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CRITICAL MULTITYPE BRANCHING PROCESSES

By John M. Holte

Gustavus Adolphus College

A general multitype branching process in which individuals are counted according to some possibly type-dependent characteristic may be defined along the lines laid out by Jagers (1969, 1974) for the single type process. In the critical case, the probability of nonextinction at time t is shown to be $O(t^{-1})$, and, conditioned on nonextinction at time t, the totals of the characteristic counts, normalized by t, are shown to satisfy an exponential limit law, under weak (essentially, second moment) hypotheses.

1. Introduction. The multitype branching process model that we consider in this paper is a natural extension of the single type branching process counted according to a random characteristic, or function of age, that was introduced by Jagers (1974). In the critical case we find the limiting distribution, as time $t \to \infty$, of the population counted according to a characteristic, conditioned on nonextinction by time t, and we give the asymptotic behavior of the nonextinction probability. Our results generalize the multitype results of Ney (1974) and Goldstein (1971), who dealt with the Bellman-Harris branching process, and they also generalize the results for the single type process obtained by Green (1977) and Holte (1974). Our proofs were inspired by the arguments of Green and Goldstein. Following Green, we analyze the behavior of the Laplace-Stieltjes transform of the distribution of the conditioned process described above by means of renewal theory—in this case the renewal theory developed by Ryan (1976)—and following Goldstein we base our analysis of the asymptotic nonextinction probability on a comparison with a Markov renewal process.

Informally speaking, the multitype process is initiated by a single individual of age 0. Then offspring of various types are born to this parent at random ages in the course of a random lifespan. Every individual subsequently appearing in the population also lives and reproduces, independently of all other individuals and according to probabilistic laws that depend only on the individual's type. The individuals in the population are "counted" by means of a possibly random function of age, such as the function which assigns a count of 1 to each individual who is alive but has not given birth and which assigns a count of 0 otherwise. The total counts of individuals of each type at time t are the quantities of interest. The critical case is, roughly speaking, the "zero population growth" case.

The formal formulation of the general branching process due to Jagers (1969, 1974) may easily be extended to encompass the case of a process with d types of individuals. Let N denote the positive integers. Let $S = \{1, \cdots, d\} \times N$, $S^0 = \{1, \cdots, d\} \times \{0\}$, $S^n =$ the nth Cartesian power of S, and $\mathscr{I} = \cup_{n=0}^{\infty} S^n$. $\mathscr{I}$ is the set of possible individuals: $\langle i_1, n_1; \cdots; i_k, n_k \rangle \in \mathscr{I}$ labels the n_kth type i_k offspring of $\cdots$ the n_1th type i_1, offspring of the initial individual. $\mathscr{I}$ may be decomposed as $\mathscr{I}_1 \cup \cdots \cup \mathscr{I}_d$ where $\mathscr{I}_i$ is the set of possible type i individuals.

We assume that to each $I \in \mathscr{I}$ is associated a triple $(\lambda_I, \xi_I, \chi_I)$ of random entities: λ_I, a nonnegative random variable, represents the lifespan of I; ξ_I, a vector of d point processes $\xi_{I,j}$ on $[0, \infty)$, represents the birth process ($\xi_{I,j}([0, t]) = \xi_{I,j}(t) =$ the number of type j offspring born to I during age interval $[0, t]$); and χ_I is a stochastic process giving a "measure" or "count" of I as a function of age ($\chi_I(t) = 0$ for $t < 0$). We assume that the triples $(\lambda_I, \xi_I, \chi_I)$, $I \in \mathscr{I}$, are stochastically independent, and that for $i = 1, \cdots, d$ the

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triples $(\lambda_I, \xi_I, \chi_I)$, $I \in \mathscr{I}_i$, are identically distributed. Then we may introduce the following notation. Let P^i denote probability, given that the initial individual is of type i, and let E^i denote the corresponding expectation. For random variables, write λ^i for $\lambda_{\langle i,0\rangle}$, ξ^i_j or ξ_j for $\xi_{\langle i,0\rangle,j}$, $\chi^i(t)$ for $\chi_{\langle i,0\rangle}(t)$, etc. Assume that $P^i(\chi^j(t) = 0) = 1$ if $i \neq j$, so that the type j count of a type i individual is 0. We also make the following assumptions (which, it seems, could be weakened somewhat): each χ_I is a nonnegative stochastic process; $P^i(\chi^i(t) = 0 \mid \lambda^i \leq t) = 1$, so that we do not count individuals after their deaths; and $P^i(\xi_j([\lambda^i, \infty)) = 0) = 1$, so that individuals cannot give birth after death.

The process we study is $\mathbf{X}'(t) = (X_1(t), \cdots, X_d(t))$, where $'$ denotes transpose, and

$$X_j(t) = \textstyle\sum_{I \in \mathscr{I}_j} \chi_I(t - \sigma_I),$$

where σ_I is the time of birth of I. If $\langle 0\rangle$ is of type i, i.e. if the original individual is $\langle i, 0\rangle$, then we may write $X^i_j(t)$ for $X_j(t)$. One important special case is the case where χ_I is the indicator of $[0, \lambda_I)$; then $\mathbf{X}(t) = \mathbf{Z}(t)$, the population size process. Another example arises when we take χ_I to be the indicator of $[0, \lambda_I) \cap [a, b)$; then $X_j(t)$ is the number of type j individuals who are alive and in the age interval $[a, b)$ at time t.

2. Statement of main results. Let $m^i_j(t) = E^i(\xi_j(t))$, the mean number of type j offspring born to a type i parent in the age interval $[0, t]$, and let $\mathbf{m}(t) = [m^i_j(t)]$, a $d \times d$ matrix of distributions (or measures). Our main theorem makes the following hypotheses concerning $\mathbf{m}$.

(M.1) $\mathbf{m}(\infty)^p > 0$ (componentwise) for some $p \in \mathbb{N}$.
(M.2) $\mathbf{m}(\infty)$ has largest eigenvalue 1.
(M.3) $m^i_j(0) < m^i_j(\infty)$ for some i, j.
(M.4) $\mathbf{m}(\infty) - \mathbf{m}(t) = o(t^{-2})$ as $t \to \infty$.
(M.5) Each $m^i_j(t)$ is a nonlattice distribution.

Assumption (M.1) allows us to invoke Perron-Frobenius theory, and assumption (M.2) is the criticality assumption. Together they imply that $\exists$ strictly positive left and right eigenvectors $\mathbf{v}'$ and $\mathbf{u}$ corresponding to the eigenvalue 1 which we may assume to be normalized so that $\mathbf{v}\cdot\mathbf{u} = \mathbf{v}'\mathbf{u} = 1$ and $\mathbf{v}\cdot\mathbf{1} = 1$. Assumption (M.3) excludes the case where the entire population is born at time 0. Assumption (M.4) is slightly weaker than the assumption that the distributions $m^i_j(\cdot)$ have finite second moments; when (M.4) holds, we do have finite *first* moments $\mu^i_j = \int_0^\infty t m^i_j(dt)$; we write $\boldsymbol{\mu} = [\mu^i_j] = \int_0^\infty t\mathbf{m}(dt)$.

Let $G^i(t) = P^i(\lambda^i \leq t)$ and $\mathbf{G}(t) = (G^1(t), \cdots G^d(t))'$. Our assumption on $\mathbf{G}$ is like (M.4):

$$(\Lambda) \qquad \mathbf{1} - \mathbf{G}(t) = o(t^{-2}) \quad \text{as} \quad t \to \infty.$$

Let $\zeta_i(n)$ be the number of type i individuals in the nth generation of the branching process. Then $\boldsymbol{\zeta}(n)$ is a multitype Galton-Watson process, a fact that will be exploited in the proof of the asymptotic nonextinction probability result. Notice that $\boldsymbol{\zeta}(1) = \boldsymbol{\xi}(\infty)$. Our assumption

$$(\text{GW.1}) \qquad \mathbf{E}\{\mathbf{s}^{\boldsymbol{\xi}(\infty)}\} \neq \mathbf{m}(\infty)\mathbf{s},$$

where the ith component of the left side is $E^i(\prod_{j=1}^d s_j^{\xi_j(\infty)})$, rules out the case where each parent gives birth to exactly one offspring (in which case $\mathbf{m}(\infty)$ would be the transition matrix of a Markov chain on the space of types). We also assume finiteness of the second moments:

$$(\text{GW.2}) \qquad q^i_{jk} \equiv E^i(\xi_k(\infty)\xi_j(\infty) - \delta_{jk}\xi_k(\infty)) \ \forall\ i, j, k.$$

Our main result is the following exponential limit law.

