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STOCHASTIC EQUATIONS OF HYPERBOLIC TYPE AND A TWO-PARAMETER STRATONOVICH CALCULUS[1]

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Existence, uniqueness, and a Markov property are proved for the solutions of a hyperbolic equation with a white Gaussian noise driving term. A two-parameter analog of the Stratonovich stochastic integral is introduced and is used to formulate integral versions of the hyperbolic equation. The stochastic calculus associated with the Stratonovich integral formally agrees with ordinary calculus. A class of two-parameter semimartingales is found which is closed under all the operations of a complete stochastic calculus. The class of processes which are solutions to the type of hyperbolic equation studied is closed under smooth state space transformations.

1. Introduction. The purpose of this paper is to formulate a class of quasi-linear partial differential equations driven by two-parameter Gaussian white noise and to demonstrate the existence and uniqueness of solutions. Our primary tool is a two-parameter analog of the Stratonovich stochastic calculus, which is also introduced in this paper.

The partial differential equation of interest is

$$\frac{\partial^2 Y}{\partial t_1 \partial t_2} - a(Y)\frac{\partial Y}{\partial t_1}\frac{\partial Y}{\partial t_2} - b(Y) - c(Y)\eta = 0 \qquad t_1, t_2 \geq 0$$

(1.1)

$$Y(t_1, t_2) = 0 \quad \text{if} \quad t_1 t_2 = 0.$$

This is a special case of the following quasi-linear initial value problem in two independent variables

$$\frac{\partial^2 Y}{\partial t_1 \partial t_2} = f\left(t_1, t_2, Y, \frac{\partial Y}{\partial t_1}, \frac{\partial Y}{\partial t_2}\right)$$

(1.2)

$$\left(Y, \frac{\partial Y}{\partial t_1}, \frac{\partial Y}{\partial t_2}\right)\bigg|_{\gamma} = (u, p, q),$$

where u, p and q are specified along a given curve γ and satisfy certain consistency conditions. In the deterministic setting, the equation (1.2) is well understood [4]. The initial curve for the equation (1.1) consists of the positive axes which are characteristic curves of the hyperbolic operator $\partial^2/(\partial t_1 \partial t_2)$. Thus (1.1) is a characteristic initial value problem with zero boundary conditions. It is well-posed although the derivatives $\partial Y/\partial t_1$ and $\partial Y/\partial t_2$ are not specified along the initial curve.

In this paper we study the case when the driving term η in (1.1) is a two-parameter Gaussian white noise, which is formally a Gaussian process with mean zero and autocovariance $E[\eta(s_1, s_2)\eta(t_1, t_2)] = \delta(s_1 - t_1)\delta(s_2 - t_2)$. The first and second order derivatives

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of the solution Y must then be somewhat singular, so an important problem is to give meaning to the equation. This will be accomplished by interpreting (1.1) as a stochastic integral equation.

Two-parameter white noise can be used to model small independent local disturbances in a spatially distributed system. For example, white noise can be used to represent dispersed energy sources in models of turbulence, or external disturbances to a transmission line. A stochastic calculus is needed to analyze such problems, especially when the systems are nonlinear, and it is hoped that the two-parameter calculus introduced in this paper will prove to be a useful tool. An important feature of the calculus which is exploited in the study of (1.1) is that, unlike the Wong-Zakai-Ito calculus, it formally obeys the rules of ordinary calculus.

When η is a two-parameter white noise, equation (1.1) represents a nonlinear wave equation in two space-time dimensions with distributed random wave sources—it could well model, for example, the evolution of ocean waves or ripples in a (one-dimensional) pool of water exposed to rain. However, our main reason for studying hyperbolic equations first is that their natural causality structure allows us to apply two-parameter martingale methods which, so far, require a partially-ordered parameter set. In addition, the causality structure of (1.1) gives rise to a generalized Markov property of the solutions.

In Section II the stochastic calculus of Wong and Zakai [9] through [14] for two-parameter processes is reviewed. In addition, a class of processes is found which is closed under all operations of the calculus. In Section III the analog of the Stratonovich integral for two-parameter processes is defined and investigated. Finally, Section IV contains the main results concerning the stochastic equation (1.1) (see Theorem 4.2).

