) = P(Z(0) > x)$ for $x > 0$. Assumption 3.I implies, for $t = 0$, that

$$P(Z(0) > x) = 1, \qquad \text{for } x \leq 0 \tag{4.6}$$

and

$$Q(x) = \lim_{u\to\infty} \frac{1 - F(u + x/w)}{1 - F(u)}, \qquad \text{for } x > 0. \tag{4.7}$$

Put $u' = u + y/w$ for fixed $y > 0$. Then, by (4.4) and (4.5), and by the continuity and monotonicity of Q, we obtain

$$\lim_{u\to\infty} \frac{1 - F(u + y/w + x/w)}{1 - F(u)} = \lim_{u\to\infty} \frac{1 - F(u' + x/w(u))}{1 - F(u')} \cdot \frac{1 - F(u + y/w(u))}{1 - F(u)}$$

$$= Q(x)Q(y), \qquad \text{for } x > 0, y > 0,$$

so that

$$Q(x + y) = Q(x)Q(y), \qquad \text{for } x, y > 0. \tag{4.8}$$

The continuity of the distribution of $Z(0)$ (see 3.I) and (4.6) imply that $Q(x) > 0$ for all sufficiently small $x > 0$. Repeated application of (4.8) gives $Q^n(x) = Q(nx)$ for every n, so that $Q(x) > 0$ for all $x > 0$. Define

$$\begin{aligned} K(x) &= Q(x) && \text{for } x > 0 \\ &= (Q(-x))^{-1} && \text{for } x < 0 \\ &= 1 && \text{for } x = 0; \end{aligned} \tag{4.9}$$

then K is nonincreasing and continuous and satisfies

$$K(x + y) = K(x)K(y) \tag{4.10}$$

for all x and y. Therefore $K(x)$ is necessarily of the form e^{-ax} for some $a > 0$. The case $a = 0$ is excluded because, by definition, $Q(x) \to 0$ for $x \to \infty$.

The relation (4.7) can be extended to

$$\lim_{u\to\infty} \frac{1 - F(u + x/w)}{1 - F(u)} = K(x) \tag{4.11}$$

for all x by employing the argument leading to (4.8): for $x > 0$,

$$\left(\frac{1 - F(u)}{1 - F(u - x/w)}\right)^{-1} = \left(\frac{1 - F(u - x/w + x/w)}{1 - F(u - x/w)}\right)^{-1} \to (K(x))^{-1} = K(-x).$$

If (4.11) holds for $K(x) = e^{-ax}$ for some function $w(u)$, then (4.1) necessarily holds for the function $aw(u)$ in the place of w.

Next we indicate how the function $v = v(u)$ depends on the bivariate distributions of the process. The hypothesis 3.I implies

$$P(w(X(t/v) - u) > 0 \mid X(0) > u) = \frac{P(X(t/v) > u, X(0) > u)}{P(X(0) > u)} \to P(Z(t) > 0),$$

for every t. This may be sufficient to determine v in some cases. For example, if the bivariate distributions satisfy

$$\lim_{u\to\infty} \frac{P(X(t) > u, X(0) > u)}{P(X(0) > u)} = 0$$

for every $t > 0$, and if X is stochastically continuous, so that

$$\lim_{t\to 0} \frac{P(X(t) > u, X(0) > u)}{P(X(0) > u)} = 1$$

for every u, then we may define $v = v(u)$ as the largest solution of the equation

$$\frac{P(X(1/v) > u, X(0) > u)}{P(X(0) > u)} = P(Z(1) > 0)$$

provided the right hand member is positive.

We close with a comment about the meaning of Assumption 3.II. The latter may be expressed as

$$\lim_{d\to\infty} \operatorname{limsup}_{u\to\infty} E(vL_t | X(0) > u) - E(vL_{d/v} | X(0) > u) = 0.$$

This signifies that, under the condition $X(0) > u$, the time spent above u in $[0, t]$ is nearly exhausted in an interval starting at 0 and of length $O(1/v)$. Thus, if the process is known to exceed u at a fixed time point, then it will tend to fall quickly to some level below u.

5. Sufficient conditions for Assumption 3.I. In this section we show that the convergence of the conditional distributions specified in Assumption 3.I is, under certain general conditions, implied by the convergence of conditional distributions of another kind, namely, where the conditioning is by the value $X(0) = u + y/w$ instead of the event $X(0) > u$.

LEMMA 5.1. *Suppose that*

i) *F is in the domain of attraction of Λ;*

ii) *$f(x) = F'(x)$ exists and is nonincreasing for all sufficiently large x;*

iii) *$w(u)$ is defined as*

(5.1) $$w(u) = \frac{f(u)}{1 - F(u)}$$

for all sufficiently large u, and satisfies (4.5). *Then*

(5.2) $$\lim_{u\to\infty} \frac{f(u + x/w)}{w\cdot(1 - F(u))} = e^{-x}, \qquad \textit{for all } x,$$

and

(5.3) $$\lim_{u\to\infty} \int_M^\infty \left| \frac{f(u + x/w)}{w\cdot(1 - F(u))} - e^{-x} \right| dx = 0, \qquad \textit{for all } M.$$

PROOF. It is known that i) implies (4.4); indeed, otherwise (4.1) could not hold for all $x < 0$. (5.2) follows from (4.1), (4.4), (4.5) and (5.1):

$$\lim_{u\to\infty} \frac{f(u + x/w(u))}{w(u)(1 - F(u))} = \lim_{u\to\infty} \frac{w(u + x/w(u))(1 - F(u + x/w(u)))}{w(u)(1 - F(u))}$$
$$= e^{-x}.$$

Condition ii) implies that the function appearing under the limit in the left hand member of (5.2) is, for all large u, a density function on $x > 0$. It follows from the convergence theorem of Scheffe [38] that (52) implies (5.3) for $M = 0$. This obviously implies (5.3) for $M > 0$. For $M < 0$ we note that the convergence (5.2) is uniform on $[M, 0]$ because f is monotonic in x for all sufficiently large u, and the convergence of a monotone function to a continuous function is uniform. We conclude that (5.3) holds also for $M < 0$ because the integral can be written as the sum of integrals over $[M, 0]$ and $[0, \infty)$.

THEOREM 5.1. *Let $X(t)$ be a stationary process with the marginal distribution function F satisfying the conditions of Lemma* 5.1. *Suppose there exists a process $W(t)$, $-\infty < t < \infty$, with continuous finite dimensional distributions such that the finite dimensional distributions of the process*

$$w\cdot\left(X\left(\frac{t}{v}\right) - X(0)\right), \qquad -\infty < t < \infty, \tag{5.4}$$

conditioned by

$$X(0) = u + y/w \tag{5.5}$$

converge to those of $W(t)$, for any fixed real y. Then the conditions of Assumption 3.I *hold, and $\{Z(t)\}$ is equivalent in distribution to the process*

$$W(t) + \eta \tag{5.6}$$

where η is a random variable with an exponential distribution, and is independent of the process W.

PROOF. Let $t_1, \cdots, t_k, x_1, \cdots, x_k$ be arbitrary, and consider the conditional probability

$$P\left(w\cdot\left(X\left(\frac{t_i}{v}\right) - u\right) \le x, \qquad i = 1, \cdots, k \,|\, X(0) > u\right).$$

It may be written as

$$(1 - F(u))^{-1} \int_u^\infty P\left(w\cdot\left(X\left(\frac{t_i}{v}\right) - u\right) \le x_i, \qquad i = 1, \cdots, k \,|\, X(0) = y\right) f(y)\, dy,$$

or, equivalently, as

$$\int_0^\infty P\left(w\cdot\left(X\left(\frac{t_i}{v}\right) - X(0)\right) + z \le x_i,\, i = 1, \cdots, k \,\middle|\, X(0) = u + z/w\right) \frac{f(u + z/w)}{w\cdot(1 - F(u))}\, dz. \tag{5.7}$$

The hypothesis of the conditional convergence in distribution and the result (5.3) imply that (5.7) converges to

$$\int_0^\infty P(W(t_i) + z \le x_i, \qquad i = 1, \cdots, k) e^{-z}\, dz,$$

which completes the proof.

We also have this variation of Theorem 5.1:

THEOREM 5.2. *Let $X(t)$ be a stationary process with the marginal distribution function F satisfying the conditions of Lemma* 5.1. *Suppose that there is a measurable function $\mu(t, x)$ of (t, x) such that*

i) *there exists a process $V(t)$, $-\infty < t < \infty$, with continuous finite dimensional distributions such that the finite dimensional distributions of the process*

$$w\cdot[X(t/v) - \mu(t/v; u + y/w)], \qquad -\infty < t < \infty, \tag{5.8}$$

conditioned by (5.5), *converge to those of $V(t)$, for any fixed real y; and*

ii) *there is a function $m(t)$ not depending on y such that*

$$\lim_{u\to\infty} w\cdot[\mu(t/v, u + y/w) - (u + y/w)] = m(t), \tag{5.9}$$

for $-\infty < t, y < \infty$.

Then the finite dimensional distributions of process (5.4), *conditioned by* (5.5), *converge to those of the process* $V(t) + m(t)$, *and so* $Z(t)$ *in the Sojourn Limit Theorem is of the form*

(5.10) $$Z(t) = V(t) + m(t) + \eta.$$

PROOF. This is obtained directly from Theorem 5.1 by writing the process (5.4) as the sum of the terms in (5.8) and (5.9).

We remark that Theorem 5.2 is particularly well suited to the analysis of Gaussian processes. We put $\mu(t, x) = E(X(t) \mid X(0) = x) = xr(t)$, and note that the process (5.8) is independent of the condition (5.5).

6. Application to processes with a stochastic derivative. The hypothesis 3.I and the second parts of the hypotheses of Theorems 5.1 and 5.2 contain conditions on the k-dimensional distributions of $X(t)$, for every $k \geq 1$. Now we show that if the process has a stochastic derivative $X'(t)$ at each point, then there are sufficient conditions for the hypotheses above which are expressible in terms of only the joint distribution of $X(t)$, $X(0)$ and $X'(0)$. The derivative at a point is defined as the limit in probability of the standard difference quotient. Thus the existence of this derivative is certainly implied by the existence of the mean square derivative.

THEOREM 6.1. *Let* $X(t)$ *have a marginal distribution function* F *satisfying* (4.1), *and have a stochastic derivative* $X'(t)$ *at each point. Assume that*

i) *There is a random variable* ξ *such that the conditional distribution of* $X'(0)$, *given* $X(0) > u$, *converges to that of* ξ; *and*

ii) *there is a continuous function* $M(t)$ *such that, for each* t, *the random variable* $w\cdot[X(t/w) - X(0)] - tX'(0)$ *converges in conditional probability, given* $X(0) > u$, *to the constant* $M(t)$.

Then the conditions stated in Assumption 3.I *hold with* $v = w$ *and the process* $Z(t)$ *is of the form*

(6.1) $$Z(t) = t\xi + \eta + M(t),$$

where η *is exponentially distributed and* η *and* ξ *are independent.*

PROOF. Write the process (3.4) with $w = v$ as the sum of three terms,

(6.2) $$w\cdot[X(t/w) - X(0)] - tX'(0),$$

(6.3) $$w\cdot[X(0) - u],$$

(6.4) $$tX'(0);$$

and condition by $X(0) > u$, and let $u \to \infty$. By condition ii) above, the random variable (6.2) converges in conditional probability to $M(t)$.

Let us determine the joint conditional limiting distribution of the random variables (6.3) and (6.4). For any x in the continuity set of ξ and any $y \leq 0$, i) implies

$$P(X'(0) \leq x, w\cdot[X(0) - u] > y \mid X(0) > u) = P(X'(0) \leq x \mid X(0) > u) \to P(\xi \leq x).$$

For any such x and for $y > 0$, we have

$$P(X'(0) \leq x, w\cdot[X(0) - u] > y \mid X(0) > u)$$
$$= P(X'(0) \leq x \mid X(0) > u + y/w) \frac{P(X(0) > u + y/w)}{P(X(0) > u)},$$

and the latter, by i) and by (4.1), converges to $P(\xi \leq x)e^{-y}$. Therefore, the random variables are conditionally asymptotically independent with limiting distributions identical with those of $t\xi$ and η, respectively.

Finally we observe that $Z(t)$ has continuous finite dimensional distributions because η is exponentially distributed and $M(t)$ is, by hypothesis, continuous.

We also have the following version of Theorem 6.1 for conditioning $X(0)$ at $u + y/w$.

THEOREM 6.2. *Let $X(t)$ have a marginal distribution function F satisfying the conditions of Lemma* 5.1. *If the conditioning by $X(0) > u$ in the hypothesis of Theorem* 6.1 *is replaced by conditioning by $X(0) = u + y/w$ for arbitrary y, then the conclusion of Theorem* 5.1 *holds with $v = w$, and the process $W(t)$ is of the form*

$$W(t) = \xi t + M(t), \tag{6.5}$$

where ξ and η are independent.

The proof is similar to that of Theorem 6.1.

The conditions of these theorems may be summarized as follows: When $X(0)$ is conditioned at a high level, the difference quotient $w\cdot[X(t/w) - X(0)]$ is roughly equal to $tX'(0)$ plus a constant which may depend on t.

7. Application to stationary Gaussian processes. Let $X(t)$, $-\infty < t < \infty$, be a stationary Gaussian process with mean 0, variance 1, and continuous covariance function $r(t)$. Here F is the standard normal distribution function Φ, and $\phi(x)$ is the density function. Φ is in the domain of attraction of the extreme value distribution function Λ, and, according to (5.1) and the well known asymptotic formula for the tail of the normal distribution, we have

$$w(u) = \frac{\phi(u)}{1 - \Phi(u)} \sim u, \qquad \text{for } u \to \infty. \tag{7.1}$$

We will show that the Sojourn Limit Theorem is valid for stationary Gaussian processes of a very general type on an interval $[0, t]$ where t is sufficiently small.

We will assume that $1 - r(t)$ is regularly varying of index α, for some $0 < \alpha \le 2$, for $t \to 0$. Define $v = v(u)$ as the largest solution of the equation $u^2(1 - r(1/v)) = 1$; then it follows from the regularly varying property that

$$u^2(1 - r(t/v)) \to t^\alpha, \qquad \text{for } u \to \infty. \tag{7.2}$$

It also follows from Karamata's representation of regularly varying functions that for every $\alpha' > \alpha$ there exists $t_0 > 0$ such that

$$u^2(1 - r(s/v)) \ge \tfrac{1}{2}\, s^{\alpha'}, \qquad \text{for all } 0 \le s \le 1 \quad \text{and} \quad s/v < t_0 \tag{7.3}$$

(see [20] [21]; also [7], page 1015).