THEOREM. *Assume* (M.1)–(M.5), (Λ), *and* (GW.1)–(GW.2). *Also assume for* $i = 1, \cdots, d$ *that* $E^i(\chi^i(\cdot))$ *is directly Riemann integrable,* $E^i((\chi^i(t))^2)$ *is bounded and* $\to 0$ *as*

$t \to \infty$, and $\chi^i(t)$ is almost surely almost everywhere continuous. Then for $i = 1, \cdots, d$

$$\lim_{t\to\infty} P^i\{t^{-1}\mathbf{X}(t) > \mathbf{x} \mid \mathbf{Z}(t) \neq \mathbf{0}\} = \exp\{-\max_{1\le j\le d} x_j/c_j\},$$

where

$$c_j = \frac{1}{2} v_j Q \int_0^\infty E^j(\chi^j(y))\, dy/\beta^2,$$

$$\beta = \mathbf{v}'\boldsymbol{\mu}\mathbf{u} = \sum\sum v_i \mu^i_j u^j,$$

$$Q = \mathbf{v}\cdot\mathbf{q}[\mathbf{u}],$$

$$q^i[\mathbf{u}] = \sum\sum q^i_{jk} u^j u_k.$$

An interpretation of this conclusion is offered in Section 5.

The proof of this theorem is based in part on the following result on the probability of nonextinction as $t \to \infty$.

PROPOSITION. *Assume* (M.1)–(M.4), (Λ), *and* (GW.1)–(GW.2). *Then for $i = 1, \cdots, d$*

$$\lim_{t\to\infty} tP^i\{\mathbf{Z}(t) \neq \mathbf{0}\} = 2\beta Q^{-1} u_i.$$

3. Renewal theory. Ryan's multidimensional renewal theorem [12] plays a key role in our proofs. The following notation is used: if **A** and **B** are matrices of measures and **f** is a vector function, then

$$\mathbf{A}*\mathbf{B}(t) = [\textstyle\sum_k a^i_k * b^k_j(t)], \quad a^i_k * b^k_j(t) = \displaystyle\int_{[0,t]} b^k_j(t-y)\, a^i_k(dy),$$

and

$$\mathbf{A}*\mathbf{f}(t) = \langle \textstyle\sum_j a^i_j * f_j(t)\rangle, \quad a^i_j * f_j(t) = \displaystyle\int_{[0,t]} f_j(t-y)\, a^i_j(dy).$$

RENEWAL THEOREM. *Let $\mathbf{A} = [a^i_j]$ be a $d \times d$ matrix of measures satisfying hypotheses* (M.1)–(M.3), (M.5) (*with* **m** *replaced by* **A**) *and $\int_0^\infty t\mathbf{A}(dt) < \infty$. Let $\mathbf{f}(t)$ be a directly Riemann integrable d-vector function. Then the renewal equation*

$$\mathbf{r}(t) = \mathbf{f}(t) + \mathbf{A}*\mathbf{r}(t)$$

has a unique solution $\mathbf{r}(t)$ which is bounded on finite intervals.

This solution satisfies

$$\mathbf{r}(t) \to \mathbf{B}\int_0^\infty \mathbf{f}(y)\, dy \quad \text{as} \quad t \to \infty,$$

where

$$\mathbf{B} = \beta^{-1}[u^i v_j],$$

where **u** and **v**$'$ are the unique right and left eigenvectors of $\mathbf{A}(\infty)$ with eigenvalue 1 such that $\mathbf{v}\cdot\mathbf{1} = 1$ and $\mathbf{v}\cdot\mathbf{u} = 1$, and $\beta = \mathbf{v}'\int_0^\infty t\mathbf{A}(dt)\mathbf{u}$.

Let $\mathbf{U}(t) = \sum_{n\ge 0}\mathbf{A}^{*n}(t)$, the renewal function. $\mathbf{U}(t) < \infty\ \forall\ t$ by Lemma 4 of Ryan [12].

COROLLARY. *Under the assumptions of the Renewal Theorem*

(1) $$\mathbf{U}(t) - \mathbf{U}(t-h) \to h\mathbf{B} \quad \textit{as} \quad t \to \infty\ \forall\ h > 0, \quad \textit{and}$$

(2) $$t^{-1}\mathbf{U}(t) \to B \quad \textit{as} \quad t \to \infty.$$

The Renewal Theorem was proved by Ryan (1976). (The $\mathscr{A}_{1k}^{1k}(\infty)$ appearing in his theorem should be $\mathscr{A}_{1k}^{1}(\infty)$ due to a corresponding correction of his Lemma 5.) Assuming the existence part of Ryan's theorem, the limit formula may be derived in a different form. We show this in the following lemma, which we shall use in Section 5, but not in the main proofs.

RENEWAL LEMMA. *Assume that* $\mathbf{A}$ *is a matrix of measures on* $[0, \infty)$ *satisfying* (M.1)–(M.3), (M.5), *and* $\int_0^\infty t\mathbf{A}(dt) \equiv \boldsymbol{\mu} < \infty$. *Assume that* $\mathbf{f}$ *is a bounded measurable vector function on* $[0, \infty)$ *such that* $\mathbf{f}(t) \to \mathbf{0}$ *at* $t \to \infty$ *and* $\lim_{t\to\infty} \int_0^t \mathbf{f}(y)\, dy$ *exists. Finally, assume that* $\mathbf{r}(t)$ *is the solution of the renewal equation*

$$\mathbf{r}(t) = \mathbf{f}(t) + \mathbf{A} * \boldsymbol{r}(t) \qquad (t \geq 0)$$

that is bounded on finite intervals, and that $\lim_{t\to\infty} \mathbf{r}(t)$ *exists finitely. Then*

$$\lim_{t\to\infty} \mathbf{r}(t) = (\mathbf{v}'\boldsymbol{\mu}\mathbf{u})^{-1}\mathbf{v}' \lim_{t\to\infty} \int_0^t \mathbf{f}(y)\, dy\, \mathbf{u}.$$

COROLLARY. *Under the same hypotheses as above, but with the vectors* $\mathbf{f}$ *and* $\mathbf{r}$ *replaced by matrices* $\mathbf{F}$ *and* $\mathbf{R}$ *so that*

$$\mathbf{R}(t) = \mathbf{F}(t) + \mathbf{A} * \mathbf{R}(t) \qquad (t \geq 0)$$

we have

$$\lim_{t\to\infty} \mathbf{R}(t) = \mathbf{B} \lim_{t\to\infty} \int_0^t \mathbf{F}(y) dy,$$

where

$$\mathbf{B} = [u^i v_j]/v'\boldsymbol{\mu}\mathbf{u}.$$

PROOF OF THE RENEWAL LEMMA. Let $\mathbf{r}(\infty) = \lim \mathbf{r}(t)$. From the renewal equation we get $\mathbf{r}(\infty) = \mathbf{A}(\infty)\mathbf{r}(\infty)$, whence $\mathbf{r}(\infty) = k\mathbf{u}$ for some constant k. From the renewal equation again we get $\mathbf{v}'(\mathbf{r}(t) - \mathbf{A} * \mathbf{r}(t)) = \mathbf{v}'\mathbf{f}(t)$. Writing $\mathbf{A}(t) = \mathbf{A}(\infty) - (\mathbf{A}(t) - \mathbf{A}(\infty))$ and using $\mathbf{v}'\mathbf{A}(\infty) = \mathbf{v}'$, we get $\mathbf{v}'\mathbf{f}(t) = \mathbf{v}'(\mathbf{A}(\infty) - \mathbf{A}(\cdot)) * r(t)$. By integration,

$$\int_0^t \mathbf{v}'\mathbf{f}(y)\, dy = \int_0^t \mathbf{v}'(\mathbf{A}(\infty) - \mathbf{A}(\cdot)) * \mathbf{r}(y)\, dy$$

$$= \mathbf{v}' \int_0^{(\cdot)} (\mathbf{A}(\infty) - \mathbf{A}(x))\, dx * \mathbf{r}(t).$$

Therefore,

$$\mathbf{v}' \lim_{t\to\infty} \int_0^t \mathbf{f}(y)\, dy = \mathbf{v}' \int_0^\infty (\mathbf{A}(\infty) - \mathbf{A}(x))\, dx\, \mathbf{r}(\infty) = \mathbf{v}' \int_0^\infty x\, \mathbf{A}(dx)\mathbf{r}(\infty) = \mathbf{v}'\boldsymbol{\mu}\mathbf{r}(\infty),$$

upon integration by parts. Since $\mathbf{r}(\infty) = k\mathbf{u}$, this implies

$$k = \mathbf{v}' \lim_{t\to\infty} \int_0^t \mathbf{f}(y)\, dy / \mathbf{v}'\boldsymbol{\mu}\mathbf{u},$$

whence the formula for $\mathbf{r}(\infty)$ follows. □

4. First and second moments. The starting point for our analysis is the following equality in distribution:

$$X_j^i(t) = \delta_j^i \chi^j(t) + \sum_{k=1}^d \sum_{n=1}^{\xi_k(t)} X_{j,n}^k(t - \tau_k(n)), \tag{4.1}$$

where $\tau_k(n) = \inf\{t: \xi_k^i(t) \geq n\}$ is the age of the original parent at the birth of the nth type k offspring, and $X_{j,n}^k (n = 1, 2, 3, \cdots)$ are i.i.d. copies of X_j^k. This equation asserts that the total χ-count of type j individuals is equal to the χ-count of the initial individual plus the sum of the χ-counts of type j individuals in the lines of descent emanating from each offspring of the initial individual. In this section we shall deduce from (4.1) the basic equations for the first and second moments of $\mathbf{X}(t)$ (only the first moment is used in the sequel), and in Section 7 we deduce the basic equation for the Laplace-Stieltjes transform of $\mathbf{X}(t)$.