2. The stochastic calculus. The basic definitions of [2] will be used, and are summarized as follows. Let $R_+ = [0, \infty) \times [0, \infty)$ denote the positive quadrant of the plane. For two points $s = (s_1, s_2)$ and $s' = (s'_1, s'_2)$ in R_+, $s > s'$ will denote the condition $s_1 \geq s'_1$ and $s_2 \geq s'_2$, $s \gg s'$ will denote the condition $s_1 > s'_1$ and $s_2 > s'_2$, $s \wedge s'$ will denote the condition $s'_1 \geq s_1$ and $s_2 \geq s'_2$, $s' \times s$ will denote the point (s'_1, s_2), and $s \vee s'$ will denote the point $(\max(s_1, s'_1), \max(s_2, s'_2))$. $I(s \wedge s')$ will denote the indicator function of the set $\{s \wedge s'\}$. 0 will denote the origin in R_+. For $z, z' \in R_+$, $(z, z']$ is the set $\{s : z \ll s < z'\}$ and if f is a function on R_+ then $f(z, z'] = X_{z'} - X_{z \times z'} - X_{z' \times z} + X_z$. $R_z = \{s : 0 < s < z\}$ and $R_z \otimes R_z$ is the set $\{(s, s') : s \in R_z, s' \in R_z, s \wedge s'\}$.

Fix a point $z_0 \in R_+$. Throughout this paper $(\Omega, \mathscr{F}, P)$ will be a complete probability space with a family of sub-σ-fields $\mathscr{F}_. = \{\mathscr{F}_z : z \in R_{z_0}\}$ such that

(F1) if $z < z'$ then $\mathscr{F}_z \subset \mathscr{F}_{z'}$,
(F2) $\mathscr{F}_0$ contains all null sets of $\mathscr{F}$,
(F3) for each z, $\mathscr{F}_z = \bigcap_{z \ll z'} \mathscr{F}_{z'}$,
(F4) for each z, $\mathscr{F}_{z \times z_0}$ and $\mathscr{F}_{z_0 \times z}$ are conditionally independent given $\mathscr{F}_z$.

Let $\mathscr{G}_z = \mathscr{F}_{z \times z_0} \vee \mathscr{F}_{z_0 \times z}$. A stochastic process $\{X_z : z \in R_{z_0}\}$ is adapted if X_z is $\mathscr{F}_z$ measurable for each z.

A two-parameter Wiener process (or "Brownian sheet") is a sample continuous Gaussian process $W = \{W_z : z \in R_{z_0}\}$ with mean zero and covariance $E[W_z W_{z'}] = \mu(R_z \cap R_{z'})$, where μ denotes Lebesgue measure on R_+. Formally, W_z is the integral of white Gaussian noise over the rectangle R_z. An $\mathscr{F}_.$-Wiener process is an adapted Wiener process W such that "the future increments of W" are independent of $\mathscr{G}_z$, i.e., for each $z \in R_{z_0}$, $\{W(s, s'] : s' > s > z\}$ is independent of $\mathscr{G}_z$. For example, a Wiener process W will be an $\mathscr{F}_.$-Wiener process if $\mathscr{F}_z = \sigma(W_s : s < z)$ for each z. Throughout this paper it is assumed that there is an $\mathscr{F}_.$-Wiener process W on $(\Omega, \mathscr{F}, P)$.

Let $\mathscr{A}$ consist of all sets of the form $R_{t_1} \cup \cdots \cup R_{t_n}$ for some $n < +\infty$ and $t_1, \cdots, t_n \in R_{z_0}$. For $a \in \mathscr{A}$, δa will denote the boundary of a, and a^c the complement of a as a subset of $(0, z_0]$. If $a \in \mathscr{A}$, define $\mathscr{F}_a = \bigvee_{z \in a} \mathscr{F}_z$. For any subset $A \subset R_{z_0}$ and random process $Y = \{Y_z : z \in R_{z_0}\}$, define $\mathscr{O}_Y(A) = \sigma(Y_z : z \in A)$. By a stretch of notation, $\mathscr{O}_{dW}(A) = \sigma(W(z, z'] : (z, z'] \in A)$ so that $\mathscr{O}_{dW}(A)$ is the σ-field generated by "white noise" in A.