By virtue of the Gaussian character of the process, the convergence of the finite dimensional distributions stipulated by Assumption 3.I or the hypotheses of Theorems 5.1 and 5.2 is implied by the convergence of the conditional means and incremental variances. Let us verify the conditions in the hypothesis of Theorem 5.2. We have already noted that the conditions on the marginal distribution are known to hold for Φ; thus, we have to verify conditions i) and ii) of the latter theorem. Here $w \sim u$, $\mu(t, x) = E[X(t) \mid X(0) = x]$, and

$$E[X(t/v) \mid X(0) = u + y/u] = (u + y/u)r(t/v) \tag{7.4}$$

and

$$\operatorname{Var}\{X(t/v) - X(s/v) \mid X(0) = u + y/u\} = 2\left(1 - r\left(\frac{t-s}{v}\right)\right) - \left(r\left(\frac{t}{v}\right) - r\left(\frac{s}{v}\right)\right)^2. \tag{7.5}$$

The relations (7.2) and (7.4) imply that the limit (5.9) exists, and that

$$m(t) = -t^\alpha. \tag{7.6}$$

The process (5.8) has conditional mean 0, and, by (7.2) and (7.5), its incremental conditional variance function converges to

$$E|V(t) - V(s)|^2 = 2|t-s|^\alpha. \tag{7.7}$$

Theorem 5.2 has been shown to hold, so that Assumption 3.I also does.

Next we show that Assumption 3.II holds as long as $t < t_0$, where t_0 is given in condition (7.3). According to an estimate of Pickands [34], we have

$$P(X(s) > u \mid X(0) > u) \leq \frac{3\phi(u)}{u(1-\Phi(u))}\left\{1 - \Phi\left(u\left[\frac{1-r(s)}{1+|r(s)|}\right]^{1/2}\right)\right\}.$$

Therefore, Assumption 3.II holds if

$$\lim_{d\to\infty} \operatorname{limsup}_{u\to\infty} v\int_{d/v}^{t}\left\{1 - \Phi\left(u\left[\frac{1-r(s)}{1+|r(s)|}\right]^{1/2}\right)\right\} ds = 0.$$

By a change of variable of integration and the inequality $1 + |r| \leq 2$, the relation above is implied by

$$\lim_{d\to\infty} \operatorname{limsup}_{u\to\infty} \int_{d}^{tv}\left\{1 - \Phi\left(\frac{u}{\sqrt{2}}\left[1 - r\left(\frac{s}{v}\right)\right]^{1/2}\right)\right\} ds = 0. \tag{7.8}$$

If $t < t_0$, then, by (7.3), the integral above is at most equal to

$$\int_d^\infty \left\{1 - \Phi\left(\frac{1}{2}s^{\alpha'/2}\right)\right\} ds,$$

so that (7.8) holds.

The particular case $\alpha = 2$, where the Gaussian process has a mean square derivative $X'(t)$, is also covered by Theorem 6.2. For simplicity, let us first consider the particular case where the second derivative r'' of r satisfies $r''(0) = -2$. $X(0)$ and $X'(0)$ are independent and normally distributed with common mean 0 and with variances 1 and 2, respectively. Therefore we may take $\xi = X'(0)$ in the hypothesis of Theorem 6.2. Furthermore, it follows from (7.4) and (7.5), and the relation (7.2) with $u = w = v$ and $\alpha = 2$, that the random variable $w[X(t/w) - X(0)] - tX'(0)$ has a conditional normal distribution, given $X(0) = u + y/w$, with mean

$$(u + y/u)u(1 - r(t/u)) \to \tfrac{1}{2}t^2 r''(0) = -t^2$$

and variance dominated by the unconditional variance

$$E[u[X(t/u) - X(0)] - tX'(0)]^2,$$

which tends to 0 because $X'(0)$ exists in mean square. Therefore the conclusion of Theorem 6.2 holds with ξ normally distributed and $M(t) = -t^2$. The function $\Gamma(x)$ in (3.8) takes the form

$$\Gamma(x) = P\left(\int_0^\infty I_{[t\xi+\eta-t^2>0]}\, dt > x\right). \tag{7.9}$$

We claim that $\Gamma(x)$ is equivalent to the expression in the formula

$$\Gamma(x) = P(x\xi + \eta - x^2 > 0). \tag{7.10}$$

To see this we note that the function of t, $t\xi + \eta - t^2$, is positive for $t = 0$ because $\eta > 0$. Furthermore, the equation $t\xi + \eta - t^2 = 0$ has exactly two real roots τ_1 and τ_2, and these satisfy $\tau_1 < 0 < \tau_2$. Therefore, the indicator function in the integrand in (7.9) is equal to 1 if and only if $0 < t < \tau_2$, and so the integral has the value τ_2. It follows that $\tau_2 > x$ if and only if $t\xi + \eta - t^2 > 0$ for $t = x$; therefore, the formula (7.10) follows from (7.9).

We claim that

(7.11) $$-\frac{d}{dx}P(x\xi + \eta - x^2 > 0) = \pi^{-1/2}\exp\left(-\frac{1}{4}x^2\right).$$

Indeed, we note that

$$P(x\xi + \eta - x^2 > 0) = \int_0^\infty \left[1 - \Phi\left(\frac{x}{\sqrt{2}} - \frac{y}{x\sqrt{2}}\right)\right]e^{-y}\,dy,$$

and take the negative of the derivative with respect to x.

Our conclusion from (7.9), (7.10) and (7.11) is

(7.12) $$-\Gamma'(x) = \pi^{-1/2}\exp(-\tfrac{1}{4}x^2).$$

This coincides with the result in [2], where we found $1 - \Gamma'(x)/\Gamma'(0)$ to be $1 - \exp(-\tfrac{1}{8}x^2)$, where the factor $\tfrac{1}{8}$ occurs because the scaling was different. The result (7.12) can be easily extended to the general case where $r''(0)$ is not necessarily equal to -2.

8. Application to a cosine wave process with a general amplitude distribution. In this section we apply the results of Section 6 to a class of simple processes which are, generally, non-Gaussian, but which includes a particular Gaussian process. Let Y be a nonnegative random variable, and let $G(y)$ be the tail of its distribution function:

(8.1) $$G(y) = P(Y > y), \qquad y > 0.$$

Let Z be uniformly distributed on $[0, 2\pi]$, and be distributed independently of Y. Form the process

(8.2) $$X(t) = Y\cos(t - Z), \qquad -\infty < t < \infty.$$

It is easily verified that $X(t)$ is stationary, and that $EX(0) = 0$. Put

$$U = Y\cos Z, \qquad V = y\sin Z;$$

then, by an elementary identity, $X(t)$ is representable as

(8.3) $$X(t) = U\cos t + V\sin t, \qquad -\infty < t < \infty.$$

We make the following assumptions about G:

8.I The distribution function $1 - G(y)$ is in the domain of attraction of Λ.

8.II $$\lim_{u\to\infty}\frac{G(u)}{u\int_u^\infty G(y)\,dy} = 1.$$

Define

(8.4) $$w(u) = \frac{G(u)}{\int_u^\infty G(y)\,dy};$$

then, according to (4.1) and (4.2), condition 8.I implies

(8.5) $$\lim_{u\to\infty}\frac{G(u + x/w)}{G(u)} = e^{-x}, \qquad -\infty < x < \infty;$$

and 8.II implies $w \sim u$, so that

(8.6) $$\lim_{u\to\infty}\frac{G(u + x/u)}{G(u)} = e^{-x}, \qquad -\infty < x < \infty.$$

Applying Scheffe's theorem as in the proof of Lemma 5.1, we conclude from (8.5) that

$$\lim_{u\to\infty}\int_0^\infty \left|\frac{G(u+x/w)}{G(u)}-e^{-x}\right| dx = 0. \tag{8.7}$$

By the change of variable $x = \frac{1}{2} y^2$ we also obtain

$$\lim_{u\to\infty}\int_0^\infty y\left|\frac{G(u+y^2/2w)}{G(u)}-e^{-y^2/2}\right| dy = 0,$$

which implies

$$\lim_{u\to\infty}\int_0^\infty \left|\frac{G(u+y^2/2w)}{G(u)}-e^{-y^2/2}\right| dy = 0. \tag{8.8}$$

The Taylor expansion of the cosine and the relation (8.6) also imply

$$\lim_{u\to\infty}\frac{G\left(\dfrac{u}{\cos(z/u)}\right)}{G(u)} = e^{-z^2/2}, \qquad -\infty < z < \infty. \tag{8.9}$$

Finally, the definition (8.4) and the condition 8.II imply

$$\lim_{u\to\infty} u^2\int_1^\infty \frac{G(ux)}{G(u)}\,dx = 1. \tag{8.10}$$

LEMMA 8.1.

$$\lim_{u\to\infty}\frac{uP(U>u)}{P(Y>u)} = (2\pi)^{-1/2}. \tag{8.11}$$

PROOF. The definition of U implies

$$P(U>v) = \pi^{-1}\int_0^{\pi/2} G(u/\cos z)\,dz. \tag{8.12}$$

Therefore, for arbitrary $\epsilon > 0$ the ratio in (8.11) may be expressed as the sum of two terms,

$$\pi^{-1}\int_\epsilon^{\pi/2}\frac{uG(u/\cos z)}{G(u)}\,dz, \tag{8.13}$$

and

$$\pi^{-1}\int_0^{u\epsilon}\frac{G(u/\cos(z/u))}{G(u)}\,dz. \tag{8.14}$$

Change the variable of integration in (8.13) to $x = (\cos z)^{-1}$; then, it can be seen by a direct calculation that the integral is at most a constant multiple of

$$u\int_{1/\cos\epsilon}^\infty \frac{G(ux)}{G(u)}\,dx,$$

which, by (8.10), converges to 0.

By a direct computation with the Taylor formula for the secant function, we can show that (8.14) is asymptotically unchanged if the integrand is replaced by

$$\frac{G\left(u+\dfrac{1}{2}\dfrac{x^2}{w}\right)}{G(u)}.$$

The relation (8.8) then implies that (8.14) converges to

$$\pi^{-1}\int_0^\infty \exp(-\tfrac{1}{2}x^2)\,dx = (2\pi)^{-1/2}.$$

LEMMA 8.2. *The limiting conditional distribution of V, given $U > u$, is, for $u \to \infty$, standard normal.*

PROOF. Since (U, V) has the same joint distribution as $(U, -V)$, the conditional distribution of V, given $U > u$, is symmetric; thus, it suffices to evaluate the limit of

$$P(V > x \mid U > u) \tag{8.15}$$

only for $x > 0$.

It is a consequence of the definitions of U and V that

$$\begin{aligned} P(V > x, U > u) &= \frac{1}{2\pi}\int_0^{\pi/2} P(Y > x/\sin z, y > u/\cos z)\,dz \\ &= \frac{1}{2\pi}\int_0^{\tan^{-1}(x/u)} G(x/\sin z)\,dz \\ &\quad + \frac{1}{2\pi}\int_{\tan^{-1}(x/u)}^{\pi/2} G(u/\cos z)\,dz, \end{aligned} \tag{8.16}$$

which, by a change of variable, is

$$\frac{1}{2\pi u}\int_0^{u\tan^{-1}(x/u)} G\left(\frac{x}{\sin(z/u)}\right)dz + \frac{1}{2\pi u}\int_{u\tan^{-1}(x/u)}^{u\pi/2} G\left(\frac{u}{\cos(z/u)}\right)dz,$$

which is asymptotic to

$$\frac{1}{2\pi u}\int_0^{x} G(ux/z)\,dz + \frac{1}{2\pi u}\int_x^{u\pi/2} G\left(\frac{u}{\cos(z/u)}\right)dz.$$

Therefore, by Lemma 8.1, the conditional probability (8.15) is asymptotic to

$$(2\pi)^{-1/2}\left\{\int_0^x \frac{G(ux/z)}{G(u)}\,dz + \int_x^{u\pi/2}\frac{G(u/\cos(z/u))}{G(u)}\,dz\right\}. \tag{8.17}$$

The first term in the braces above converges to 0 for $u \to \infty$. In fact, by a change of variable, we have

$$\int_0^x \frac{G(ux/z)}{G(u)}\,dz = x\int_1^\infty \frac{G(uy)}{G(u)}\frac{dy}{y^2} \le x\int_1^\infty \frac{G(uy)}{G(u)}\,dy,$$

and the last expression converges to 0 for $u \to \infty$ in accordance with (8.10). The second term in the braces in (8.17) converges to $\int_x^\infty e^{-y^2/2}\,dy$. Indeed, the proof is the same as that used to estimate the sum of the terms (8.13) and (8.14) with the exceptional point that the lower limit here is $x \ge 0$ in place of $x = 0$. This completes the proof of the lemma.

LEMMA 8.3. *The process* (8.3) *satisfies the conditions of Theorem* 6.1 *with $w = u$ and*

$$M(t) = -\tfrac{1}{2}t^2, \tag{8.18}$$

and with ξ having a standard normal distribution.

PROOF. Since $X(0) = U$, Lemma 8.1 and Assumption 8.I imply that the distribution of $X(0)$ is in the domain of attraction of Λ.

Since $X'(0) = V$, Lemma 8.2 implies condition i) of Theorem 6.1 with a standard normal ξ.

The random variable $u[X(t/u) - X(0)] - tX'(0)$ assumes the particular form

$$uU(\cos(t/u) - 1) + V(u \sin(t/u) - t). \tag{8.19}$$

The first term in (8.19) converges in conditional probability, given $U > u$, to $-\frac{1}{2}t^2$. On the one hand, for every $x < 1$, $P(U/u < x \mid U > u) = 0$; and, on the other hand, for every $x > 1$, Lemma 8.1 implies

$$P(U/u > x \mid U > u) \sim G(ux)/G(u),$$

and the right hand member, by (8.10) and a weak compactness argument, converges to 0 for $u \to \infty$.

By Lemma 8.2 and the fact that $u \sin(t/u) \to t$ for $u \to \infty$, the second term in (8.19) converges in conditional probability to 0. This completes the verification of condition ii) of Theorem 6.1.

THEOREM 8.1. *Under* 8.I *and* 8.II *and for* $T \leq \pi/2$, *the process* (8.3) *satisfies the conditions in the hypothesis of Theorem* 3.1, *and the conclusion of the latter holds with* $Z(t)$ *specified by* (6.1) *and* (8.18).

PROOF. Lemma 8.3 ensures that assumption 3.I is satisfied and furnishes the form of $Z(t)$.