Taking expectations in (4.1) leads to a renewal equation for $\mathbf{M}(t) \equiv [E^i(X_j^i(t))]$:

$$\mathbf{M}(t) = [\delta_j^i E^j(\chi^j(t))] + \mathbf{m} * \mathbf{M}(t). \tag{4.2}$$

Accordingly, when $\mathbf{m}$ satisfies the hypotheses on $\mathbf{A}$ in the Renewal Theorem,

$$M_j^i(t) \to \beta^{-1} u^i v_j \int_0^\infty E^j(\chi^j(y))\, dy \quad \text{as} \quad t \to \infty \tag{4.3}$$

where now $\beta = \sum \sum v_i \mu_j^i \mu^j \mu_j^i = \int_0^\infty t m_j^i(dt)$, and $\mathbf{v}'$ and $\mathbf{u}$ are the left and right eigenvectors of $\mathbf{m}(\infty)$ corresponding to eigenvalue 1 normalized so that $\mathbf{v} \cdot \mathbf{1} = 1$ and $\mathbf{v} \cdot \mathbf{u} = 1$.

Equations for higher moments may also be derived from equation (4.1). For example, the second factorial moment,

$$Q_{j_1 j_2}^i(t_1, t_2) \equiv E^i\{X_{j_1}(t_1) X_{j_2}(t_2) - \delta_{j_1 j_2} X_{j_1}(t_1 \wedge t_2)\},$$

satisfies

$$Q_{j_1 j_2}^i(t_1, t_2) = f_{j_1 j_2}^i(t_1, t_2) + \textstyle\sum_{k_1} \sum_{k_2} \displaystyle\int_0^{t_1} \int_0^{t_2} M_{j_1}^{k_1}(t_1 - y_1) M_{j_2}^{k_2}(t_2 - y_2) q_{k_1 k_2}^i(dy_1, dy_2)$$

$$+ \textstyle\sum_k \displaystyle\int_0^{t_1 \wedge t_2} Q_{j_1 j_2}^k(t_1 - y, t_2 - y) m_k^i(dy),$$

where

$$f_{j_1 j_2}^i(t_1, t_2) \equiv \delta_{ij_1} \delta_{ij_2} E^i\{\chi^i(t_1)\chi^i(t_2) - \chi^i(t_1 \wedge t_2)\}$$

$$+ \delta_{ij_1} \textstyle\sum_k \displaystyle\int_0^{t_1} \int_0^{t_2} M_{j_2}^k(t_2 - y_2) q_{j_1 k}^i(dy_1, dy_2)$$

$$+ \delta_{ij_1} \int_0^{t_1 \wedge t_2} M_{j_2}^{j_1}(t_2 - y) m_{j_1}^i(dy)$$

$$+ \delta_{ij_2} \textstyle\sum_k \displaystyle\int_0^{t_1} \int_0^{t_2} M_{j_1}^k(t_1 - y_1) q_{kj_2}^i(dy_1, dy_2)$$

$$+ \delta_{ij_2} \int_0^{t_1 \wedge t_2} M_{j_2}^{j_1}(t_1 - y) m_{j_2}^i(dy).$$

If $t_1 = t$ and $t_2 - t_1 = \Delta > 0$ is fixed, then this is a system of renewal equations in t.

5. The embedded 1-type processes. The results of this section permit us to interpret the conclusion of our theorem, but are not used in the main line of proof. The quantities $c_1, \cdots, c_d$ that appear in the exponential limit law may be interpreted with reference to 1-type critical branching processes $\tilde{X}_j(t)$ associated with the birth processes $\tilde{\xi}_1(t), \cdots, \tilde{\xi}_d(t)$, where $\tilde{\xi}_j(t)$ = the number of type j individuals born during $[0, t]$ which have no type j ancestor, with the possible exception of the initial ancestor. (The occurrence of these embedded 1-type processes was noted by Doney in [1].) To analyze the $\tilde{\xi}_j(\cdot)$ processes, we start with

$$\tilde{\xi}_j(t) =_{\mathscr{D}} \xi_j(t) + \sum_{k\neq j} \sum_{l=1}^{\xi_k(t)} \tilde{\xi}_{j,l}(t - \tau_k(l)), \tag{5.1}$$

where $\tilde{\xi}_{j,1}, \tilde{\xi}_{j,2}, \cdots$ are independent copies of the $\tilde{\xi}_j$ process. This equation says that the number of type j individuals born during $[0, t]$ without a type j ancestor other than (possibly) $\langle 0\rangle$ is equal to the number of type j offspring born to $\langle 0\rangle$ during $[0, t]$ plus the number of such individuals descended from the offspring of $\langle 0\rangle$ that are not of type j.

Taking expectations in (5.1) leads to

$$\tilde{m}_j^i(t) \equiv E^i(\tilde{\xi}_j(t)) = m_j^i(t) + \sum_{k\neq j} m_k^i * \tilde{m}_j^k(t),$$

which may be written

$$\tilde{\mathbf{m}}(t) = \mathbf{m} * (\mathbf{I} - \mathbf{m}^{dg}(t)) + \mathbf{m} * \tilde{\mathbf{m}}(t), \tag{5.2}$$

where $\tilde{\mathbf{m}}^{dg}(t) = [\delta_j^i \mathbf{m}_j^j(t)]$. By the monotone convergence theorem,

$$\tilde{m}_j^i(\infty) = \lim_{t\to\infty} \tilde{m}_j^i(t) = m_j^i(\infty) + \sum_{k\neq j} m_k^i(\infty)\tilde{m}_j^k(\infty),$$

or, $\tilde{\mathbf{m}}(\infty) = \mathbf{m}(\infty)(\mathbf{I} - \tilde{\mathbf{m}}^{dg}(\infty)) + \mathbf{m}(\infty)\tilde{\mathbf{m}}(\infty)$. Left multiplying by $\mathbf{v}'$ and using $\mathbf{v}'\mathbf{m}(\infty) = \mathbf{v}'$, we deduce $\tilde{\mathbf{m}}^{dg}(\infty) = \mathbf{I}$. Consequently $\tilde{\mathbf{m}}(\infty) = \mathbf{m}(\infty)\tilde{\mathbf{m}}(\infty)$. But here every column of $\tilde{\mathbf{m}}(\infty)$ must be a right eigenvector of $\mathbf{m}(\infty)$ corresponding to the eigenvalue 1, and hence must be proportional to $\mathbf{u}$. So $\tilde{\mathbf{m}}_j^i(\infty) = u^i k_j$, say, and $1 = \tilde{m}_j^j(\infty) = u^j k_j$, whence $\tilde{m}_j^i(\infty) = u^i/u_j$ $(i, j = 1, \cdots, d)$. In particular, $\tilde{m}_j^j(\infty) = 1$ $(j = 1, \cdots, d)$, and so the d resulting 1-type branching processes $\tilde{X}_j(t)$ are critical.

Next we shall calculate the so-called mean age at reproduction, $\tilde{\mu}_j^j = \int_0^\infty t\tilde{m}_j^j\,(dt) = \int_0^\infty (1 - \tilde{m}_j^j(t))\,dt$. As we have noted, $\tilde{\mathbf{m}}(t)$ satisfies the renewal equation $\tilde{\mathbf{m}}(t) = \mathbf{m}*(\mathbf{I} - \tilde{\mathbf{m}}^{dg}(t)) + \mathbf{m}*\tilde{\mathbf{m}}(t)$ and converges to $\tilde{\mathbf{m}}(\infty) = [u^i/u_j]$. On the other hand,

$$\lim_{t\to\infty} \int_0^t \mathbf{m}*(\mathbf{I} - \tilde{\mathbf{m}}^{dg}(y))\,dy = \lim_{t\to\infty}\mathbf{m}* \int_0^t (\mathbf{I} - \tilde{\mathbf{m}}^{dg}(y))\,dy$$

$$= \mathbf{m}(\infty) \int_0^\infty (\mathbf{I} - \tilde{\mathbf{m}}^{dg}(y))\,dy = \mathbf{m}(\infty)\tilde{\boldsymbol{\mu}}^{dg},$$

and so, by the Renewal Lemma,

$$\lim_{t\to\infty}\tilde{\mathbf{m}}(t) = \mathbf{B}\mathbf{m}(\infty)\tilde{\boldsymbol{\mu}}^{dg} = \mathbf{B}\tilde{\boldsymbol{\mu}}^{dg} = [u^i v_j \tilde{\mu}_j^j]/\mathbf{v}'\boldsymbol{\mu}\mathbf{u}.$$

As this also equals $[u^i/u_j]$, it follows that

$$\tilde{\mu}_j^j = \mathbf{v}'\boldsymbol{\mu}\mathbf{u}/(u_j v^j), \quad \text{for } j = 1, \cdots, d. \tag{5.3}$$

A lengthy calculation based on (5.1) leads to a system of equations for the second factorial moments:

$$\begin{aligned}\tilde{q}_{jk}^i(t_1, t_2) &\equiv E^i(\tilde{\xi}_j(t_1)\tilde{\xi}_k(t_2) - \delta_{jk}\tilde{\xi}_k(t_1 \wedge t_2)) \\ &= \sum_\ell \sum_r \int_0^{t_1} \int_0^{t_2} f_{j,k}^{\ell,r}(t_1 - y_1, t_2 - y_2) q_{\ell r}^i\,(dy_1, dy_2) \\ &\quad + (1 - \delta_{jk}) \int_0^{t_1\wedge t_2} (\tilde{m}_j^k(t_1 - y) + \tilde{m}_k^j(t_2 - y)) m_j^i(dy) \\ &\quad + \sum_{r\notin\{j,k\}} \int_0^{t_1\wedge t_2} \tilde{q}_{jk}^r(t_1 - y, t_2 - y) m_r^i(dy),\end{aligned}$$

where

$$f^{\ell r}_{j,k}(t_1, t_2) = \begin{cases} \tilde{m}^{\ell}_j(t_1)\tilde{m}^r_k(t_2) & \text{if } \ell \neq j, r \neq k \\ \tilde{m}^{\ell}_j(t_1) & \text{if } \ell \neq j, r = k \\ \tilde{m}^r_k(t_2) & \text{if } \ell = j, r \neq k \\ 1 & \text{if } \ell = j, r = k. \end{cases}$$

When $t_1 \wedge t_2 \to \infty$, we get

$$\tilde{q}^i_{jk} = \textstyle\sum_\ell \sum_r q^i_{\ell r} u^\ell u^r/(u_j u_k) + (1 - \delta_{jk}) m^i_j u^j/u_k$$
$$+ (1 - \delta_{jk}) m^i_k u^k/u_j + \textstyle\sum_{r \notin \{j,k\}} m^i_r \tilde{q}^r_{jk},$$

where $\tilde{q}^i_{jk} = \lim \tilde{q}^i_{jk}(t_1, t_2)$, etc. Let

$$q^i[\mathbf{u}] = \textstyle\sum_\ell \sum_r q^i_{\ell r} u^\ell u^r \tag{5.4}$$

and $\mathbf{q}[\mathbf{u}] = (q^1[\mathbf{u}], \cdots, q^d[\mathbf{u}])'$. Then when $j = k$, the limiting equation becomes

$$\tilde{q}^i_{jj} = q^i[\mathbf{u}]/u_j^2 + \textstyle\sum_{r \neq j} m^i_r \tilde{q}^r_{jj}.$$

Left multiplying by v_i and summing over i leads to an equation which can be solved for $\tilde{q}^j_{jj}$:

$$\tilde{q}^j_{jj} = \tilde{q}^j_{jj}(\infty, \infty) = \mathbf{v}\cdot\mathbf{q}[\mathbf{u}]/(v_j u_j^2) = Q/(v_j u_j^2). \tag{5.5}$$

We conclude from equations (5.3) through (5.5) that

$$c_j = \tfrac{1}{2}\, \mu^j_j \bar{\chi}^j \tilde{q}^j_{jj}/(\tilde{\mu}^j_j)^2,$$

where

$$\bar{\chi}^j \equiv \int_0^\infty E^j(\chi^j(y))\, dy/\mu^j_j \, (j = 1, \cdots, d),$$

and so the jth marginal distribution is exponential with this mean. But this is exactly the limit law that holds for each 1-type process $\tilde{X}_j(t)$ when the *conclusion* of [4] for the critical case is true.

6. Proof of the proposition on the asymptotic nonextinction probability. We prove the proposition via a sequence of lemmas like those in Goldstein (1971). The basic lemma is the asymptotic nonextinction probability result for critical multitype Galton-Watson processes due to Joffe and Spitzer (1967). To state it, we need some additional notation. Let $f^i_n(\mathbf{s}) = E^i(s_1^{\zeta_1(n)} \cdots s_d^{\zeta_d(n)})$, where $\zeta_i(n)$ = the number of type i individuals in the nth generation of the embedded multitype Galton-Watson process. Then $f^i_n(\mathbf{0}) = P^i(\zeta(n) = \mathbf{0})$, the probability of extinction by the nth generation.

Throughout this section we assume that the hypotheses (M.1) through (M.4), (Λ), and (GW.1) through (GW.2) are in force.

LEMMA 1. $\lim n(\mathbf{1} - \mathbf{f}_n(\mathbf{0})) = \tfrac{1}{2}\,(\mathbf{v}\cdot\mathbf{q}[\mathbf{u}])^{-1}\mathbf{u}$, *where* $\mathbf{q}[\mathbf{u}]$ *is given by equation* (5.4).

NOTE. (GW.1) implies that $\mathbf{v}\cdot\mathbf{q}[\mathbf{u}] > 0$.

The remaining lemmas relate the population sizes at time t, $\mathbf{Z}(t)$, to the generation sizes, $\zeta(n)$. Let $F^i(\mathbf{s}, t) = E^i(s_1^{Z_1(t)} \cdots s_d^{Z_d(t)})$. Next we give the counterpart of Goldstein's "main lemma."

LEMMA 2. *For* $t \geq 0$, $n = 1, 2, 3, \cdots$,

$$-\mathbf{m}^{*n} * \mathbf{G}(t) \leq \mathbf{f}_n(\mathbf{0}) - \mathbf{F}(\mathbf{0}, t) \leq \textstyle\sum_{j=0}^{n-1} \mathbf{m}^{*j} * (\mathbf{1} - \mathbf{G}(t)).$$

PROOF. The ith component of $\mathbf{f}_n(\mathbf{0}) - \mathbf{F}(\mathbf{0}, t)$ is $P^i(\zeta(n) = \mathbf{0}) - P^i(\mathbf{Z}(t) = \mathbf{0}) = P^i(\mathbf{Z}(t) \neq \mathbf{0}) - P^i(\zeta(n) \neq \mathbf{0}) = P^i(\mathbf{Z}(t) \neq \mathbf{0}\ \&\ \zeta(n) = \mathbf{0}) - P^i(\zeta(n) \neq \mathbf{0}\ \&\ \mathbf{Z}(t) = \mathbf{0})$. Let $\alpha_i(n, t)$ be the number of type i individuals from generations $0, 1, \cdots, n-1$ alive at time t. Also, let $\beta_i(n, t)$ be the number of type i individuals born into the nth generation by time t, and $\gamma_i(n, t)$ the number of those who are also alive at time t. Now $\mathbf{Z}(t) \neq \mathbf{0}$ & $\zeta(n) = \mathbf{0}$ implies $\boldsymbol{\beta}(n, t) = \mathbf{0}$ & $\mathbf{Z}(t) = \boldsymbol{\alpha}(n, t) \neq \mathbf{0}$, and so

$$P^i(\mathbf{Z}(t) \neq \mathbf{0} \;\&\; \zeta(n) = \mathbf{0}) \leq P^i(\boldsymbol{\alpha}(n, t) \neq \mathbf{0}) \leq \sum_j P^i(\alpha_j(n, t) \geq 1) \leq \sum_j E^i(\alpha_j(n, t))$$
$$= \sum_{j=1}^{d} \sum_{k=0}^{n-1} (\mathbf{m}^{*k})^i_j * (1 - G^j(t)),$$

the ith component of $\sum_{k<n} \mathbf{m}^{*k} * (\mathbf{1} - \mathbf{G}(t))$. On the other hand, $\zeta(n) \neq \mathbf{0}$ & $\mathbf{Z}(t) = \mathbf{0}$ implies $\boldsymbol{\beta}(n, t) \neq \mathbf{0}$ & $\boldsymbol{\gamma}(n, t) = \mathbf{0}$. Therefore

$$P^i(\zeta(n) \neq \mathbf{0} \;\&\; \mathbf{Z}(t) = \mathbf{0}) \leq P^i(\boldsymbol{\beta}(n, t) \neq \mathbf{0} \;\&\; \boldsymbol{\gamma}(n, t) = \mathbf{0})$$
$$\leq \sum_j P^i(\beta_j(n, t) \geq 1 \;\&\; \gamma_j(n, t) = 0)$$
$$\leq \sum_j E^i(\beta_j(n, t); \gamma_j(n, t) = 0) = \sum_j (\mathbf{m}^{*n})^i_j * G^j(t),$$

the ith component of $\mathbf{m}^{*n} * \mathbf{G}(t)$. □

The following lemma, which is the counterpart of Lemma (2.12) of Goldstein, may be proved by direct calculation and induction.