The proof is completed by verifying assumption 3.II with $v = u$. Let us estimate the conditional probability $P(X(s) > u \mid X(0) > u)$ with $2s$ in the place of s:

$$\begin{aligned}\frac{P(X(2s) > u, X(0) > u)}{P(X(0) > u)} &= \frac{P(X(s) > u, X(-s) > u)}{P(X(0) > u)} \\ &= \frac{P(U \cos s + V \sin s > u,\ U \cos s - V \sin s > u)}{P(X(0) > u)} \\ &\leq P(U \cos s > u)/P(U > u).\end{aligned}$$

By Lemma 8.1 the latter is asymptotic to $G(u/\cos s)/G(u)$. Therefore, the expression following the limit sign in (3.6) is asymptotic to

$$\int_d^{Tu} \frac{G(u/\cos(s/u))}{G(u)}\, ds,$$

which, by the same argument as in the proof of Lemma 8.1, converges, for $u \to \infty$, to

$$\int_d^\infty \exp\left(-\frac{x^2}{2}\right) dx.$$

This tends to 0 for $d \to \infty$. (Note that in order to use the argument in the proof of Lemma 8.1 we have to assume that $T \leq \pi/2$.)

We note that if Y has the Rayleigh distribution then $X(t)$ in (8.2) is a stationary Gaussian process with mean 0 and covariance function $r(t) = \cos t$. It can be shown that the Rayleigh distribution satisfies the conditions 8.I and 8.II, so that Theorem 8.1 holds. The process with $M(t)$ specified by (8.18) differs from the corresponding process in (7.9) only by a scale factor $1/\sqrt{2}$ in the time parameter t. Indeed, the latter process was derived under the assumption $r''(0) = -2$, but the former has $r''(0) = -1$.

The cosine wave process studied in this section is of interest because it is the typical term of a random Fourier series with independent coefficients and independent phases. Preliminary results on the extension of Theorem 8.1 to a finite Fourier sum have already been obtained, and are expected to appear in a future publication of the author.

9. Application to Markov processes. Let $X(t)$, $-\infty < t < \infty$, be a stationary Markov process. Assume that it has a transition density function $p(t; x, y)$ defined as the conditional density of $X(t+s)$ at y given $X(s)=x$, for any s. Let $F(x)$ be the stationary marginal distribution function with the density $f=F'$.

THEOREM 9.1. *Suppose that F satisfies the conditions in the hypothesis of Lemma 5.1, and $w=w(u)$ is defined by (5.1). Let the following three conditions hold:*

i) *the process is reversible:*

$$f(x)p(t;x,y)=f(y)p(t;y,x) \quad \text{for all } x, y, t. \tag{9.1}$$

ii) *there exists a function $v=v(u)\to\infty$ and a Markov transition density function $q(t;x,y)$ such that*

$$\lim_{u\to\infty} w^{-1}p(t/v;u+x/w,u+y/w)=q(t;x,y), \tag{9.2}$$

for all x, y, t.

iii) *For $t>0$*

$$\lim_{d\to\infty}\limsup_{u\to\infty}\left\{\frac{v\int_{d/v}^{t}\int_u^\infty\int_u^\infty f(x)p(s;x,y)\,dx\,dy\,ds}{1-F(u)}\right\}=0. \tag{9.3}$$

Then the conditions of the Sojourn Limit Theorem are satisfied and the process $Z(t)$ in (3.7) is the Markov process with the transition density $q(t;x,y)$ and an initial exponential distribution.

PROOF. In order to verify Assumption 3.I, let us, for simplicity, restrict the time parameter set to $t\geq 0$. For arbitrary $0<t_1<\cdots<t_k$, the conditional joint density of $(X(t_i), i=1,\cdots,k)$ at $(x_1,\cdots,x_k)$, given $X(0)=y$, is

$$p(t_1;y,x_1)\prod_{i=2}^k p(t_i-t_{i-1};x_{i-1},x_i).$$

Therefore, the conditional joint density of $[w(X(t_i/v)-u), i=1,\cdots,k]$, given $X(0)>u$, is

$$w^{-k-1}(1-F(u))^{-1}\int_0^\infty f(u+y/w)p\left(t_1/v;u+\frac{y}{w},u+\frac{x_1}{w}\right)dy \cdot\prod_{i=2}^k p\left(\frac{t_i-t_{i-1}}{v};u+\frac{x_{i-1}}{w},u+\frac{x_i}{w}\right) \tag{9.4}$$

Apply (9.1) to the integrand in (9.4), and rewrite the entire displayed expression as

$$\frac{f(u+x_1/w)}{w(1-F(u))}\cdot\int_0^\infty w^{-1}p\left(\frac{t_1}{v};u+\frac{x_1}{w},u+\frac{y}{w}\right)dy \cdot\prod_{i=2}^k w^{-1}p\left(\frac{t_i-t_{i-1}}{v};u+\frac{x_{i-1}}{w},u+\frac{x_i}{w}\right). \tag{9.5}$$

By (5.2) the first factor in (9.5) converges to e^{-x}. For each t_1 and x_1, $w^{-1}p$ is a density function in y, which, by condition ii) above, converges pointwise to q. Therefore, by Scheffe's theorem (see proof of Lemma 5.1), $\int_0^\infty w^{-1}p\,dy\to\int_0^\infty q\,dy$. Finally, condition ii) above also implies the convergence of the product in (9.5) to the corresponding product of q-functions. Therefore, (9.5) converges to

$$e^{-x_1}\int_0^\infty q(t_1;x_1,y)\,dy\cdot\prod_{i=2}^k q(t_i-t_{i-1};x_{i-1},x_i).$$

The latter is equal to

$$\int_0^\infty e^{-y}q(t_1;y,x_1)\,dy\cdot\prod_{i=2}^k q(t_i-t_{i-1};x_{i-1},x_i) \tag{9.6}$$

because (5.2), (9.1) and (9.2) imply that the integrand in (9.6) is a symmetric function of (x_1, y). This completes the proof of the validity of Assumption 3.I and establishes the form of the process $Z(t)$ stated in the theorem.

Finally we note that the condition (9.3) is simply a statement of Assumption 3.II in the Markovian context.

10. Application to the stationary χ^2-process. Several authors have recently investigated the high level crossings and the limiting distribution of the maximum for the stationary χ^2-process. It is a non-Gaussian process derived from Gaussian processes in the same way that the Chi squared distribution is obtained from the normal. Let $X_1(t), \cdots, X_k(t)$ be independent stationary Gaussian processes with means 0, variance 1, and common covariance function $r(t)$; then $\|\mathbf{X}(t)\|^2 = \sum_{i=1}^k X_i^2(t)$ is the stationary χ^2-process. Sharpe [39] and Lindgren [27] have studied the process under the assumption that $r''(0)$ is finite. We will apply our Sojourn Limit Theorem to the more general case where we assume the conditions of Section 7 above.

THEOREM 10.1. *Let the function* $1 - r(t)$ *of the component Gaussian process* $X_i(t)$ *be regularly varying of index* α, *for some* α, $0 < \alpha \le 2$. *Let* $v = v(u)$ *be the function defined in Section* 7 *in terms of* $r(t)$, *and put*

$$L_t(u) = \int_0^t I_{[\|\mathbf{X}(s)\|^2 > ku^2]}\, ds. \tag{10.1}$$

Let $V(t)$, $-\infty < t < \infty$, *be a Gaussian process with mean* 0, *and covariance function*

$$4k(|s|^\alpha + |t|^\alpha - |t-s|^\alpha); \tag{10.2}$$

and let η *be an exponential random variable, independent of* $V(t)$. *Then the conclusions of the Sojourn Limit Theorem and Theorem* 5.2, *with the process* $Z(t)$ *in* (5.10) *of the form* $V(t) - 2k|t|^\alpha + \eta$, *hold for* $0 < t < t_0$, *where* t_0 *is defined by* (7.3).

PROOF. We will show that the process $\|X(t)\|^2$ satisfies the conditions of the Sojourn Limit Theorem. Consider the process:

$$\|\mathbf{X}(t/v\|^2 - ku^2, \qquad -\infty < t < \infty, \tag{10.3}$$

given the condition $\|X(0)\|^2 > ku^2$. Let $\psi_k(y)$ be the Chi squared density with k degrees of freedom:

$$\psi_k(y) = (2^{k/2}\Gamma(k/2))^{-1} y^{\frac{k}{2}-1} e^{-y/2}, \qquad y > 0. \tag{10.4}$$

Then, as is well known, $\|\mathbf{X}(0)\|^2$ has the density ψ_k, and so, for any event A,

$$P(A \mid \|\mathbf{X}(0)\|^2 > ku^2) = \frac{\displaystyle\int_{ku^2}^\infty P(A \mid \|\mathbf{X}(0)\|^2 = y)\psi_k(y)\, dy}{\displaystyle\int_{ku^2}^\infty \psi_k(y)\, dy}. \tag{10.5}$$

The right hand member of (10.5) is transformed into

$$\int_0^\infty P(A \mid \|\mathbf{X}(0)\|^2 = ku^2 + y) \left\{ \frac{\psi_k(ku^2 + y)}{\displaystyle\int_{ku^2}^\infty \psi_k(x)\, dx} \right\} dy. \tag{10.6}$$

The inequality (for $k \ge 2$)

$$\int_y^\infty \psi_k(x)\, dx \ge \frac{1}{2}\psi_k(y), \qquad y > 0, \tag{10.7}$$

follows, by integration by parts, from (10.4). From (10.4) and (10.7) it may be inferred that the expression in the braces in (10.6) is dominated by

$$(1+y)^{\frac{k}{2}-1} e^{-y/2} \qquad \text{for} \qquad u^2 > k^{-1}$$

and converges to $\frac{1}{2} e^{-y/2}$ for $u \to \infty$. Therefore, the limit of (10.6) is equal to

$$\int_0^\infty \lim_{u\to\infty} P(A \mid \|\mathbf{X}(0)\|^2 = ku^2 + y)\ \tfrac{1}{2} e^{-y/2}\, dy \tag{10.8}$$

if the limit above exists.

Let A in (10.8) above be a finite-dimensional rectangle in the sample space of the process (10.3). The latter process may be written as the sum

$$\begin{aligned}(10.9)\quad & \|\mathbf{X}(t/v) - r(t/v)\mathbf{X}(0)\|^2 \\ & - 2 \textstyle\sum_{i=1}^k [X_i(t/v) - r(t/v)X_i(0)]X_i(0) + r^2(t/v)\|\mathbf{X}(0)\|^2 - ku^2.\end{aligned}$$

It follows from the assumed independence of the component processes, and from the independence of $X_i(0)$ and $X_i(t/v) - E[X_i(t/v) \mid X_i(0)]$ that the $2k$ random variables

$$\begin{aligned}(10.10)\qquad & X_i(t/v) - r(t/v)X_i(0), \qquad i = 1, \cdots, k, \\ & X_i(0), \qquad i = 1, \cdots, k,\end{aligned}$$

are mutually independent. Therefore, an elementary computation yields

$$E\{\|\mathbf{X}(t/v) - r(t/v)\mathbf{X}(0)\|^2 \,|\, \|\mathbf{X}(0)\|^2 = ku^2 + y\} \to 0.$$

Therefore, by (10.9), the process (10.3), conditioned by $\|\mathbf{X}(0)\|^2 = ku^2 + y$, is asymptotically distributed as the conditioned process

$$-2 \textstyle\sum_{i=1}^k [X_i(t/v) - r(t/v)X_i(0)]X_i(0) + ku^2(r^2(t/v) - 1) + yr^2(t/v). \tag{10.11}$$

This conditioned process is Gaussian with mean

$$ku^2(r^2(t/v) - 1) + yr^2(t/v) \tag{10.12}$$

and covariance function

$$4\,(ku^2 + y)\left[r\left(\frac{t-s}{v}\right) - r\left(\frac{s}{v}\right) r\left(\frac{t}{v}\right)\right]. \tag{10.13}$$

Indeed, the mutual independence of the random variables (10.10) implies that the conditional distribution of the first term in (10.11), given $\mathbf{X}(0)$, is Gaussian with mean 0 and covariance function

$$4\left[r\left(\frac{t-s}{v}\right) - r\left(\frac{s}{v}\right) r\left(\frac{t}{v}\right)\right] \textstyle\sum_{i=1}^k X_i^2(0).$$

Since this depends on $\mathbf{X}(0)$ only through $\|\mathbf{X}(0)\|^2$, the conditioning by $\mathbf{X}(0)$ is equivalent to the coarser conditioning by $\|\mathbf{X}(0)\|^2$. Replacing the latter by $ku^2 + y$ in the expression for the covariance displayed above, we obtain (10.13). The last two terms of (10.11) are constants, and contribute the mean (10.12). The relation (7.2) implies that (10.12) converges to $-2kt^\alpha + y$, and that (10.13) converges to (10.2). We conclude that the conditional probability in the integrand in (10.8) converges to the probability for the limiting process $V(t) - 2kt^\alpha + y$. This completes the verification of Assumption 3.I.