LEMMA 3. *Let* $\mathbf{P}(t) = [P_{ij}(t)] = [u_i^{-1} m_{ij}(t) u_j]$ *and let* $\mathbf{P} = \mathbf{P}(\infty)$. *Then*
(i) $\mathbf{P}$ *is a stochastic matrix*;
(ii) $\mathbf{uv} \equiv (u_1 v_1, \cdots, u_d v_d) = (\mathbf{uv})\mathbf{P}$, i.e. $\mathbf{uv}$ *is a stationary measure for the Markov chain defined by* $\mathbf{P}$;
(iii) $\mathbf{P}^{*n}(t) = [u_i^{-1} m_{ij}^{(*n)}(t) u_j]$ *and* $\mathbf{m}^{*n}(t) = [u_i p_{ij}^{(*n)}(t) u_j^{-1}]$, *where* $m_{ij}^{(*n)}(t)$ is the i, j *component of* $\mathbf{m}^{*n}(t)$ *and* $p_{ij}^{(*n)}(t)$ *is the* i, j *component of* $\mathbf{P}^{*n}(t)$. *Here and in the remainder of this section we write* m_{ij} *for* m^i_j, *etc., to leave the superscript position open for powers and convolution powers.*

In order to study the possible 1-step transitions of the Markov chain Z_n corresponding to $\mathbf{P}$, we introduce the "expanded process" Y_n of S. Hudson that is described in Section 6.5 of Kemeny and Snell (1960). If $i \to j$ and $k \to l$ are possible one step transitions, i.e. if $p_{ij} p_{k\ell} > 0$, then the probability of $k \to l$, given that $i \to j$ was the previous transition, is $P(Y_{n+1} = (k, l) \mid Y_n = (i, j)) = \delta_{jk} p_{k\ell}$. Since $\mathbf{P}$ gives rise to a "regular" process, i.e. some power of $\mathbf{P}$ is strictly positive, and since $\mathbf{P}$ has stationary measure $\mathbf{uv}$, we have the following.

LEMMA 4. *The expanded process for* $\mathbf{P}$ *is regular and has stationary measure* $\mathbf{w}$, *where* $w_{(i,j)} = u_i v_i p_{ij} = v_i m_{ij}(\infty) u_j$.

PROOF. See pages 141–142 of Kemeny and Snell (1960).

LEMMA 5. *Let* $S_n(i, j)$ *be the number of times the one step transition* $i \to j$ *occurs in* $n + 1$ *steps of the Markov chain for* $\mathbf{P}$. *Then* $\forall\, \delta > 0\ \exists\, c > 0, \lambda \in (0, 1)$ *such that*

$$P\{|n^{-1} S_n(i, j) - u_i v_i p_{ij}| \geq \delta \quad \text{for some} \quad (i, j) \mid Z_0 = k\} \leq c\lambda^n$$

for $n = 1, 2, 3, \cdots$.

PROOF. Form the expanded chain Y_n whose states are the possible one step transitions. Let $\delta > 0$, and let B_n be the event that $|n^{-1} S_n(i, j) - u_i v_i p_{ij}| \geq \delta$ for some (i, j). By the theorem of Katz and Thomasian (1961) via Lemma (2.13) of Goldstein, $\exists\, c > 0, \lambda \in (0, 1)$ such that $P(B_n \mid Y_0 = (k, l)) \leq c\lambda^n$. Then $P(B_n \mid Z_0 = k) = \sum_l P(B_n \mid Y_0 = (k, l)) P(Z_1 = l \mid Z_0 = k) \leq c\,\lambda^n$. □

The next five lemmas relate the hypothesis that $m(\infty) - m(t) = o(t^{-2})$ to the asymptotic behavior of the error bounds in Lemma 2.

LEMMA 6. *Let* $\rho_{ij}(t) = m_{ij}(t)/m_{ij}(\infty) = p_{ij}(t)/p_{ij}$ *if* $m_{ij}(\infty) \neq 0$ *and* $\rho_{ij}(t) = 0$ *otherwise, and let* $\bar{\rho}_{ij} = \int_0^\infty t\rho_{ij}(dt)$. *For* n_t *defined as in* (i) *or* (ii) *below, let* $\{n_{ij}\}$ *be positive integers depending on* n_t *such that* $\sum\sum n_{ij} = n_t$, *and let* $\{b_{ij}\}$ *be positive numbers such that* $\sum\sum b_{ij} = 1$. (*Then* $\sum\sum b_{ij}\bar{\rho}_{ij} > 0$.) *Let* $\varepsilon > 0$.

(i) *If* $n_t = \lfloor t(1+\varepsilon)/\sum\sum b_{ij}\bar{\rho}_{ij}\rfloor$, *then* $\exists\ \delta > 0$ *such that*

$$\lim_{t\to\infty} t\cdot\rho_{11}^{*n_{11}} * \rho_{12}^{*n_{12}} * \cdots * \rho_{dd}^{*n_{dd}}(t) = 0$$

whenever $|n_{ij}/n_t - b_{ij}| < \delta\ \forall\ i, j$. (*Here* $\lfloor x\rfloor$ *denotes "the greatest integer* $\leq x$.")

(ii) *If* $n_t = \lfloor t(1-\varepsilon)/\sum\sum b_{ij}\bar{\rho}_{ij}\rfloor$, *then* $\exists\ \delta > 0$ *such that*

$$\lim_{t\to\infty} t\{\rho_{11}^{*n_{11}} * \cdots * \rho_{dd}^{*n_{dd}}(\infty) - \rho_{11}^{*n_{11}} * \cdots \rho_{dd}^{*n_{dd}}(t)\} = 0$$

whenever $|n_{ij}/n_t - b_{ij}| < \delta\ \forall\ i, j$.

PROOF. Notice that each $\rho_{ij}(\cdot)$ is a probability distribution or is identically zero. Thus this lemma follows directly from Goldstein's Lemma (2.15) (which is based on Theorem 2 of Franck and Hanson (1966)). □

LEMMA 7. *Let* $\varepsilon > 0$. *Recall that* $\beta = \mathbf{v}'\boldsymbol{\mu}\mathbf{u} > 0$.

(i) *If* $n_t = \lfloor t(1+\varepsilon)/\beta\rfloor$, *then* $\lim t\mathbf{m}^{*n_t}(t) = \mathbf{0}$.

(ii) *If* $n_t = \lfloor t(1-\varepsilon)/\beta\rfloor$, *then* $\lim t(u_i v_j - m_{ij}^{(*n_t)}(t)) = 0$.

PROOF. Fix i, j. Using the notation of Lemmas 3 and 6 we write

$$m_{ij}^{(*n)}(t) = u_i p_{ij}^{(*n)}(t) u_j^{-1}$$
$$= u_i \sum_{\ell\in A(n-1)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}\rho_{i\ell_1} * \rho_{\ell_1\ell_2} * \cdots * \rho_{\ell_{n-1}j}(t) u_j^{-1},$$

where $A(k)$ is the set of all k-tuples from $\{1, \cdots, d\}$. Let $n_{i'j'} = n_{i'j'}(i, \ell, j)$ be the number of times in the sequence $(i, l_1 \cdots, l_{k-1}, j)$ that the transition (i', j') occurs. Let

$$A(k, \delta) = \{\ell\in A(k-1): |k^{-1}n_{i'j'}(i, \ell, j) - u_i v_i p_{ij}| \leq \delta\ \forall\ i', j'\}.$$

Let $\tilde{A}(k, \delta) = A(k) - A(k, \delta)$. Now

$$\sum_{\ell\in A(n-1,\delta)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}\rho_{11}^{*n_{11}} * \rho_{12}^{*n_{12}} * \cdots * \rho_{dd}^{*n_{dd}}(t) = o(t^{-1})$$

for $n = n_t = \lfloor t(1+\varepsilon)/\beta\rfloor$, by Lemma 6(i). (Notice that $\beta = \sum\sum v_i\mu_{ij}u_j = \sum\sum v_i u_i \bar{\rho}_{ij}$.) Also,

$$0 \leq \sum_{\ell\in\tilde{A}(n-1,\delta)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}\rho_{11}^{*n_{11}} * \rho_{12}^{*n_{12}} * \cdots * \rho_{dd}^{*n_{dd}}(t)$$
$$\leq \sum_{\ell\in\tilde{A}(n-1,\delta)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j} = P^i(|n^{-1}S_n(i', j') - u_i v_i p_{ij}| > \delta \text{ for some } i', j') \leq c\lambda^n$$

for some $c > 0$, $\lambda \in (0, 1)$, by Lemma 5. When $n = n_t$, this bound is easily $o(t^{-1})$ as $t \to \infty$. Combining the above estimates we get that $m_{ij}^{(*n)}(t) = o(t^{-1})$.