Finally, we shall verify Assumption II:

$$\lim_{d\to\infty} \limsup_{u\to\infty} \int_d^{Tv} P\left(\left\|\mathbf{X}\left(\frac{t}{v}\right)\right\|^2 > ku^2 \,\Big|\, \|\mathbf{X}(0)\|^2 > ku^2\right) dt = 0 \tag{10.14}$$

for all $T > 0$ sufficiently small. By the calculation following (10.5), the integrand in (10.14) is bounded, for $u^2 > k^{-1}$, by

$$(10.15)\qquad \int_0^\infty P(\|\mathbf{X}(t/v)\|^2 > ku^2 \mid \|\mathbf{X}(0)\|^2 = ku^2 + y)(1+y)^{\frac{k}{2}-1} e^{-y/2}\, dy.$$

For fixed t, put

$$\xi_i = \frac{X_i(t/v) - r(t/v)X_i(0)}{(1 - r^2(t/v))^{1/2}}, \qquad i = 1, \cdots, k,$$

and $\boldsymbol{\xi} = (\xi_i)$. Then $\xi_i, X_i(0), i = 1, \cdots, k$ are mutually independent random variables with standard normal distributions; see (10.10). Write $\mathbf{X} = \mathbf{X}(0)$ and $X_i = X_i(0)$ and $r = r(t/v)$; then the conditional probability in the integrand in (10.15) may be written as

$$P(\|\boldsymbol{\xi}(1 - r^2)^{1/2} + r\mathbf{X}\|^2 > ku^2 \mid \|\mathbf{X}\|^2 = ku^2 + y).$$

By an application of the Chebyshev inequality to the random variable $\exp(\frac{1}{4}\|\boldsymbol{\xi}(1 - r^2)^{1/2} + r\mathbf{X}\|^2)$, the conditional probability above is at most equal to

$$(10.16)\qquad \exp(-\tfrac{1}{4}ku^2)E(\exp[\tfrac{1}{4}\|\boldsymbol{\xi}(1 - r^2)^{1/2} + r\mathbf{X}\|^2] \mid \|\mathbf{X}\|^2 = ku^2 + y).$$

We claim that this expression is equal to

$$(10.17)\qquad \exp\left(-\frac{1}{4}ku^2\right)\left(\frac{2}{1+r^2}\right)^{k/2} \exp\left[\frac{1}{2}\frac{r^2}{1+r^2}(ku^2 + y)\right].$$

Indeed the independence of $(\xi_i, X_i, i = 1, \cdots, k)$ implies

$$(10.18)\qquad E\{\exp(\tfrac{1}{4}\|\boldsymbol{\xi}(1 - r^2)^{1/2} + r\mathbf{X}\|^2) \mid \mathbf{X}\} = \textstyle\prod_{i=1}^k E\{\exp[\tfrac{1}{4}(\xi_i(1 - r^2)^{1/2} + rX_i)^2] \mid X_i\}.$$

By the identity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp[\tfrac{1}{4}(x(1 - r^2)^{1/2} + rX)^2 - \tfrac{1}{2}x^2]\, dx = \left(\frac{2}{1+r^2}\right)^{1/2} \exp\left[\frac{r^2X^2}{2(1+r^2)}\right]$$

(obtained by completion of the square), the right hand member of (10.18) is equal to

$$\left(\frac{2}{1+r^2}\right)^{k/2} \exp\left[\frac{r^2\|\mathbf{X}\|^2}{2(1+r^2)}\right].$$

Passing to the coarser condition $\|\mathbf{X}\|^2 = ku^2 + y$ in (10.18), we obtain (10.17).

We may now use the bound (10.17) for the conditional probability in the integrand of (10.15). Since $r^2/(1 + r^2) \le \frac{1}{2}$, (10.15) has the bound

$$(10.19)\qquad 2^{k/2}\exp[-\tfrac{1}{4}u^2(1 - r^2(t/v))]\int_0^\infty (1+y)^{\frac{k}{2}-1} e^{-y/4}\, dy.$$

If T is smaller than the number t_0 in (7.3), then the latter inequality implies that (10.19) is at most equal to

$$\text{constant} \times \exp(-t^{\alpha'}/8).$$

Replacing the integrand in (10.14) by the function above, we conclude that (10.14) holds.

11. The distribution tail of the maximum: a general lower bound. The object of the next several sections is to establish the formula

$$(11.1)\qquad \lim_{u\to\infty}\frac{P(\sup_{[0,t]}X(s) > u)}{vt(1 - F(u))} = -\Gamma'(0)$$

under general conditions, where $\Gamma'(x)$ is the right hand derivative of the function $\Gamma(x)$ in

(3.7). If X has continuous sample functions then $L_t(u) > 0$ if and only if $\max_{[0,t]} X(s) > u$; therefore, it is plausible that (11.1) follows from (3.21) by putting $x = 0$. The challenge is to formulate the conditions under which this can be done.

Our first result is that a lower bound for the left hand member can always be obtained in this way:

THEOREM 11.1. *If the Sojourn Limit Theorem holds, then*

$$-\Gamma'(0) = \lim_{x\downarrow 0} x^{-1}(\Gamma(0) - \Gamma(x)) \tag{11.2}$$

exists and is nonnegative, and

$$-\Gamma'(0) \leq \liminf_{u\to\infty} \frac{P(\sup_{[0,t]} X(s) > u)}{vt(1 - F(u))}. \tag{11.3}$$

PROOF. According to Corollary 3.1, $-\Gamma'(y)$ is defined as the negative of the Radon-Nikodym derivative of $\Gamma(y)$. With the possible exception of a set of measure 0, $-\Gamma'(y)$ is nonnegative and nonincreasing for $y > 0$. As a nonincreasing function, $\Gamma(x)$ has a limit $\Gamma(0)$ for $x \downarrow 0$, and, since $-\Gamma'(y)$ has a monotonic version, the limit in (11.2) exists. We define $-\Gamma'(0)$ to be this limit.

If X spends positive time above the level u, then its supremum exceeds u; thus,

$$P(vL_t > x) \leq P(\sup_{[0,t]} X(s) > u), \tag{11.4}$$

for every $x > 0$. Averaging x over an arbitrary interval $[y, y']$, $0 < y < y'$, we obtain

$$(y' - y)^{-1} \int_y^{y'} \frac{P(vL_t > x)}{E(vL_t)}\, dx \leq \frac{P(\sup_{[0,t]} X(s) > u)}{E(vL_t)}.$$

Let $u \to \infty$ and apply the Sojourn Limit Theorem:

$$-\frac{\Gamma(y') - \Gamma(y)}{y' - y} \leq \liminf_{u\to\infty} \frac{P(\sup_{[0,t]} X(s) > u)}{E(vL_t)}.$$

Let $y \downarrow 0$, and then $y' \downarrow 0$ to get (11.3).

12. An exact estimate of the distribution of the maximum for a stochastically differentiable process. We now find the exact form of $-\Gamma'(0)$ and show that (11.1) holds for the stochastically differentiable processes discussed in Sections 6, 7, and 8.

LEMMA 12.1. *Suppose that the process $Z(t)$ in the statement of the Sojourn Limit Theorem is of the form*

$$Z(t) = \xi t + \eta + M(t), \tag{12.1}$$

where ξ and η are independent random variables, and the latter has an exponential distribution; and that $M(0) = 0$ and $M(t) \leq 0$ for all t. Then

$$E\xi^- \leq -\Gamma'(0), \tag{12.2}$$

where $\xi^- = -\min(\xi, 0)$.

PROOF. By the definition (3.7) of Γ, we have

$$\Gamma(0) - \Gamma(x) = P\left(0 < \int_0^\infty I_{[\xi t + \eta + M(t) > 0]}\, dt < x\right). \tag{12.3}$$

The integral above is positive because $M(0) = 0$ and $\eta > 0$ almost surely. Therefore, the right hand member above is equal to

(12.4) $$P\left(\int_0^\infty I_{[\xi t+\eta+M(t)>0]}\,dt < x\right).$$

Since $M(t) \le 0$, the removal of the term $M(t)$ from the indicator function in (12.4) cannot decrease the integral, and this, in turn, cannot increase the probability in (12.4). Therefore, the expression (12.4) is at least equal to

(12.5) $$P\left(\int_0^\infty I_{[\eta>-\xi t]}\,dt < x\right).$$

The integral above has the values given by

$$\int_0^\infty I_{[\eta>-\xi t]}\,dt = \infty \qquad \text{if } \xi \ge 0$$

$$= \eta/(-\xi) \qquad \text{if } \xi < 0;$$

therefore, (12.5) is equal to

$$P(\eta < x\xi^-),$$

which, by the assumptions on ξ and η, is equal to

(12.6) $$E(1 - e^{-x\xi^-}).$$

From (12.3), (12.4), (12.5) and (12.6) we infer

$$x^{-1}E(1 - e^{-x\xi^-}) \le x^{-1}(\Gamma(0) - \Gamma(x)).$$

Letting $x \downarrow 0$, we obtain (12.2).

LEMMA 12.2. *Let $X(t)$ satisfy Assumption* 3.I *of the Sojourn Limit Theorem, and this additional condition on the convergence of the bivariate distributions of* (3.4):

(12.7) $$\lim_{u\to\infty} h^{-1}P(w\cdot(X(h/v)-u) \le 0 \mid X(0) > u) = h^{-1}P(Z(h) \le 0)$$

uniformly in $h > 0$. Then

(12.8) $$\limsup_{u\to\infty} \frac{P(\sup_{[0,t]}X(s) > u)}{tv(1 - F(u))} \le \limsup_{h\to 0} h^{-1}P(Z(h) \le 0).$$

PROOF. Decompose the interval $[0, t]$ into $m = [tv]$ subintervals of equal length. Without loss of generality and for the purpose of simplicity, we shall suppose that tv is an integer, so that the subintervals are actually of equal length $1/v$. If $\sup_{[0,t]}X(s) > u$, then either $X(t) > u$, or else for at least one of the m subintervals

$$X(b) \le u \quad \text{and} \quad \sup_{[a,b]}X(s) > u,$$

where the subinterval is of the form $[a, b]$ with $b - a = 1/v$. Thus, by Boole's inequality and stationarity, we have

(12.9) $$P(\sup_{[0,t]}X(s) > u) \le 1 - F(u) + tvP(X(1/v) \le u, \sup_{[0,v^{-1}]}X(s) > u).$$

By the separability of the process, the second term on the right hand side of (12.9) is equal to

$$tv \lim_{n\to\infty} P(X(1/v) \le u, \sup_{1\le k\le n-1}X(k/nv) > u),$$

which, by reasoning similar to that leading to (12.9), is at most

$$tv \limsup_{n\to\infty} nP(X(0) \ge u, X(1/vn) \le u).$$

Thus, it follows upon division of each member of (12.9) by $tv(1 - F(u))$, that

$$\limsup_{u\to\infty} \frac{P(\sup_{[0,t]} X(s) > u)}{tv(1-F(u))} \le \limsup_{u\to\infty} \limsup_{n\to\infty} \frac{nP(X(0) > u, X(1/vn) \le u)}{P(X(0) > u)}.$$

Condition (12.7) permits the interchange of the order of taking limits above, and (12.8) follows.

LEMMA 12.3. *If $Z(t)$ is of the form* (12.1) *and $M(t) = o(t)$ for $t \to 0$, then*

$$\limsup_{h\to\infty} h^{-1}P(Z(h) \le 0) \le E\xi^{-}.$$

PROOF. Since η is exponentially distributed, it follows that

$$h^{-1}P(Z(h) \le 0) = h^{-1}E\{1 - \exp[-(\xi h + M(h))^{-}]\} \le E(\xi + h^{-1}M(h))^{-} \to E\xi^{-}.$$

Combining the three lemmas above, we find our first set of conditions under which (11.1) holds.

THEOREM 12.1. *Under the conditions of Lemmas* 12.1, 12.2 *and* 12.3, (11.1) *holds with* $-\Gamma'(0) = E\xi^{-}$.

PROOF. If $-\Gamma'(0) = \infty$, then Theorem 11.1 trivially implies (11.1). Thus we now assume $-\Gamma'(0) < \infty$. Then (12.2) implies $E\xi^{-} < \infty$, and Lemmas 12.2 and 12.3 imply

$$\limsup_{u\to\infty} \frac{\mathrm{P}(\sup_{[0,t]} X(s) > u)}{tv(1-F(u))} \le E\xi^{-}.$$

By (12.2), the lim sup above is at most $-\Gamma'(0)$. Combining this with (11.3) we obtain (11.1).

We remark that condition (12.7) implies that the process has a finite expected number of downcrossings of any level u. Indeed, by the standard method of computing the latter expectation, we find that it is equal to

$$\lim_{n\to\infty} 2^n P[X(0) > u, X(2^{-n}) < u]$$

which is at most equal to

$$\lim_{n\to\infty} 2^n P[X(2^{-n}) \le u \mid X(0) > u].$$

Condition (12.7) holds for the stationary Gaussian process with $-r''(0) < \infty$ and $u = v = w$.

Condition (12.7) is closely related to one used by Leadbetter, Lindgren and Rootzen [26] to derive an estimate similar to (12.8). Our additional assumptions lead to the explicit form of the tail of the distribution of the supremum. Note that we have not required the existence of the second moment of the number of crossings (see [15]).

The assumption of stochastic differentiability is not explicitly stated in the hypothesis of Theorem 12.1. We require that the limiting process $Z(t)$ be of the form (12.1) which arises as the limit in the stochastically differentiable case (Section 6). Condition (12.7) seems to be related to the existence of a stochastic derivative through the result about level crossings stated above, but we have not been able to determine the exact relation.

Theorem 12.1 may be applied to the derivation of the limiting distribution of the supremum over $[0, t]$ for $t \to \infty$ by using Corollaries 1 and 2 of Theorem 4.2 of [23]. If F is in the domain of attraction of Λ, and the function w satisfies (4.5), and v satisfies a similar condition, then the distribution function of the random variable $\sup[X(t): 0 \le t \le 1]$ is also in the domain of Λ. Indeed, Theorem 12.1 implies that

$$\frac{P[\sup_{[0,1]} X(t) > u + x/w]}{P[\sup_{[0,1]} X(t) > u]} \sim \frac{w\left(u + \dfrac{x}{w(u)}\right)\left(1 - F\left(u + \dfrac{x}{w(u)}\right)\right)}{w(u)[1 - F(u)]} \to e^{-x}$$

for $u \to \infty$, so that (4.1) holds for the tail of the distribution of the supremum. Under the mixing hypotheses in [23], it follows that $\sup(X(s)\colon 0 \le s \le t)$ has, after appropriate normalization, the limiting distribution Λ.

We will consider this question again in Section 19 without the restriction of $Z(t)$ to the form (12.1). Our mixing conditions will be different from those of [23].

13. General results on the distribution of the maximum of a stochastic process. In previous work on the maximum of a Gaussian process, we made use of some of the well known methods and results of Fernique [12] and others (see, for example, Marcus [29]). The methods are actually refinements of the fundamental method of Kolmogorov used to obtain a general criterion for the continuity of the sample functions of a general stochastic process (see Slutsky [40]). One of the contributions of Fernique was to find a remarkably efficient extension of Kolmogorov's method in the Gaussian case. Since it is our purpose now to extend some of our results in the Gausian case to the general case, we will return to the original Kolmogorov estimates instead of using those of Fernique as we did in the Gaussian case.

THEOREM 13.1. *Let $Y(t)$, $t \ge 0$, be a real, separable stochastic process. Suppose that there is a nondecreasing, continuous function $q(t)$, $t \ge 0$, with $q(0) = 0$, and a nonincreasing continuous function $H(x)$, $x > 0$, with $H(\infty) = 0$ such that for all $z > 0$ and $t \ge 0$,*

$$\sup_{|s-s'|\le t} P(|Y(s) - Y(s')| > z) \le H(z/q(t)). \tag{13.1}$$

Let $(b_n, n \ge 0)$ be a sequence of positive numbers such that

$$\textstyle\sum_{n=0}^{\infty} q(2^{-n}) b_n < \infty. \tag{13.2}$$

Define

$$Q(t) = \textstyle\sum_{n=0}^{\infty} q(t2^{-n}) b_n, \qquad 0 \le t \le 1. \tag{13.3}$$

$$J(x) = \textstyle\sum_{n=0}^{\infty} 2^n H(x b_n), \qquad x > 0. \tag{13.4}$$

Then for any interval $[a, b]$ with $b - a \le 1$, and any $z > 0$,

$$P(\sup_{a\le t\le b} |Y(t) - Y(a)| > z) \le J(z/Q(b-a)), \tag{13.5}$$

where the right hand side may be finite or infinite.

PROOF. (13.1) implies that $Y(t)$ is stochastically continuous, so that the dyadic rationals form a separability subset. Therefore, the events

$$\{|Y(t) - Y(a)| > z, \text{ for some } a \le t \le b\}$$

and

$$\{|Y((b-a)k2^{-n} + a) - Y(a)| > z, \qquad \text{for some } n \ge 0 \text{ and some } 1 \le k \le 2^n\}$$

have the same probability. By the covering principle (see Loève [28], page 515), the probability of the latter is at most equal to

$$\textstyle\sum_{n=0}^{\infty} \sum_{k=1}^{2^n} P(|Y((b-a)k2^{-n} + a) - Y((b-a)(k-1)2^{-n} + a)| > p_n z)$$

for any sequence $\{p_n\}$ such that $p_n \ge 0$ and $\sum p_n = 1$. By (13.1) the sum above is at most equal to

$$\textstyle\sum_{n=0}^{\infty} 2^n H\left(\dfrac{z p_n}{q((b-a)2^{-n})}\right).$$

If we choose the sequence $\{p_n\}$ as

$$p_n = \frac{q((b-a)2^{-n}) b_n}{Q(b-a)},$$

then the preceding series is exactly the right hand member of (13.5).

COROLLARY 13.1. *If*

$$\lim_{h\to 0} h^{-1}J(z/Q(h)) = 0, \qquad \textit{for every } z > 0, \tag{13.6}$$

then the sample functions of Y are almost surely continuous.

PROOF. For $z > 0$ and $0 < h < 1$, (13.5) implies

$$h^{-1} \sup_{[a,b],|b-a|\le h} P(\sup_{t\in[a,b]}| Y(t) - Y(a) | > z) \le h^{-1}J(z/Q(h)),$$

which, by (13.6), converges to 0 with h. The covering principle now implies sample function continuity.

Now we turn to the problem of obtaining a bound for the tail of the distribution of the maximum itself, not of the maximum of the increment. We use a hypothesis like that of Theorem 13.1 but in conditional form.

THEOREM 13.2 *Let $X(t)$, $t \ge 0$, be a real, separable stochastic process. Suppose there exist functions q and H of the type in Theorem* 13.1 *such* (13.2) *holds for an appropriate sequence $\{b_n\}$; and let Q and J be defined by* (13.3) *and* (13.4), *respectively. Suppose also that there is a continuous function $\mu(t, x)$ on the plane such that*

$$| \mu(t, x) | \le | x |, \qquad \mu(0, x) = x \tag{13.7}$$

and such that, for each z,

$$\sup_{|s-s'|\le t} P[| X(s) - X(s') - \mu(s, X(0)) + \mu(s', X(0)) | > z | X(0)] \le H(z/q(t)) \tag{13.8}$$

almost surely. Then, for every $t > 0$ and $-\infty < u' \le u$,

$$P(\sup_{[0,t]}| X(s) | > u, X(0) < u') \le \int_{-\infty}^{u'} J\left(\frac{u-y}{Q(t)}\right) dP(X(0) \le y). \tag{13.9}$$

PROOF. If $\sup_{[0,t]}| X(s) | > u$, then either

$$\sup_{[0,t]}| \mu(s, X(0)) | > | X(0) |, \tag{13.10}$$

or

$$\sup_{[0,t]}| X(s) - \mu(s, X(0)) | > | u - X(0) |. \tag{13.11}$$

The former is excluded by (13.7) and the separability of X. To estimate the probability of (13.11), we condition by $X(0) = y$, and apply Theorem 13.1 on the strength of the hypothesis (13.8), which is the conditional form of (13.1). Then the conditional probability of (13.11), given $X(0) = y$, is at most equal to $J((u - y)/Q(t))$. The unconditional probability bound (13.9) is then obtained by integration with respect to the distribution of $X(0)$.

We may think of μ as a conditional centering function. In fact, in the Gaussian case we use $\mu(t, x) = r(t)x$.

COROLLARY 13.2. *Let $X(t)$, $t \ge 0$, be a stationary process satisfying the conditions of Theorem* 13.2; *and let F and f be the marginal distribution and density function, respectively. Let Q^{-1} be the inverse function of Q, and let $w = w(u)$ be a positive function such that $w \to \infty$ for $u \to \infty$. Then*

$$\limsup_{u\to\infty} \frac{Q^{-1}(1/w)P(\sup_{[0,t]}X(s) > u)}{t[1 - F(u)]} \le \limsup_{u\to\infty} \int_0^\infty J(x)\frac{f(u - x/w)}{w(1 - F(u))}\, dx, \tag{13.12}$$

for every $t > 0$.

PROOF. Put $v = 1/Q^{-1}(1/w)$; then $v \to \infty$ for $w \to \infty$. If $\sup X(s) > u$, then either $X(0) > u$ or $\sup X(s) > u$ *and* $X(0) \le u$; therefore, the ratio in the left hand member of (13.12) is at most equal to the sum of the terms $1/vt$ (which tends to 0 for $w \to \infty$) and

$$\frac{P(\sup_{[0,t]} X(s) > u, X(0) \le u)}{vt(1 - F(u))}. \tag{13.13}$$

Split the interval $[0, t]$ into $[(v + 1)]$ subintervals, each of length $[(v + 1)]^{-1}$. If the event in the numerator of (13.13) holds, then for some subinterval $[a, b]$ of the indicated form,

$$\sup_{[a,b]} X(s) > u \quad \text{and} \quad X(a) \le u.$$

Therefore, by Boole's inequality and stationarity, we find that (13.13) is at most equal to

$$\frac{P(\sup_{[0,1/v]} X(s) > u, X(0) \le u)}{1 - F(u)}$$

which, by (13.9), is at most

$$\int_{-\infty}^{u} J\left(\frac{u - y}{Q(1/v)}\right) \frac{f(y)}{1 - F(u)}\, dy,$$

which, by the relation $w = 1/Q(1/v)$ and the change of variable $x = w(u - y)$, yields the integral in (13.12).

We remark that if the function $\mu(t, x)$ in the hypothesis of Theorem 3.2 can be chosen to be independent of t, then (13.8) reduces to a simpler condition on the conditional increment,

$$\sup_{|s-s'| \le t} P[\,|X(s) - X(s')| > z \mid X(0)] \le H(z/q(t)).$$

We close this section with an observation about earlier estimates of the type (13.5) for non-Gaussian processes. Dudley [11] has observed that the estimates used to prove continuity in the Gaussian case also hold for more general processes whose increments have distribution tails of the Gaussian type; such processes have been said to have "sub-Gaussian" increments. The statement of Theorem 13.1 is not covered by such a remark because the theorem permits a variety of possible relations between the decay of the tail of H and the growth of q near the origin.

14. The tail of the distribution of the maximum without stochastic differentiability. Our aim in this section is to state conditions under which

$$-\Gamma''(0) < \infty \tag{14.1}$$

and

$$\limsup_{u \to \infty} \frac{P(\sup_{[0,t]} X(s) > u)}{vt(1 - F(u))} \le -\Gamma''(0), \tag{14.2}$$

in the case where $Z(t)$ is more general than (12.1). In view of (11.3), this will establish (11.1). Our main result is:

THEOREM 14.1. (*Maximum Limit Theorem*). *Let $X(t)$, $-\infty < t < \infty$, be a real separable, stationary stochastic process satisfying the following conditions*:

(14.3) *There exist functions $v(u)$ and $\mu(t, x)$ such that the hypothesis of Theorem* 5.2 *holds* (*which implies* 3.I) *and such that the convergence in* (5.9) *is uniform on compact intervals.*

(14.4) *Assumption* 3.II *of the Sojourn Limit Theorem holds for some $T > 0$ and for $v(u)$ specified in* (14.3).

(14.5) *There are functions H and q, and a sequence (b_n) such that the hypothesis of Theorem* 13.2 *holds with μ given in* (14.3).

(14.6) *If Q is defined by* (13.3) *in terms of the corresponding function q and sequence (b_n) in* (14.5), *then*

$$\limsup_{u\to\infty} w\, Q(1/v) < K$$

for some $K < \infty$.

(14.7) *If J is defined by* (13.4) *in terms of the functions in* (14.5), *then, for every $z > 0$,*

$$\lim_{h\to 0} h^{-1} J\left(\frac{z}{w\, Q(h/v)}\right) = 0$$

uniformly in all large u.

(14.8) *For every $K' < K$,*

$$\lim_{M\to\infty} \limsup_{u\to\infty} \int_M^\infty \mathrm{J}(z/K') \frac{f(u - z/w)}{w\cdot(1 - F(u))} dz = 0.$$

Then the inequalities (14.1) *and* (14.2) *hold, and the sample functions are continuous, so that*

$$\lim_{u\to\infty} \frac{P(\max_{[0,t]} X(s) > u)}{vt(1 - F(u))} = -\Gamma'(0). \tag{14.9}$$

We note that conditions (14.4) – (14.8) can be put in terms of the three dimensional distributions of the process. The only conditions involving the k-dimensional distributions for arbitrary k are stationarity and the part of (14.3) involving Theorem 5.2.

The proof will be given in a series of lemmas.

LEMMA 4.1. *Conditions* (14.3), (14.5) *and* (14.7) *imply that the sample functions of the process $X(t)$, $0 \leq t \leq t_0$, conditioned by $X(0)$, are continuous for any $t_0 > 0$. Furthermore the convergence of the conditional distributions in the hypothesis of Theorem* 5.2 *is accompanied by tightness over the space of continuous functions $C[0, t_0]$ for every $t_0 > 0$.*

PROOF. Apply Theorem 13.1 to the conditioned process $Y(t) = w\cdot[X(t/v) - \mu(t/v, X(0))]$ in Theorem 13.2:

$$P\{\sup_{a\leq t\leq b} | Y(t) - Y(a) | > z | X(0)\} \leq J\left(\frac{z}{w\, Q((b - a)/v)}\right), \quad \text{for } 0 < a < b. \tag{14.10}$$

Divide each side of the inequality above by $h = b - a$, and let $h \to 0$. According to Corollary 13.1, condition (14.7) implies the continuity of the sample functions of $Y(t)$, which, by the continuity of μ, implies the continuity of the sample functions of $X(t)$.

Conditions (14.3), (14.7) and (14.10) also imply the tightness of the conditioned measures defined in the hypothesis of Theorem 5.2. This follows by an application of a well-known criterion (see Billingsley [9], page 56). We observe that the first part of the criterion is automatically satisfied because the conditioned process has the value 0 for $t = 0$.

LEMMA 14.2. *Under the same conditions as Lemma* 14.1, *for each fixed y,*

$$\lim_{x\downarrow 0} \limsup_{u\to\infty} P\left\{\max_{[0,1]} X\left(\frac{t}{v}\right) > u,\ vL_{1/v} \leq x \,|\, X(0) = u + \frac{y}{w}\right\} = 0. \tag{14.11}$$

PROOF. Let $\max X(t/v)$ stand for the maximum over $0 \leq t \leq 1$. The conditional probability in (14.11) may be written as

$$P\left\{\max w(X(t/v) - X(0)) > -y, \quad \int_0^1 I_{[w(X(t/v)-X(0))>-y]}\, dt \leq x \,|\, X(0) = u + \frac{y}{w}\right\}.$$

Let $g(x)$ be an arbitrary monotone continuous function equal to 0 for $x \leq 0$, and strictly positive and bounded by 1 for $x > 0$. Then the conditional probability above is at most equal to

$$(14.12)\qquad P\{\max w(X(t/v) - X(0)) > -y,$$
$$\int_0^1 g\left(w\left(X\left(\frac{t}{v}\right) - X(0)\right) + y\right) dt \leq x \,|\, X(0) = u + y/w\}.$$

According to Theorem 5.2, which implies convergence in distribution, and according to Lemma 14.1 and the uniform convergence assumed in (14.3), the joint distribution of the continuous functionals $\max w(X(t/v) - X(0))$ and $\int_0^1 g(w(X(t/v) - X(0)) + y)\, dt$ converges to the joint distribution of the corresponding functionals on the process $Z(t)$, for each y. Therefore, the conditional probability (14.12) converges for almost all y and almost all $x > 0$ to

$$P\left\{\max Z(t) > -y, \int_0^1 g(Z(t) + y)\, dt \leq x\right\}.$$

This converges to 0 for $x \downarrow 0$ because, by the tightness, $Z(t)$ has continuous sample functions, and so they must spend positive time above $-y$ if they exceed $-y$.

LEMMA 14.3. *Under* (14.5), (14.6), *and* (14.8),

$$(14.13)\qquad \lim_{M\to\infty} \limsup_{u\to\infty} \frac{P\{X(0) \leq u - M/w, \max_{[0,1]} X(t/v) > u\}}{1 - F(u)} = 0.$$

PROOF. According to Theorem 13.2 with $u' = u - M/w$, and since $\max |X| \geq \max X$, the numerator in (14.13) is bounded by

$$\int_{-\infty}^{u-M/w} J\left(\frac{u-y}{Q(1/v)}\right) f(y)\, dy.$$

By the change of variable $z = w(u - y)$, this is equal to

$$w^{-1} \int_M^\infty J\left(\frac{z}{wQ(1/v)}\right) f(u - z/w)\, dz.$$

By taking u to be large and noting (14.6), we infer that there exists some $K' < K$ such that $wQ(1/v)$ may, in the integral above, be replaced by K'. Upon dividing the integral by $1 - F(u)$, we find, by (14.8), that the ratio converges to 0 under the limit operation $u \to \infty$ and then $M \to \infty$.