For (ii) we write $u_i v_j - \mathbf{m}_{ij}^{(*n)}(t)$ as

$$u_i u_j^{-1}\{u_j v_j - \sum_{\ell\in A(n-1)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}$$
$$+ \sum_{\ell\in A(n-1)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}(1 - \rho_{11}^{*n_{11}} * \cdots * \rho_{dd}^{*n_{dd}}(t))\}.$$

The first sum in the braces is $p_{ij}^{(n)}$, the i, j component of $\mathbf{P}^n$. By Corollary 4.1.5 of Kemeny and Snell, $\exists\ c_1 > 0$, $\lambda_1 \in (0, 1)$ such that $|p_{ij}^{(n)} - u_j v_j| \leq c_1\lambda_1^n$, which is easily $o(t^{-1})$ when $n_t = \lfloor t(1-\varepsilon)/\beta\rfloor$. The other sum is bounded by

$$\sum_{\ell\in A(n-1,\delta)} p_{i\ell_1}\cdots p_{\ell_{n-1}j}(1 - \rho_{11}^{*n_{11}} * \cdots * \rho_{dd}^{*n_{dd}}(t)) + \sum_{\ell\in\tilde{A}(n-1,\delta)} p_{i\ell_1}p_{\ell_1\ell_2}\cdots p_{\ell_{n-1}j}.$$

The first of these sums is $o(t^{-1})$ when $n = \lfloor t(1-\varepsilon)/\beta\rfloor$, by Lemma 6, and the second is $\leq c\lambda^n$, by Lemma 5, and hence is also $o(t^{-1})$ when $n = \lfloor t(1-\varepsilon)/\beta\rfloor$. □

As preparation for the last major lemma we need the next two elementary lemmas.

LEMMA 8. *Let* $\mathbf{A}, \mathbf{B}$ *be matrices of finite measures, and assume* $\mathbf{A}(\infty) - \mathbf{A}(t) = o(t^{-2})$ *and* $\mathbf{B}(\infty) - \mathbf{B}(t) = o(t^{-2})$ *as* $t \to \infty$. *Then* $\mathbf{A}*(\mathbf{B}(\infty) - \mathbf{B}(t)) = o(t^{-2})$ *and* $\mathbf{A}(\infty)\mathbf{B}(\infty) - \mathbf{A}*\mathbf{B}(t) = o(t^{-2})$ *as* $t \to \infty$.

The proof is straightforward.

COROLLARY. $\mathbf{m}(\infty)^n - \mathbf{m}^{*n}(t) = o(t^{-2})$ *and* $\mathbf{m}^{*n}*(\mathbf{1} - \mathbf{G}(t)) = o(t^{-2})$ *as* $t \to \infty$ *for* $n = 1, 2, 3 \cdots$.

The next lemma gives a useful consequence of the assumptions that $m_{ij}(0) < m_{ij}(\infty)$ for at least one pair ij and that $\mathbf{m}(\infty)^k > 0$ for some integer k.

LEMMA 9. $\exists$ *positive integer* ν *for which* $\mathbf{m}(0)^\nu < \mathbf{m}(\infty)^\nu$ *and* $\mathbf{m}(\infty)^\nu > 0$.

PROOF. It is easy to check that if $m_{ij}(0) < m_{ij}(\infty)$ for some pair ij, then $m_{ij}^{(k)}(0) < m_{ij}^{(k)}(\infty)$, where $\mathbf{m}(\infty)^k > 0$, for some pair ij. Then by Proposition 1 of Ryan, $(\mathbf{m}(0)^k)^r < (\mathbf{m}(\infty)^k)^r$ for $r = 3, 4, 5, \cdots$. So, we may take $\nu = 3k$. □

LEMMA 10. *If* $n_t = \lfloor t(1-\varepsilon)/\beta \rfloor$, *then*

$$\textstyle\sum_{k<n_t} \mathbf{m}^{*k}*(\mathbf{1} - \mathbf{G}(t)) = o(t^{-1}) \quad as \quad t \to \infty.$$

PROOF. The argument here is a generalization of that given in [5]. Choose ν according to Lemma 9 so that $\mathbf{m}(\infty)^\nu > 0$ and $\mathbf{m}(0)^\nu < \mathbf{m}(\infty)^\nu$. By right continuity $\exists\, h > 0$ such that $\mathbf{m}^{*\nu}(h) < \mathbf{m}(\infty)^\nu$. Given $n = n_t$ and ν, let q and r be integers such that $n = \nu q + r$, $0 \le r < \nu$. Then

$$\textstyle\sum_{j<n} \mathbf{m}^{*j}*(\mathbf{1} - \mathbf{G}(t)) \le \sum_{k=0}^{q} \mathbf{m}^{*\nu k}* \sum_{r=0}^{\nu-1} \mathbf{m}^{*r}*(\mathbf{1} - \mathbf{G}(t)).$$

By the Corollary to Lemma 8,

$$\boldsymbol{\varphi}(t) \equiv \textstyle\sum_{r=0}^{\nu-1} \mathbf{m}^{*r}*(\mathbf{1} - \mathbf{G}(t)) = o(t^{-2}) \quad \text{as } t \to \infty.$$

It remains to check that $\sum_{k=0}^{q} \mathbf{m}^{*\nu k}*\boldsymbol{\varphi}(t) = o(t^{-1})$.

Let $\varepsilon > 0$. Let $c = (1-\varepsilon)/(1-\varepsilon/2)$, so that $0 < c < 1$ and $n_t = \lfloor t(1-\varepsilon)/\beta \rfloor = \lfloor ct(1-\varepsilon/2)/\beta \rfloor$. Write

$$\textstyle\sum_{k=0}^{q} \mathbf{m}^{*\nu k}*\boldsymbol{\varphi}(t) = \sum \displaystyle\int_0^{ct} \mathbf{m}^{*\nu k}(dy)\boldsymbol{\varphi}(t-y) + \sum \int_{ct}^{t} \mathbf{m}^{*\nu k}(dy)\boldsymbol{\varphi}(t-y) = \mathbf{S}_1 + \mathbf{S}_2.$$

For the ith component of $\mathbf{S}_1$, we have

$$\begin{aligned}
&\textstyle\sum_{k=0}^{q} \sum_{j=1}^{d} \displaystyle\int_0^{ct} \varphi_j(t-y) m_{ij}^{(*\nu k)}(dy) \\
&\le \nu^{-1} n_t \textstyle\sum_j \sup\{\varphi_j(x) : x \ge (1-c)t\} m_{ij}^{(*\nu k)}(\infty) \\
&= t^{-1}(n_t/\nu t) \textstyle\sum_j t^2 \sup\{\varphi_j(x) : x \ge (1-c)t\} m_{ij}^{(\nu k)}(\infty) \\
&= t^{-1} O(1) o(1) = o(t^{-1}) \quad \text{as } t \to \infty.
\end{aligned}$$

Turning our attention to $\mathbf{S}_2$, we let

$$\Phi_l = \sup\{\varphi_j(x) : lh \le x \le (l+1)h,\ 1 \le j \le d\} \quad \text{and} \quad L = \lfloor (t-ct)/h \rfloor.$$

Then for the ith component of $\mathbf{S}_2$ we have

$$\begin{aligned}
\textstyle\sum_{k=0}^{q} \sum_{j=1}^{d} \displaystyle\int_{ct}^{t} \varphi_j(t-y) m_{ij}^{(*\nu k)}(dy) &\le \textstyle\sum_k \sum_j \sum_{l=0}^{L} \displaystyle\int_{t-(l+1)h}^{t-lh} \varphi_j(t-y) m_{ij}^{(*\nu k)}(dy) \\
&\le \textstyle\sum_l \Phi_l \sum_k \sum_j \{m_{ij}^{(*\nu k)}(t-lh) - m_{ij}^{(*\nu k)}(t-(l+1)h)\}.
\end{aligned}$$

An upper bound for the inner sums is gotten as follows (recall that $\mathbf{u} = \mathbf{m}(\infty)\mathbf{u} = \mathbf{m}(\infty)^{\nu}\mathbf{u}$).

$$\begin{aligned}
(\mathbf{I} - \mathbf{m}^{*\nu(q+1)}(t))\mathbf{u} &= \textstyle\sum_{k=0}^{q} \mathbf{m}^{*\nu k} * (\mathbf{I} - \mathbf{m}^{*\nu}(t))\mathbf{u} \\
&= \textstyle\sum_{k=0}^{q} \mathbf{m}^{*\nu k} * (\mathbf{m}(\infty)^{\nu} - \mathbf{m}^{*\nu}(t))\mathbf{u} \\
&= \textstyle\sum_{k=0}^{q} \langle \sum_{l=1}^{d} \sum_{j=1}^{d} \displaystyle\int_0^t \{m_{lj}^{(\nu)}(\infty) - m_{lj}^{(*\nu)}(t-y)\} m_{il}^{(*\nu k)}(dy) u_j \rangle \\
&\geq \textstyle\sum_k \langle \sum_l \sum_j \displaystyle\int_{t-h}^t \{m_{lj}^{(\nu)}(\infty) - m_{lj}^{(*\nu)}(t-y)\} m_{il}^{(*\nu k)}(dy) u_j \rangle \\
&\geq \textstyle\sum_k \langle \sum_l \sum_j \displaystyle\int_{t-h}^t \{m_{lj}^{(\nu)}(\infty) - m_{lj}^{(*\nu)}(h)\} m_{il}^{(*\nu k)}(dy) u_j \rangle \\
&= \textstyle\sum_k \langle \sum_l \sum_j \{m_{il}^{(*\nu k)}(t) - m_{il}^{(*\nu k)}(t-h)\}\{m_{lj}^{(\nu)}(\infty) - m_{lj}^{(*\nu)}(h)\} u_j \rangle \\
&\geq \textstyle\sum_{k=0}^{q} \{\mathbf{m}^{*\nu k}(t) - \mathbf{m}^{*\nu k}(t-h)\}\eta \mathbf{1}
\end{aligned}$$

where η denotes the least (necessarily positive) component of $\mathbf{m}(\infty)^{\nu} - \mathbf{m}^{*\nu}(h)$. Therefore

$$\begin{aligned}
\textstyle\sum_{k=0}^{q} \sum_{j=1}^{d} \langle m_{ij}^{(*\nu k)}(t) - m_{ij}^{(*\nu k)}(t-h) \rangle &\leq \eta^{-1}(\mathbf{I} - \mathbf{m}^{*\nu(q+1)}(t))\mathbf{u} \\
&= \eta^{-1}(\mathbf{I} - \mathbf{u}\mathbf{v}' + \mathbf{u}\mathbf{v}' - \mathbf{m}^{*\nu(q+1)}(t))\mathbf{u} \\
&= \eta^{-1}(\mathbf{u}\mathbf{v}' - \mathbf{m}^{*\nu(q+1)}(t))\mathbf{u},
\end{aligned}$$

since $(\mathbf{I} - \mathbf{u}\mathbf{v}')\mathbf{u} = \mathbf{u} - \mathbf{u}(\mathbf{v}'\mathbf{u}) = \mathbf{u} - \mathbf{u}(1) = 0$. Returning to $\mathbf{S}_2$ we now have

$$\begin{aligned}
\mathbf{S}_2 &\leq \textstyle\sum_{l=0}^{L} \Phi_l \eta^{-1}(\mathbf{u}\mathbf{v}' - \mathbf{m}^{*\nu(q+1)}(t - Lh))\mathbf{u} \\
&\leq \textstyle\sum_{l=0}^{\infty} \Phi_l \eta^{-1}(\mathbf{u}\mathbf{v}' - \mathbf{m}^{*\nu(q+1)}(ct))\mathbf{u}.
\end{aligned}$$

Here $\sum \Phi_l < \infty$ because $\{\Phi_l\}$ is bounded and $\Phi_l = o(l^{-2})$ as $l \to \infty$. Also, since $\mathbf{m}^{*n}(t)\mathbf{u}$ is decreasing in n, $(\mathbf{u}\mathbf{v}' - \mathbf{m}^{*\nu(q+1)}(ct))\mathbf{u} \leq (\mathbf{u}\mathbf{v}' - \mathbf{m}^{*r_t}(ct))\mathbf{u} = o(t^{-1})$, by Lemma 7(ii), where $\mathbf{r}_t = \lfloor ct(1 - \varepsilon/4)/\beta \rfloor$, and the lemma now follows. □

PROOF OF THE PROPOSITION. By Lemma 2,

$$1 - \mathbf{f}_n(\mathbf{0}) - \mathbf{m}^{*n} * \mathbf{G}(t) \leq 1 - \mathbf{F}(\mathbf{0}, t) \leq 1 - \mathbf{f}_n(\mathbf{0}) + \textstyle\sum_{k<n} \mathbf{m}^{*k} * (1 - \mathbf{G}(t)).$$

Let $\varepsilon > 0$. Put $n = \lfloor t(1+\varepsilon)/\beta \rfloor$ in the left inequality and multiply through by t. By Lemmas 1 and 7(i),

$$\frac{2\beta\mathbf{u}}{(1+\varepsilon)\mathbf{v}\cdot\mathbf{q}[\mathbf{u}]} - 0 \leq \liminf_{t\to\infty} t(1 - \mathbf{F}(\mathbf{0}, t)).$$

Put $n = \lfloor t(1-\varepsilon)/\beta \rfloor$ in the right inequality and multiply through by t. By Lemmas 1 and 10,

$$\limsup_{t\to\infty} t(1 - \mathbf{F}(\mathbf{0}, t)) \leq \frac{2\beta\mathbf{u}}{(1-\varepsilon)\mathbf{v}\cdot\mathbf{q}[\mathbf{u}]} + 0.$$

Since ε is arbitrary,

$$\lim_{t\to\infty} t(\mathbf{1} - \mathbf{F}(\mathbf{0}, t)) = 2\beta(\mathbf{v}\cdot\mathbf{q}[\mathbf{u}])^{-1}\mathbf{u}. \qquad \square$$

7. **Proof of the exponential limit law.** The proof of the theorem will follow quickly from the proposition on the asymptotic nonextinction probability and the following lemma, which was abstracted and generalized from the proof given by Green (1977). It is remarkable how well Green's proof extends in a natural way to the multitype case, and so

the proof in our case needs merely to be sketched; it is in these omitted details, which may be filled in by reference to Green's paper, that the hypotheses on χ are used.

To begin, we introduce 1—the Laplace-Stieltjes transform of $\mathbf{X}(t)$:

$$\mathbf{H}(\boldsymbol{\theta}, t) \equiv \mathbf{E}(1 - e^{-\boldsymbol{\theta}\cdot\mathbf{X}(t)}),$$

where $\mathbf{E}(\cdot) = (E^1(\cdot), \cdots, E^d(\cdot))'$. Then using equation (4.1), conditioning on $(\lambda^i, \xi^i, \chi^i)$, and taking expectations, we find that

$$\mathbf{H}(\boldsymbol{\theta}, t) = \mathbf{1} - \mathbf{E}(e^{-\boldsymbol{\theta}\cdot\chi(t)} \textstyle\prod_{k=1}^{d} \prod_{n=1}^{\xi_k(t)} \{1 - H^k(\boldsymbol{\theta}, t - \tau_k(n))\}). \tag{7.1}$$

We also introduce the norm $\|\boldsymbol{\theta}\| = |\sum_j \theta_j v_j \mu_j^j \bar{\chi}^j/\beta|$, where $\bar{\chi}^j = \int_0^\infty E^j(\chi^j(y))\, dy/\mu_j^j$.

LEMMA.

$$\lim_{t\to\infty} \frac{t}{\|\boldsymbol{\theta}\|} \mathbf{H}\left(\frac{\boldsymbol{\theta}}{t}, t\right) = (1 + \mathbf{c}\cdot\boldsymbol{\theta})^{-1}\mathbf{u}$$

for $\|\boldsymbol{\theta}\|$ *in some interval* $(0, \alpha)$, *where* $\mathbf{c} = (c_1, \cdots, c_d)'$, $c_j = ½\, v_j \mu_j^j \bar{\chi}^j(\mathbf{v}\cdot\mathbf{q}[\mathbf{u}])/\beta^2$.

PROOF. The basic equation (7.1) may be written as

$$\mathbf{H}(\boldsymbol{\theta}, t) = \mathbf{E}(\boldsymbol{\theta}\cdot\chi(t)) - \textstyle\sum_{r=1}^{3} \mathbf{E}(Q_r(\boldsymbol{\theta}, t)) - ½\mathcal{Q}(\boldsymbol{\theta}, t) + \mathbf{m}*\mathbf{H}(\boldsymbol{\theta}, t), \tag{7.2}$$

where

$$Q_1(\boldsymbol{\theta}, t) = e^{-\boldsymbol{\theta}\cdot\chi(t)} - 1 + \boldsymbol{\theta}\cdot\chi(t),$$
$$Q_2(\boldsymbol{\theta}, t) = (1 - e^{-\boldsymbol{\theta}\cdot\chi(t)})(1 - \textstyle\prod_{k=1}^{d} \prod_{l=1}^{\xi_k(t)} (1 - H^k(\boldsymbol{\theta}, t - \tau_k(l)))),$$
$$Q_3(\boldsymbol{\theta}, t) = \textstyle\sum_{k=1}^{d} \sum_{l=1}^{\xi_k(t)} (1 - H^k(\boldsymbol{\theta}, t - \tau_k(l))) - 1 + \sum_k \sum_l H^k(\boldsymbol{\theta}, t - \tau_k(l))$$
$$-½ \textstyle\sum_{k_1} \sum_{k_2} \sum\sum_{l_1\neq l_2} H^{k_1}(\boldsymbol{\theta}, t - \tau_{k_1}(l_1))H^{k_2}(\boldsymbol{\theta}, t - \tau_{k_2}(l_2)),$$

and the ith component of $\mathcal{Q}(\boldsymbol{\theta}, t)$ is

$$\textstyle\sum_{k_1} \sum_{k_2} \displaystyle\iint_{[0,t]^2} H^{k_1}(\boldsymbol{\theta}, t - y_1)H^{k_2}(\boldsymbol{\theta}, t - y_2)q^i_{k_1k_2}(dy_1, dy_2).$$