LEMMA 14.4. *Under the assumptions of Theorem* 14.1,

$$(14.15)\qquad \lim_{x\downarrow 0} \limsup_{u\to\infty} \frac{P\{\max_{[0,1]} X(t/v) > u, vL_{1/v} \leq x\}}{1 - F(u)} = 0.$$

PROOF. According to Lemma 14.3, it suffices to show that

$$(14.16)\qquad \lim_{x\downarrow 0} \limsup_{u\to\infty} \frac{P\{X(0) > u - M/w, \max X(t/v) > u, vL_{1/v} \leq x\}}{1 - F(u)} = 0,$$

for arbitrary $M > 0$. The ratio in (14.16) may be written as

$$(1 - F(u))^{-1} \int_{u-M/w}^\infty P\left\{\max X\left(\frac{t}{v}\right) > u, vL_{1/v} \leq x \,|\, X(0) = z\right\} f(z)\, dz,$$

which, by the change of variable $y = w(z - u)$, is

$$\int_{-M}^{\infty} P\left\{\max X\left(\frac{t}{v}\right) > u,\ vL_{1/v} \le x \,|\, X(0) = u + y/w\right\} \frac{f(u + y/w)}{w(1 - F(u))}\, dy.$$

According to Lemma 5.1 the ratio $f/w(1 - F)$ may be replaced by e^{-y} in the evaluation of the limit of the integral above. Therefore, Lemma 14.2 implies the conclusion (14.15) of the lemma.

We now prove the statement of our theorem. Under condition (14.6) there exists a number $K' < K$ such that $Q(1/v) < K'/w$ for all sufficiently large u, so that $1/v \le Q^{-1}(K'/w)$. Corollary 13.2, with w/K' in place of w, implies

$$\text{(14.17)} \quad \text{limsup}_{u\to\infty} \frac{P\{\max_{[0,t]} X(s) > u\}}{vt(1 - F(u))} \le \text{limsup}_{u\to\infty} \int_0^{\infty} J(x) \frac{f(u - K'x/w)}{w(1 - F(u))/K'}\, dx.$$

Since H and J are used as bounds in our theorems only for large values of x, we may assume without loss of generality that H has been selected so that J is bounded on bounded intervals. Then (5.3) implies

$$\lim_{u\to\infty} \int_0^M J\left(\frac{x}{K'}\right) \frac{f(u - x/w)}{w \cdot 1 - F(u)}\, dx = \int_0^M J\left(\frac{x}{K'}\right) e^x\, dx < \infty$$

for every $M > 0$, so that (14.8) implies that the right hand member of (14.17) is finite. Theorem 11.1 then implies that $-\Gamma'(0) < \infty$ and so (14.1) holds.

The ratio in (14.9) is dominated, for every $x > 0$, by the sum of two terms,

$$\text{(14.18)} \quad \frac{x^{-1} \int_0^x P\{vL_t > y\}\, dy}{vt(1 - F(u))}$$

and

$$\text{(14.19)} \quad \frac{P\{\max_{[0,t]} X(s) > u,\ vL_t \le x\}}{vt(1 - F(u))};$$

indeed, $vL_t > y$ implies $\max_{[0,t]} X(s) > u$ for every $y > 0$. By the Sojourn Limit Theorem, the ratio (14.18) converges, for $u \to \infty$, to $x^{-1}(\Gamma(0) - \Gamma(x))$, which, by (11.2), converges to $-\Gamma'(0)$ for $x \to 0$.

We now use the decomposition of $[0, t]$ which was constructed following (13.13); then, by the argument used there, the ratio (14.19) is at most equal to

$$\text{(14.20)} \quad \frac{P\{\max_{[0,1]} X(s/v) > u,\ vL_{1/v} \le x\}}{1 - F(u)},$$

which, by Lemma 14.4, converges to 0 for $u \to \infty$ and then $x \to 0$. This completes the proof of (14.2), and the theorem is established.

15. Comments on the Maximum Limit Theorem. The sufficient conditions for the Maximum Limit Theorem are stronger than those for the Sojourn Limit Theorem because they include (14.5)–(14.8). Here we discuss the latter conditions in some detail.

Suppose that the function $H(x)$ in condition (14.5) is assumed to be of the form

$$\text{(15.1)} \quad H(x) = e^{-Kx}$$

for all sufficiently large x, for some $K > 1$, and that the function q satisfies (13.2) with

$$\text{(15.2)} \quad b_n = n + 1$$

It follows that $J(x)$, defined by (13.4), is of the form

$$J(x) = 2e^{-Kx} \tag{15.3}$$

for all large x. It can be shown that a sufficient condition for (14.5) is that

$$E\left[\exp\left\{K\frac{X(t)-X(s)-\mu(t,x)+\mu(s,x)}{q(t-s)}\right\}\middle| X(0)=x\right]$$

is bounded in s, t and x. For example, if $X(t)$ is the stationary Gaussian process with mean 0 and covariance function $r(t)$ with $r(0) = 1$, and if $q^2(t) = 2(1 - r(t))$ and $\mu(t, x) = xr(t)$, then the expectation above is equal to $\exp\{\frac{1}{2} K^2\}$.

If J is of the form (15.3), then (14.8) is implied by

$$\limsup_{u\to\infty} \int_0^\infty e^{-cx}\frac{f(u-x/w)}{w\cdot(1-F(u))}\,dx < \infty \tag{15.4}$$

for every $c > 1$. This is a supplement to the condition that F be in the domain of attraction of Λ. Indeed, if (5.2) holds, then the integrand (15.4) converges everywhere to $e^{-2x(c-1)}$. Thus, (15.4) requires that the convergence be preserved under integration. This requirement is satisfied in the normal case; indeed, $w = u$, and

$$\frac{\phi(u-x/u)}{u(1-\Phi(u))} = \frac{\Phi(u)}{u(1-\Phi(u))}\exp\left\{x-\frac{x^2}{2u^2}\right\},$$

and so, by (7.1), (15.4) is satisfied.

It is of interest to compare $-\Gamma'(0)$ to the constant H_α discovered by Pickands [34] in the version of (14.9) in the particular case of the Gaussian process. Using our notation, we write his constant H_α as

$$H_\alpha = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty P\{\max_{[0,T]} W(t) > s\}e^s\,ds.$$

By a change of variable of integration, this may be transformed into

$$H_\alpha = \lim_{T\to\infty}\frac{1}{T}E\exp\{\max_{[0,T]} W(t)\}.$$

We remark that, under the conditions of the Maximum Limit Theorem, we have

$$-\Gamma'(0) \le \limsup_{T\to\infty}\frac{1}{T}E\exp\{\max_{[0,T]} W(t)\} \tag{15.5}$$

even without the assumption of a Gaussian process. Thus, in the Gaussian case, $-\Gamma'(0) \le H_\alpha$. Further, it can even be shown that $-\Gamma'(0) = H_\alpha$; we omit the the details.

For the proof of (15.5), we adapt the proof of Corollary 13.2. The first part of the proof shows that the ratio in (14.9) is asymptotic to that in (13.13). Next we modify the decomposition of $[0, t]$. For arbitrary but fixed $T > 0$ we decompose $[0, t]$ into $[(v + 1)t/T]$ subintervals, each of length $[(v + 1)/T]^{-1}$. By the argument following (13.13), the latter is at most equal to

$$\frac{P(\max_{[0,T]} X(s/v) > u, X(0) \le u)}{T(1-F(u))},$$

which we may write as

$$\frac{1}{T}\int_{-\infty}^0 P\left\{\max_{[0,T]} w\left(X\left(\frac{s}{v}\right)-X(0)\right) > -y \,\middle|\, X(0) = u+\frac{y}{w}\right\}\frac{f(u+y/w)}{w(1-F(u))}\,dy. \tag{15.6}$$

By the calculations in the proof of the Maximum Limit Theorem, that is, by the convergence of the conditional distribution of the maximum of w $(X(s/v) - X(0))$ to that of $W(s)$, and by the convergence of $f/w(1 - F)$ to e^{-y}, together with the estimate (14.8), we conclude that the expression (15.6) converges to

$$\frac{1}{T}\int_0^\infty P(\max_{[0,T]} W(s) > y)e^y \, dy,$$

which is equal to

$$\frac{1}{T} E \exp[\max_{[0,T]} W(s)].$$

Taking the limsup over T we conclude (15.5).

Finally we comment on the role of the concept of the "ε-upcrossing" due to Pickands [34]. The conclusion (14.13) of Lemma 14.3 implies, for $\varepsilon = M/w$, that the probability of at least one ε-upcrossing of u is small relative to the tail of the marginal distribution for $u \to \infty$ and then $M \to \infty$.

16. A compound Poisson limit theorem. In this section we begin our program of deriving the forms of the Sojourn Limit Theorem and Maximum Limit Theorem for $t \to \infty$ and $u = u(t) \to \infty$. First we consider the former theorem. We assume the conditions in the hypothesis of the Sojourn Limit Theorem; these are local conditions, describing the behavior of the process on a fixed interval. In formulating the limiting distribution for $t \to \infty$ we naturally have to impose further conditions on the distributions of the process for widely separated time points.

The main tool in this section is a general limit theorem proved by the author in [8]. It states conditions under which the sums of nonnegative random variables taken from a "stationary array" converge in distribution to a certain infinitely divisible distribution. For the convenience of the reader, we cite the relevant form of this theorem, and refer to the original paper for the proof.

Let $\{X_{nj}: j = 1, \cdots, n\}$ be a triangular array of nonnegative random variables. We assume that, for each n, the random variables $X_{nj}, j = 1, \cdots, n$ form a stationary though finite sequence. The following four assumptions are made:

16.I) There is a nondecreasing function $H(x)$ such that

(16.1) $$\lim_{x\to\infty} H(x) = 0,$$

(16.2) $$\int_0^\infty x dH(x) > -\infty,$$

and

(16.3) $$\lim_{n\to\infty} n\int_0^y x dP(X_{n,1} > x) = \int_0^y x dH(x)$$

at every continuity point y of the limiting function; furthermore, the relation above also holds for $y = \infty$:

(16.4) $$\lim_{n\to\infty} nEX_{n,1} = -\int_0^\infty x dH(x).$$

16.II) The "local mixing" condition holds:

(16.5) $$\lim_{k\to\infty} \operatorname{limsup}_{n\to\infty}[k \textstyle\sum_{1\le i<j\le[n/k]} EX_{n,i}X_{n,j}] = 0.$$

16.III) The following "global mixing" condition holds for the k-dimensional joint distributions for each fixed $k \geq 2$ and each number q, $0 < q < 1$:

$$(16.6)\quad \lim_{n\to\infty} \sup_{1\leq j_1<\cdots<j_k\leq n, \min(j_{h+1}-j_h, 1\leq h\leq k)>qn} \left| \frac{P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k)}{P(X_{n,j_1} > x_1) \cdots P(X_{n,j_k} > x_k)} - 1 \right| = 0$$

for every k-tuple of x's such that $H(x_i) > 0$ and x_i is a point of continuity of H, $i = 1, \cdots, k$.

16.IV) The following global mixing condition hold for the joint moments of order 1, for each $k \geq 2$; each h, $1 < h < k$; and each q, $0 > fq > 1$:

$$(16.7)\quad \lim_{n\to\infty} \sup_{1\leq j_1<\cdots<j_k\leq n, j_h-j_{h-1}>qn} \left| \frac{EX_{n,j_1} \cdots X_{n,j_k}}{(EX_{n,j_1} \cdots X_{n,j_{h-1}})(EX_{n,j_h} \cdots X_{n,j_k})} - 1 \right| = 0.$$

Then we have the following result from [8] which we call

THEOREM 16.A. *Under Assumptions* 16.I)–16IV) *the distribution of* $\sum_{j=1}^{n} X_{nj}$ *converges, for* $n \to \infty$, *to the distribution with the Laplace-Stieltjes transform,*

$$(16.8)\quad \Omega(x) = \exp\left[\int_0^\infty (1 - e^{-sx})\, dH(x) \right].$$

If H is bounded at 0, then the distribution is the compound Poisson. In general it is defined even without this condition. The symbol H has been transferred from [8], and is unrelated to that in Sections 13 and 14.

In our previous work we showed that this theorem could be applied to the limiting distribution of $vL_t(u)$ when $X(t)$ is a stationary Gaussian process, and indicated our expectation that it could also be applied to more general stationary processes, not necessarily Gaussian. Let $X(t)$, $-\infty, < t < \infty$, be a stationary process with measurable sample functions, and let $L_t(u)$ be the sojourn integral (3.1). Let $v = v(u)$ be a function of u satisfying the requirements of the Sojourn Limit Theorem. We will assume also that (3.22) holds; otherwise, as we have shown in Section 3, the problem is not of interest. Without real loss of generality we also assume that the function $v = v(u)$ is continuous, and, as we have already assumed throughout much of this paper, that the marginal distribution F of $X(t)$ is continuous. It follows from (3.22) that for every $t > 0$ sufficiently large, the equation

$$(16.9)\quad v(u)t(1 - F(u)) = 1$$

has a largest solution $u = u(t)$. We assume for the rest of this work that u is defined as a function of t in this way. It follows that v is then also a function of t: $v = v(u(t))$; and w is similarly defined as $w = w(u(t))$.

We show, as in [8], that $vL_t(u)$ is representable as a sum of nonnegative random variables from a stationary array. Suppose, for simplicity, that t assumes only positive integer values n, $n = 1, 2, \cdots$ (the general case follows by a simple extension). Then $vL_t(u)$, for $t = n$, may be expressed as

$$vL_t(u) = \sum_{j=1}^{n} X_{n,j},$$

where

$$(16.10)\quad X_{n,j} = v(L_j(u) - L_{j-1}(u)), \qquad j = 1, \cdots, n,$$

where $L_0(u) = 0$.

Since the hypothesis of Theorem 16.A has four parts, we state and prove our main result by giving the sets of conditions which are sufficient for the various parts. The local conditions 16.I) and 16.II) will be studied in Section 17, and the global conditions 16.III) and 16.IV) in Section 18.

17. Conditions on X which imply the local conditions of the compound Poisson limit theorem.

LEMMA 17.1. *Let $X(t)$ obey the Sojourn Limit Theorem for some $T > 0$. If Γ satisfies $\Gamma(0+) = 1$ and $\Gamma(\infty) = 0$, then 16.I) is satisfied for $t = nT$, $vL_T(u) = X_{n,1}$ and $H(x) = -\Gamma'(x)$.*

PROOF. For simplicity let us suppose that $T = 1$, and write $L_1(u) = L$. Put $H(x) = -\Gamma'(x)$. Corollary 3.1 implies that H is nonincreasing. It converges to 0 for $x \to \infty$ because $P(vL > x) \le (1/x)E(vL)$; this implies (16.1).

Corollary 3.1 also implies

$$\int_A^B y \frac{dP(vL > y)}{E(vL)} \to \int_A^B y\, dH(y) \tag{17.1}$$

at all continuity points $0 < A < B < \infty$. Since $E(vL) = -\int_0^\infty y\, dP(vL > y)$, (17.1) implies

$$\int_A^B y\, dH(y) > -1, \qquad 0 < A < B < \infty,$$

which confirms (16.2).