Equation (7.2), the counterpart of equation (4.2) of Green, is of the form $\mathbf{H}(\boldsymbol{\theta}, t) = \mathbf{f}(\boldsymbol{\theta}, t) + \mathbf{m}*\mathbf{H}(\boldsymbol{\theta}, t)$, which for fixed $\boldsymbol{\theta}$ is a multidimenstional renewal equation. Let $\mathbf{U}(t) = \sum_{n\geq 0} \mathbf{m}^{*n}(t)$, the matrix renewal function. Then

$$\begin{aligned}\mathbf{H}(\boldsymbol{\theta}, t) &= \mathbf{U}*\mathbf{f}(\boldsymbol{\theta}, t)\\ &= \mathbf{U}*\mathbf{E}(\boldsymbol{\theta}\cdot\chi(t)) - \textstyle\sum_{r=1}^{3} \mathbf{U}*\mathbf{E}(Q_r(\boldsymbol{\theta}, t)) - ½\mathbf{U}*\mathcal{Q}(\boldsymbol{\theta}, t).\end{aligned} \tag{7.3}$$

Let $\mathbf{K}(\boldsymbol{\theta}, t) = \|\boldsymbol{\theta}\|^{-1}t\mathbf{H}(\boldsymbol{\theta}/t, t)$ and, regarding $q^j_{k_1k_2}$ as a measure, let

$${}^t\mathcal{M}^i_{k_1k_2}(w_1, w_2) = \frac{1}{2}t^{-1} \textstyle\sum_j \displaystyle\int_0^t q^j_{k_1k_2}([t(1 - w_1) - x, t - x]\times[t(1 - w_2) - x, t- x])U^i_j(dx).$$

Since $U^i_j(t)/t \to \beta^{-1}u^i v_j$ by the Corollary to the Renewal Theorem, and since

$$q^j_{k_1k_2}(t_1, t_2) \to q^j_{k_1k_2} = E^j(\xi_{k_1}(\infty)\xi_{k_2}(\infty) - \delta_{k_1k_2}\xi_{k_1}(\infty)) \quad \text{as} \quad \min(t_1, t_2) \to \infty,$$

one may check that

$${}^t\mathcal{M}^i_{k_1k_2}(w_1, w_2) \to ½\beta^{-1}u^i \textstyle\sum_j v_j q^j_{k_1k_2} \min\{w_1, w_2\} \quad \text{as} \quad t \to \infty.$$

Equation (7.3) may now be rewritten as the counterpart of Green's (4.5):

$$K^i(\boldsymbol{\theta}, t) = \|\boldsymbol{\theta}\|^{-1}\mathbf{t} \textstyle\sum_j \displaystyle\int_0^t \mathbf{f}_j(\boldsymbol{\theta}/t, t-y) U_j^i (dy)$$

$$(7.4) \qquad = u^i - \textstyle\sum_{r=1}^4 R_r^i(\boldsymbol{\theta}, t) - \|\boldsymbol{\theta}\| \sum\sum_{l_1 l_2} \displaystyle\iint_{[0,1]^2} K_{l_1}(w_1\boldsymbol{\theta}, w_1 t)$$

$$\cdot K_{l_2}(w_2\boldsymbol{\theta}, w_2 t)\, {}^t\mathcal{M}^i_{l_1 l_2}(dw_1, dw_2),$$

where, for $r = 1, 2, 3$,

$$R_r(\boldsymbol{\theta}, t) = \|\boldsymbol{\theta}\|^{-1} t \int_0^t \mathbf{U}(dy)\mathbf{E}(Q_r(\boldsymbol{\theta}/t, t-y)),$$

and

$$R_4(\boldsymbol{\theta}, t) = \mathbf{u} - \|\boldsymbol{\theta}\|^{-1}\mathbf{U} * \mathbf{E}(\boldsymbol{\theta}\cdot\chi(t)).$$

Notice that, when the $\mathbf{R}_r$ terms vanish and ${}^t\mathcal{M}^i_{k_1 k_2}$ is replaced by its limit in equation (7.4), then the solution that is independent of t is

$$\mathbf{K}(\boldsymbol{\theta}) = (1 + \mathbf{c}\cdot\boldsymbol{\theta})^{-1}\mathbf{u} = \mathbf{u} - \mathbf{c}\cdot\boldsymbol{\theta} \int_0^1 (1 + \mathbf{c}\cdot\boldsymbol{\theta} w)^{-2} dw\, \mathbf{u}.$$

Subtraction from (7.4) yields the counterpart of Green's page 460 equation:

$$(7.5) \qquad K^i(\boldsymbol{\theta}, t) - (1 + \mathbf{c}\cdot\boldsymbol{\theta})^{-1}\mathbf{u} = -\textstyle\sum_{r=1}^5 R_r^i(\boldsymbol{\theta}, t)$$

$$- \frac{1}{2}\|\boldsymbol{\theta}\| \textstyle\sum\sum_{l_1 l_2} \displaystyle\iint_{[0,1]^2} \{K_{l_1}(w_1\boldsymbol{\theta}, w_1 t) K_{l_2}(w_2\boldsymbol{\theta}, w_2 t)$$

$$- K_{l_1}(w_1\boldsymbol{\theta}) K_{l_2}(w_2\boldsymbol{\theta})\}\, {}^t\mathcal{M}^i_{l_1 l_2}(dw_1, dw_2),$$

where

$$R_5^i(\boldsymbol{\theta}, t) = \frac{1}{2} u^i \|\boldsymbol{\theta}\| \textstyle\sum\sum_{l_1 l_2} \displaystyle\iint_{[0,1]^2} K_{l_1}(w_1\boldsymbol{\theta}) K_{l_2}(w_2\boldsymbol{\theta})\, {}^t\mathcal{M}^i_{l_1 l_2}(dw_1, dw_2)$$

$$- \mathbf{c}\cdot\boldsymbol{\theta} \int_0^1 (1 + \mathbf{c}\cdot\boldsymbol{\theta} w)^{-2}\, dw\, u^i.$$

Continuing in this way, the remainder of the proof may be developed step for step along the lines given by Green. Thus, we may find constants c_0, a_0, α such that $|K_l(w\boldsymbol{\theta}, wt) - K_l(w\boldsymbol{\theta})| < c_0$, $\frac{1}{2}\beta^{-1}u^i \sum_j v_j q^j_{l_1 l_2} < a_0$, and $\alpha c_0 a_0 d^2 < 1$. Letting

$$B^i(t) = \sup\{|K_i(\boldsymbol{\theta}, t') - (1 + \mathbf{c}\cdot\boldsymbol{\theta})^{-1}u_i| : 0 < \|\boldsymbol{\theta}\| < \alpha, t' > t\}$$

and

$$R^i(t) = \sup\{\textstyle\sum_{r=1}^5 R_r^i(\boldsymbol{\theta}, t') : 0 < \|\boldsymbol{\theta}\| < \alpha, t' > t\},$$

we find, for $i = 1, \cdots, d$, that

$$B^i(t) \le R^i(t) + \alpha c_0^2 \textstyle\sum_{l_1}\sum_{l_2} \sup\{{}^{t'}\mathcal{M}^i_{l_1 l_2}(\delta, 1) : t' > t\}$$

$$+ \alpha c_0 a_0 \textstyle\sum_{l_1}\sum_{l_2} a_0^{-1} \sup\{{}^{t'}\mathcal{M}^i_{l_1 l_2}(1, 1) : t' > t\} B_{l_1}(\delta t) \quad \text{for} \quad \delta \in (0, 1).$$

By counterparts of the arguments of Green, $R^i(t) \to 0$ as $t \to \infty$ ($i = 1, \cdots, d$), and, as in Green, it follows that $B^i(t) \to 0$ as $t \to \infty$ ($i = 1, \cdots, d$), and the lemma follows. □

PROOF OF THE THEOREM. For $i = 1, \cdots, d$,

$$E^i\{e^{-\boldsymbol{\theta}\cdot\mathbf{X}(t)/t} \mid \mathbf{Z}(t) \neq \mathbf{0}\} = 1 - H^i(\boldsymbol{\theta}/t, t)/P^i(\mathbf{Z}(t) \neq \mathbf{0})$$
$$= 1 - \|\boldsymbol{\theta}\|^{-1}tH^i(\boldsymbol{\theta}/t, t)/(\|\boldsymbol{\theta}\|^{-1}tP^i(\mathbf{Z}(t) \neq \mathbf{0})).$$

By the lemma and the proposition, the limit as $t \to \infty$ is

$$1 - (1 + \mathbf{c}\cdot\boldsymbol{\theta})^{-1}u^i/(\|\boldsymbol{\theta}\|^{-1}2\beta(\mathbf{v}\cdot\mathbf{q}[\mathbf{u}])^{-1}u^i) = 1 - \mathbf{c}\cdot\boldsymbol{\theta}/(1 + \mathbf{c}\cdot\boldsymbol{\theta}) = 1/(1 + \mathbf{c}\cdot\boldsymbol{\theta})$$

for every $\boldsymbol{\theta}$ with $0 < \|\boldsymbol{\theta}\| < \alpha$. By the continuity theorem for Laplace-Stieltjes transforms, this result is equivalent to the conclusion of the theorem. □

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GUSTAVUS ADOLPHUS COLLEGE
ST. PETER, MINNESOTA 56082