Next we show that

$$\lim_{y\to 0} yH(y) = 0 \tag{17.2}$$

and

$$\lim_{y\to\infty} yH(y) = 0. \tag{17.3}$$

By integration by parts, we have

$$\int_A^B y\, dH(y) = BH(B) - AH(A) + \Gamma(B) - \Gamma(A). \tag{17.4}$$

Let $A \downarrow 0$ in (17.4). By (16.2) and the hypothesis $\Gamma(0+) = 1$, it follows that

$$\int_0^B y\, dH(y) = BH(B) - \lim_{y\to 0} yH(y) + \Gamma(B) - 1.$$

To complete the proof of (17.2) it suffices to show that

$$\lim_{y\to 0} (1/y) \int_0^y xH(x)\, dx = 0.$$

By integration by parts, the average above is equal to

$$(1/y)\left(-y\Gamma(y) + \int_0^y \Gamma(x)\, dx\right),$$

which converges to 0 for $y \to 0$. A similar argument shows that (17.3) also holds.

According to the formula (17.4) for integration by parts, the result (17.1) is equivalent to

$$\frac{BP(vL > B)}{E(vL)} - \frac{AP(vL > A)}{E(vL)} - \int_A^B \frac{P(vL > y)}{E(vL)}\, dy \tag{17.5}$$

$$\to BH(B) - AH(A) + \Gamma(B) - \Gamma(A), \qquad 0 < A < B < \infty.$$

This is extended to the case $A = 0$ by means of (17.2), (3.8), and the relation $E(vL) = \int_0^\infty P(vL > y)\,dy$:

$$\frac{BP(vL > B)}{E(vL)} - \int_0^B \frac{P(vL > y)}{E(vL)}\,dy \to BH(B) + \Gamma(B) - 1. \tag{17.6}$$

Therefore, since (17.5) holds for $A = 0$, so does (17.1):

$$\int_0^B y\frac{dP(vL > y)}{E(vL)} \to \int_0^B y\,dH(y), \qquad B > 0.$$

This implies (16.3) because, by (16.9), $nE(vL) = tE(vL) = tv(1 - F(u)) = 1$.

Finally we extend (17.6) to the case $B = \infty$. Since $L \leq 1$, the left hand member of (17.6) converges to -1 for $B \to \infty$. By (17.3) and the relation $\Gamma(\infty) = 0$, the right hand member also converges to -1 for $B \to \infty$. Therefore, we have extended (17.5), and, as a consequence, (17.1), to the case $A = 0$ and $B = \infty$, which implies (16.4) with the right hand member equal to -1.

The simplifying assumption made at the beginning of the proof that $T = 1$ is removable. If $T \neq 1$, then we define a new process $\bar{X}(t)$ as $X(t/T)$, and define the array $\{X_{n,k}\}$ for it. The new process satisfies the conditions of the Sojourn Limit Theorem for $T = 1$, and the limiting distribution of the sojourn, for $t \to \infty$, is the same, except for a time scale factor, for both $X(\cdot)$ and $\bar{X}(\cdot)$. This completes the proof.

We remark that our hypothesis on Γ means that the time spent by $Z(s)$, $0 < s < \infty$, above 0 is almost surely positive and finite. It is true, in particular, whenever $Z(0) > 0$ almost surely, and $Z(s)$ has right continuous sample functions such that $Z(s) \to -\infty$ for $s \to \infty$. In particular, if the process is of the form (5.10) with $m(t) \to -\infty$ for $t \to \infty$, and $V(t)/m(t) \to 0$ in probability, for $t \to \infty$, then the condition on Γ holds. It is true for the limiting Gaussian process in Section 7.

In the next lemma we give sufficient conditions on the bivariate distributions of the process which imply the local mixing condition 16.II) for the stationary array (16.10).

LEMMA 17.2. *The following condition on $X(t)$ is sufficient for* 16.II):

$$\operatorname{limsup}_{t\to\infty} \int_0^1 \frac{P(X(0) > u, X(ts) > u)}{(P(X(0) > u))^2}\,ds < \infty. \tag{17.7}$$

PROOF. The stationarity of the array implies

$$k \sum_{i,j=1,i\neq j}^{[n/k]} EX_{n,i}X_{n,j} \leq n \sum_{j=2}^{[n/k]} EX_{n,1}X_{n,j}.$$

The latter sum may be evaluated by the application of Fubini's theorem to the representation (16.10); it is equal to

$$nv^2 \int_0^1 \int_1^{[n/k]} P(X(s) > u, X(s') > u)\,ds\,ds',$$

which, by stationarity and a change of variable of integration, is equal to

$$nv^2 \int_0^1 \int_{1-t}^{[n/k]-t} P(X(0) > u, X(s) > u)\,ds\,dt,$$

which is at most equal to

$$nv^2 \int_0^{n/k} P(X(0) > u, X(s) > u)\,ds,$$

which, by the relation (16.9), with $n = t$, is

$$\frac{1}{t}\int_0^{t/k} \frac{P(X(0) > u, X(s) > u)}{(P(X(0) > u))^2}\, ds,$$

which, by a change of variable of integration, is

$$\frac{1}{k}\int_0^{1} \frac{P(X(0) > u, X(ts/k) > u)}{(P(X(0) > u))^2}\, ds.$$

By condition (17.17), the limsup of this expression for $t \to \infty$ is a finite constant divided by k, and so (16.5) holds.

18. Conditions on X which imply the global conditions of the compound Poisson limit theorem. Conditions 16.III) and 16.IV) are mixing conditions stated in terms of a particular kind of limiting operation. The latter is more complicated than sequential convergence. We will show that the convergence in 16.III) can actually be reduced to sequential convergence. This will enable us to carry out certain standard operations, such as the interchange of limits in an integral, in the proof of condition 16.III). There is a similar formulation for the convergence in 16.IV).

Such tools have appeared in recent years in the study of sums of independent random variables with partially ordered index sets. An example is the paper by Gabriel [13]; related work is indicated in its list of references. We could have used some of the basic results of that paper to obtain, as special cases, the tools that are needed here. However, the necessary concepts are relatively elementary so that, for the convenience of the reader, we will give a self-contained description.

Let us now formally describe the convergence taking place in the mixing relations above. For each integer $n \geq 1$, let B_n be an abstract finite set, and form the ordered pairs (n, b), $b \in B_n$. Define $\mathcal{N}$ as the array

$$\mathcal{N} = \{(n, b),\ b \in B_n,\ n \geq 1\}. \tag{18.1}$$

Introduce the partial order relation in $\mathcal{N}$: We shall say that two elements (m, b) and (n, c) of $\mathcal{N}$ satisfy $(m, b) \leq (n, c)$ if the integers m and n satisfy $m \leq n$. An elementary argument shows that $\leq$ directs the set $\mathcal{N}$ (see Kelley [22], page 65). An infinite subset of $\mathcal{N}$ is called a *sequence* if it is totally ordered.

The following result links convergence of a net on $\mathcal{N}$ to convergence on sequences in $\mathcal{N}$.

LEMMA 18.1. *A net assuming values in some topological space converges to a limit on the directed set $\mathcal{N}$ if and only if it converges to the same limit over every sequence in $\mathcal{N}$.*

PROOF. Necessity is obvious. For the proof of sufficiency, suppose that the net, which we denote as $f(\omega)$, $\omega \in \mathcal{N}$, fails to converge to a point f_0 of the space. Then there exists an infinite subset $\mathcal{N}'$ of $\mathcal{N}$ such that $f(\omega)$, $\omega \in \mathcal{N}'$, does not belong to some neighborhood of f_0. Since every set B_n in (18.1) is finite, every infinite subset of $\mathcal{N}$ contains a sequence; therefore, $f(\omega)$ does not belong to the neighborhood of f_0 for all ω in some sequence $\mathcal{N}'' \subset \mathcal{N}'$, and so the net f does not converge to f_0 over this sequence.

Lemma 18.1 implies that the basic convergence theorems of probability for sequences of distributions are also valid for nets of distributions over directed sets of the form (18.1). The latter sets belong to the class described in [13] as "filtering to the right, locally finite and countable."

Now we will show how condition 16.III) may be expressed in terms of a limit over a directed set of the form (18.1). Let q, $0 < q < 1$, be arbitrary, and let $k \geq 2$ be an arbitrary integer. Let $\mathbf{J}$ be a k-tuple of closed, disjoint intervals $J_1, \cdots, J_k$ with nonnegative, integer endpoints and of common length 1; thus each is of the form $[m-1, m]$ for some integer $m \geq 1$. Put

$B_n = \{\mathbf{J}$: Each J_i is contained in $[0, n]$ and the distance separating J_i and J_j is at least qn, for all $i \le k, j \le k$ and $i \ne j.\}$, $\quad n \ge 1$

Each set B_n is finite, and so we may form the directed set $\mathcal{N}$ in (18.1). Note that B_n is empty at least for $n \le 2k$, but is not empty for all sufficiently large n.

Let the random variable $X_{n,j}$ be associated with the interval $[j-1, j]$ as in (16.10). Then the condition (16.6) may be expressed in terms of the limit of a net over $\mathcal{N}$. We write this limiting operation as

$$\mathrm{LIM}_{n\to\infty} = \text{limit of the net over } \mathcal{N},$$

with the capital letters used to distinguish this convergence from sequential convergence. Thus (16.6) may be expressed as

$$\mathrm{LIM}_{n\to\infty} \frac{P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k)}{P(X_{n,j_1} > x_1)\cdots P(X_{n,j_k} > x_k)} = 1. \tag{18.2}$$

We now give conditions on the stationary process $X(t)$ which are sufficient for the validity of assumption 16.III) for the random variables in the array defined by (16.10). These conditions are of the global mixing type, prescribing the rate at which parts of the process which are moving apart in time are becoming mutually independent. They imply that the Sojourn Limit Theorem holds locally and independently for the various parts of the process.

LEMMA 18.2. *Assume that the hypothesis of Lemma* 17.1 *holds and, in addition, the following conditions hold for an arbitrary integer* $k \ge 2$ *and real* q, $0 < q < 1$:

i) *Let* θ_i *be the left endpoint of the typical interval* J_i *in the set* B_n. *Then the* k-*component process on* R^k,

$$w(X(\theta_i + s_i/v) - u), \qquad -\infty < s_i < \infty, \qquad i = 1, \cdots, k, \tag{18.3}$$

conditioned by

$$X(\theta_i) > u, \qquad i = 1, \cdots, k, \tag{18.4}$$

converges in distribution, as a net over $\mathcal{N}$, *to the process with* k *identical, independent components, each distributed as* $Z(s)$, $-\infty < s < \infty$.

ii) *The following extension of assumption* 3.II) *holds*: *For each* i, $i = 1, \cdots, k$,

$$\lim_{d\to\infty} \mathrm{LIM}_{n\to\infty} \sup_{(m,b)\in\mathcal{N}, n\le m} v\int_{d/v}^{T} P(X(\theta_i + s) > u \mid X(\theta_j) > u, \quad j = 1, \cdots, k)\, ds = 0. \tag{18.5}$$

iii) *The* k-*dimensional distribution splits in the tails into the product of the marginals*:

$$\mathrm{LIM}_{n\to\infty} \sup_{t_j\in J_j, j=1,\cdots,k} \left| \frac{\mathrm{P(X(t_j) > u, j = 1, \cdots, k)}}{\mathrm{P^k(X(0) > u)}} - 1 \right| = 0. \tag{18.6}$$

Then (18.2) *holds for each* $k \ge 2$ *and* q, $0 < q < 1$.

PROOF. For simplicity, assume that $T = 1$. Write the ratio in (18.2) as

$$\frac{n^k P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k)}{nP(X_{n,j_1} > x_1)\cdots nP(X_{n,j_k} > x_k)}.$$

By Corollary 3.1, the relation (16.9) and the definition of H as $-\Gamma'$, the denominator in the latter ratio converges to $H(x_1)\cdots H(x_k)$. Therefore, (18.2) is equivalent to

$$\mathrm{LIM}_{n\to\infty}\, n^k P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k) = H(x_1)\cdots H(x_k).$$

By integration and by Lemma 18.1, the latter is equivalent to

$$(18.7)\qquad \mathrm{LIM}_{n\to\infty}\, n^k \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(X_{n,j_1} > y_1, \cdots, X_{n,j_k} > y_k)\, dy_k \cdots dy_1 = \Gamma(x_1)\cdots\Gamma(x_k)$$

for all $x_i > 0$.

Let t_j be a point in J_j, and put

$$L_{t_j}^* = \int_{J_j\cap[0,t_j]} I_{[X(s)>u]}\, ds.$$

By a direct extension of the identity (3.2) to k intervals, the multiple integral in (18.7) is equal to

$$v^k \int_{J_1}\cdots\int_{J_k} P(vL_{t_i}^* > x_i, X(t_i) > u, \qquad i = 1, \cdots, k)\, dt_k \cdots dt_1,$$

which is identical with

$$(18.8)\qquad \begin{aligned} &v^k P^k(X(0) > u)\int_{J_1}\cdots\int_{J_k} \frac{P(X(t_i) > u, \qquad i = 1, \cdots, k)}{P^k(X(0) > u)} \\ &P(vL_{t_i}^* > x_i, \qquad i = 1, \cdots, k\,|\,X(t_i) > u, \quad i = 1, \cdots, k)\, dt_k \cdots dt_1. \end{aligned}$$

When multiplied by n^k, as in (18.7), the expression outside the integral converges, for $t \to \infty$ (i.e., $n \to \infty$) to 1; this follows from (16.9). The ratio following the integral sign in (18.8) converges, by (18.6), to 1. Therefore (18.8) is asymptotic to

$$(18.9)\qquad \int_{J_1}\cdots\int_{J_k} P(vL_{t_i}^* > x_i, \qquad i = 1, \cdots, k\,|\,X(t_i) > u, i = 1, \cdots, k)\, dt_k \cdots dt_1.$$

The conditions i) and ii) of our lemma imply that the Sojourn Limit Theorem holds for the individual sojourns $vL_{t_j}^*$ and independently over the set of sojourns. Therefore, the integrand in (18.9) factors, and the product of the integrals converges to the product in (18.7). The details of the proof are exactly the same as in the proof of the corresponding result in [8], which was stated for Gaussian processes. The reader is referred to the latter paper.

The mixing condition 16.IV) can also be expressed in terms of a limit over a directed set of the form (18.1), with a modified definition of the sets B_n. For each h, $1 < h \le k$, define

$B_n^h = \{$J: Each J_i is contained in $[0, n]$, and the distance separating J_i and J_j is at least qn for every $i = 1, \cdots, h-1$, and every $j = h, \cdots, k.\}$.

Let $\mathrm{LIM}(h)_{n\to\infty}$ stand for the limit over the net $\mathcal{N}$, where the sets B_n are replaced by B_n^h. Then the following result follows directly from 16.IV) by means of Fubini's theorem:

LEMMA 18.3. *Condition* 16.IV) *is satisfied if*

$$(18.10)\ \mathrm{LIM}(h)_{n\to\infty} \frac{\int_{J_1}\cdots\int_{J_k} P(X(t_i)>u, i=1,\cdots,k)\, dt_k\cdots dt_1}{\int_{J_1}\cdots\int_{J_{h-1}} P(X(t_i)>u, i=1,\cdots,h-1)\, dt_{h-1}\cdots dt_1 \int_{J_h}\cdots\int_{J_k} P(X(t_i)>u, i=1,\cdots,k)\, dt_k\cdots dt_h} = 1.$$

The main results of our previous work in the Gaussian case [8] also depend on the concept of net convergence and the result of Lemma 18.1. At the time the latter paper was written, I used the latter result without formally stating the concept. Indeed, the proof of the compound Poisson limit theorem employed the continuity theorem for the Laplace-Stieltjes transform, which is, of course, valid for sequences of distributions. Thus Lemma 18.1 furnishes the explicit point of the proof which was only tacitly used in [8].

We showed in the latter work that the compound Poisson limit held for the sojourns of a stationary Gaussian process. The local conditions were the same as in Section 7 above, namely, that $1 - r(t)$ is of regular variation for $t \to 0$. The condition on $r(t)$ which was sufficient for the local and global mixing properties was that $(\log t) r(t) \to 0$ for $t \to \infty$. The calculations in the current work represent abstractions of those that were carried out in the Gaussian case.

There is one minor difference between the calculations in the past and current work. In the former, we calculated the limiting distribution of the sojourn for $t \to \infty$ by first considering $vL_t(u) - vL_t(bu)$, for $b > 1$, and then showing that $vL_t(bu)$ was relatively negligible. The purpose of this step was to make it easier to estimate the joint distribution tails in the Gaussian case. This intermediate step is not included in the present work.

19. The limiting distribution of the maximum for t → ∞. In this section we combine our results on the distribution of $\max(X(t): 0 \le t \le 1)$ and on the compound Poisson limit to obtain the limiting distribution of the maximum over an interval of length tending to ∞. The approach is different from the one used at the end of Section 12 in the case of a differentiable process.

LEMMA 19.1. *Let ζ be a random variable whose distribution function has the Laplace-Stieltjes transform $\Omega(s)$ in* (16.8); *then $P(\zeta = 0) = \exp(H(0+))$.*

PROOF. This follows from the form (16.8) and the simple relation $P(\zeta = 0) = \lim_{s\to\infty} Ee^{-s\zeta}$. Note that $H(0+)$ may assume the value $-\infty$, in which case $P(\zeta = 0) = 0$.

Our main result is:

THEOREM 19.1. *For all sufficiently large t, let $u = u(t)$ and $v = v(t)$ be defined as in* (16.9), *and let $w = w(u(t))$ be defined by* (4.3), *and satisfy* (4.4) *and* (4.5). *Let v also satisfy* (4.5), *that is,*

$$v(u')/v(u) \to 1 \quad \text{for} \quad u'/u \to 1, \quad u \to \infty. \tag{19.1}$$

Then under the conditions of the Maximum Limit Theorem and the conditions of Sections 17 *and* 18 *which are sufficient for the compound Poisson limit, we have*

$$\lim_{t\to\infty} P\{w(\max_{[0,t]} X(s) - u) \le x\} = \exp\{\Gamma'(0)e^{-x}\}, \qquad \textit{for every } x. \tag{19.2}$$

PROOF. For arbitrary $x > 0$ we have the decomposition

$$P\{\max_{[0,t]} X(s) > u\} = P\{vL_t(u) > x\} + P\{\max_{[0,t]} X(s) > u,\ vL_t(u) \le x\}. \tag{19.3}$$

Under the conditions of Sections 17 and 18, the first term on the right side of (19.3) converges to $P(\zeta > x)$, where ζ is defined in Lemma 19.1. By the argument leading from (14.19) to (14.20), and by (16.9), the second term of the right member of (19.3) has a limsup (for $t \to \infty$) which depends on x in such a way that it tends to 0 with x. Since $x > 0$ is arbitrary, we infer from (19.3) that

$$\limsup_{t\to\infty} P\{\max_{[0,t]} X(s) > u\} \le P(\zeta > 0).$$

On the other hand, the general inequality (11.4) implies

$$P(\zeta > 0) \le \liminf_{t\to\infty} P\{\max_{[0,t]} X(s) > u\}.$$

For these inequalities and from Lemma 19.1, we infer that (19.2) holds for the particular value $x = 0$.

Now we extend (19.2) from the case $x = 0$ to the general case. If we multiply the time variable t by an arbitrary constant $c > 0$, then the only change in (19.2) for $x = 0$ is that $\Gamma'(0)$ is replaced by $c\Gamma'(0)$:

$$\lim_{t\to\infty} P\{\max_{[0,ct]} X(s) \le u\} = \exp(c\Gamma'(0)). \tag{19.4}$$

The reason for this is that the relation (19.4) for $c = 1$ was established by comparing the distribution tail of the maximum with that of $vL_t(u)$. For arbitrary $c > 0$, the distribution of $vL_{tc}(u)$ for $t \to \infty$ is asymptotically the same as that of the sum

$$\sum_{j=1}^{[tc]} X_{[t],j}$$

from the stationary array. It follows directly from the assumptions of the compound Poisson limit theorem (16.I – 16.IV) that the only change in the conclusion is that H is replaced by cH. Therefore, $\Gamma'(0)$ is replaced by $c\Gamma'(0)$ as in (19.4).

Recalling that u is a function of t, we express (19.4) in the form

$$\lim_{t\to\infty} P\{\max_{[0,t]} X(s) \le u(t/c)\} = \exp(c\Gamma'(0)). \tag{19.5}$$

We claim that we may choose the function $u(t)$ so that it satisfies

$$u(t/c) = u(t) - \frac{\log c}{w(u(t))}(1 + 0(1)). \tag{19.6}$$

Indeed, according to (4.1) and the hypothesis concerning u and v, we have for every $c > 0$,

$$\frac{t}{c} v\left(u - \frac{\log c}{w(u)}\right)\left[1 - F\left(u - \frac{\log c}{w(u)}\right)\right] \sim tv(u)[1 - F(u)], \qquad u \to \infty,$$

and the latter, by (16.9), is equal to 1. Thus the right hand member of (19.6) plays the role of $u(t/c)$ in the defining relation (16.9), so that (19.6) holds.

Let us put $c = e^{-x}$; then, we conclude from (19.5) and (19.6) that (19.2) holds.

We note that Theorem 19.1 has had several versions in the case of Gaussian processes with $1 - r(t)$ regularly varying. These are due to Pickands [35], Berman [3], and Qualls and Watanabe [36]. The papers of Leadbetter, Lindgren and Rootzen [24] and Mittal [32] furnish corrections to the results of Berman [3]. Finally we note that a new class of limit theorems in the Gaussian case was introduced by Mittal and Ylvisaker [31].

20. Directions of future research. In this section we shall indicate how the results of this paper may be extended and generalized.

Domains of attraction of other extreme value distributions. The conditions of this paper imply that the limiting distribution of the global maximum is of the double exponential type. This is a consequence of the choice of the normalization of the high level process: $w(X(t/v) - u)$, where w and v are suitable functions. The three domains of attraction for the corresponding three limiting types have different kinds of normalization [17]. The type $\exp(-x^{-\alpha})$, $x > 0$, attracts the distribution of the maximum through the normalization of multiplication by a function which is regularly varying at ∞ of index $-1/\alpha$; and the domain of functions F is characterized by

$$\lim_{u\to\infty} u(1 - F(xw(u))) = x^{-\alpha},$$

where $w(u)$ is regularly varying of index α^{-1}. This suggests that the sojourns of $X(t)$ above $w(u)$ should be analyzed through the process $X(t/v)/w(u)$ and its sojourns above the level 1. The condition for the convergence of finite dimensional distributions would then refer to the process above conditioned by $X(0) > w(u)$.

The same program may be carried out for the type

$$\exp(-(-x)^{\alpha}), \qquad x < 0; \quad 1, \quad x > 0.$$

The domain of attraction consists of distribution functions F having a finite x_0 such that $F(x_0) = 1$, $F(x) < 1$ for $x < x_0$, and such that

$$\lim_{u\to\infty} u\left(1 - F\left(x_0 - \frac{x}{w(u)}\right)\right) = x^{\alpha},$$

for $x > 0$ and $w(u)$ regularly varying of index α^{-1}. This suggests the normalization $w(u)(X(t/v) - x_0)$ for the high level process, and the analysis of its sojourns above the level $x_0 - (w(u))^{-1}$ for $u \to \infty$. We have already indicated in Section 4 that we can also consider the part of the domain of $\exp(-e^{-x})$ consisting of functions F with $F(x_0) = 1$ for some $x_0 < \infty$.

Extension of the results to sums of independent processes. Let $\{X_i(t), -\infty < t < \infty\}$, $i = 1, \ldots, k$ be independent processes satisfying the conditions of the Sojourn Limit Theorem with the same normalizing functions w and v but with possibly different limiting processes $Z_i(t)$, $i = 1, \ldots, k$. Put $Y(t) = \sum_{i=1}^k X_i(t)$. Under what conditions will $Y(t)$ also satisfy the conditions of the Sojourn Limit Theorem? Let us define the high level process

$$w\cdot\left(Y\left(\frac{t}{v}\right) - ku\right) = \sum_{i=1}^k [w\cdot(X_i(t/v) - u)], \tag{20.1}$$

conditioned by $Y(0) > ku$.

Under appropriate conditions supplementing those in Theorem 3.1, we expect the conditioned sum above to converge in distribution to a process

$$\sum_{i=1}^k Z_i(t), \tag{20.2}$$

with mutually independent $Z_i(t)$, $i = 1, \ldots, k$.

Extension to a function of several independent processes. The decomposition (20.1) suggests an extension to an arbitrary differentiable function of k variables, $f(x_1, \ldots, x_k)$. Consider the process

$$w\left(f\left(X_1\left(\frac{t}{w}\right), \ldots, X_k\left(\frac{t}{w}\right)\right) - f(u, \ldots, u)\right)$$

where the $X_i(\cdot)$, $i = 1, \ldots, k$, are independent. By the Taylor formula we expect the expression above to be approximately

$$\sum_{i=1}^k \frac{\partial f}{\partial x_i}\bigg|_{x_j=u, j=1,\ldots,k} w\left[X_i\left(\frac{t}{w}\right) - u\right].$$

By the same analysis as for (20.1), we expect the conditioned process above to converge in distribution to a sum of the form (20.2) but where each term in each sum is weighted by the corresponding constant $\partial f/\partial x_i$. In this way we would expect to generalize the results of Section 10 on the Chi squared process to an arbitrary function of k independent copies of a differentiable Gaussian process.

Extensions of the Maximum Limit Theorem. It should also be possible to extend the generalizations above to the estimate of the tail of the distribution of the maximum. The latter was based on our estimate of the tail of the distribution of the conditioned increment of the process. In order to generalize the results to a sum of independent processes, the hypotheses required to obtain similar estimates should also put similar conditions on the increments of the individual processes. As we indicated above, the results for an arbitrary differentiable function of k independent processes ought to be reducible to the sum of independent processes.

Extension of the limit theorems for $t \to \infty$. In their studies of the Chi squared process based on mean square differentiable Gaussian component processes, Sharpe [39] and Lindgren [27] showed that standard mixing conditions on the component processes were sufficient to ensure useful mixing conditions on the Chi squared process itself. In this way, they were able to extend the standard limit theorems for crossings and extremes to the latter process. This indicates that our mixing conditions, stated in Sections 17 and 18, might also carry over from the component processes of a sum to the sum itself.

Markov processes. The sufficient conditions for the Sojourn Limit Theorem stated in Theorem 9.1 are given directly in terms of the transition density. As is well known, the transition density itself is often described as the solution of a diffusion equation with stipulated diffusion coefficients. It would be of interest to find conditions on the latter coefficients which are sufficient for the conditions on the transition density stated in Theorem 9.1.

An examination of the conditions stated in the various results of this paper shows that they are of three types:

A) conditions on the k-dimensional distributions for $k = 1, 2, 3$;

B) conditions on the k-dimensional distributions at k points in the time parameter set, where the latter points are in some neighborhood of one point, and for arbitrary $k \geq 1$;

C) conditions on sets of k-dimensional distributions which are mutually distant in time but which individually are of the type B) above.

For the Markov process, any condition of the type A) may be expressed directly in terms of the transition density or a two- or three-fold product integral. As for the condition of type B) we have already shown in Theorem 9.1 that it is implied by a relatively simple condition on the transition density. Finally, in view of the large literature on mixing properties of Markov processes, we think that there exist convenient hypotheses which are sufficient for the conditions of type C) above.

In a much earlier work [1], we analyzed the maximum of the sample function of a recurrent diffusion process by the renewal principle. By breaking the process into independent parts of random length, we showed that the limiting distribution of the maximum over a long time interval is equivalent to that of the maximum of independent random variables. The result in that work depends on an estimate of the tail of the distribution of the maximum attained between two specified stopping times. Similarly, we can also show that the long term sojourn above a high level may also be broken up into a sum of mutually independent sojourns between successive pairs of stopping times. If our current result can be applied to the diffusion, then, by the theory of sums of independent random variables, it would also imply estimates of the tail of the distribution of the sojourn between successive stopping times.

Strong limit theorems for $t \to \infty$. The maximum limit theorem gives a very precise estimate of the distribution tail of the maximum; however, the conditions of this theorem are also very detailed. Nevertheless, several preliminary estimates used in the proof are valid under relatively simple conditions. These estimates offer the opportunity of extending another class of limit theorems for $\max_{[0,t]} X(s)$, for $t \to \infty$, from the Gaussian to the general case, namely strong limit theorems such as the law of the iterated logarithm and "upper and lower class" limit theorems. We refer to the work of Qualls [37], which also contains references to previous work, and to that of Weber [41] and Geman [16]. Theorem 11.1 furnishes a very general lower bound for the tail of the distribution of the maximum and Theorem 13.2 furnishes an upper bound. A suitable mixing condition and an application of the Borel-Cantelli Lemma should then provide the tools for proving strong limit theorems.

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