An adapted process $\{X_z : z \in R_{z_0}\}$ is Markov relative to $\{\mathscr{F}_z : z \in R_{z_0}\}$ if for each set $a \in \mathscr{A}$, $\mathscr{F}_a$ and $\mathscr{O}_X(a^c)$ are conditionally independent given $\mathscr{O}_X(\delta a)$. W is Markov relative to $\{\mathscr{F}_z : z \in R_{z_0}\}$ by Theorem 4.1 below.

Stochastic integrals with respect to W will be introduced next. For $1 \leq p \leq +\infty$, define $\mathscr{L}_1^p$ to be the collection of measurable functions $q(s, \omega)$ on $(R_{z_0}) \times \Omega$, $\mathscr{B}(R_{z_0} \times \mathscr{F})$ (where $\mathscr{B}$ denotes Borel subsets) such that $q(s)$ is $\mathscr{F}_s$ measurable for each s and $\int_{R_{z_0}} |q(s, \omega)|^p \, ds < +\infty$ a.s. if $p < +\infty$ and $\sup_s |q(s, \omega)| < +\infty$ a.s. if $p = +\infty$. Define $\mathscr{L}_2^p$ to be the collection of measurable functions $r(s, s', \omega)$ on $(R_{z_0} \otimes R_{z_0} \times \Omega$, $\mathscr{B}(R_{z_0} \otimes R_{z_0}) \times \mathscr{F})$ such that $r(s, s')$ is $\mathscr{F}_{s \vee s'}$ measurable for each $s, s' \in R_{z_0}$ and $\int_{R_{z_0} \otimes R_{z_0}} | r(s, s')|^p \, ds \, ds' < +\infty$ a.s. if $p < +\infty$ and $\sup_{s,s'} |r(s, s')| < +\infty$ a.s. if $p = +\infty$. Clearly $\mathscr{L}_i^p \subset \mathscr{L}_i^q$ if $p \leq q$ for $i = 1, 2$.

For $q \in \mathscr{L}_1^2$ the stochastic integral

$$q \cdot W(z) = \int_{R_z} q(s) \, dW_s$$

is defined [14] in direct analogy with the Ito integral for one parameter processes if $E[\int_{R_{z_0}} |q(s)|^2 \, ds] < +\infty$. The integral is defined in the general case by localization arguments in [10], [11]. The resulting process $q \cdot W$ is adapted, sample continuous, and $E[(q \cdot W_{z_0})^2] = E[\int_{R_{z_0}} q(s)^2 \, ds] \leq +\infty$. For $r, \alpha, \beta \in \mathscr{L}_2^2$, the multiple stochastic integrals

$$W \cdot r \cdot W(z) = \int_{R_z \otimes R_z} r(s, s') \, dW_s \, dW_{s'}$$

$$\mu \cdot \alpha \cdot W(z) = \int_{R_z \otimes R_z} \alpha(s, s') \, ds \, dW_{s'}$$

$$W \cdot \beta \cdot \mu(z) = \int_{R_z \otimes R_z} \beta(s, s') \, dW_s \, ds'$$

are also defined in [13], [10], and [11] and the resulting processes $W \cdot r \cdot W$, $\mu \cdot \alpha \cdot W$ and $W \cdot \beta \cdot \mu$ are each sample continuous, adapted processes parameterized by $z \in R_{z_0}$. For $b \in \mathscr{L}_1^p$, $b \cdot \mu$ will denote the ordinary Lebesgue integral $b \cdot \mu(z) = \int_{R_z} b_s \, ds$.

For $2 \leq p \leq +\infty$ let $\mathscr{S}^p$ be the linear space of processes of the form

$$Z = q \cdot W + W \cdot r \cdot W + \mu \cdot \alpha \cdot W + W \cdot \beta \cdot \mu + b \cdot \mu \tag{2.1}$$

where $q, b \in \mathscr{L}_1^p$ and $r, \alpha, \beta \in \mathscr{L}_2^p$. Then $\mathscr{S}^p \subset \mathscr{S}^2$ and the processes in $\mathscr{S}^2$ will be called (two-parameter) semimartingales. A semimartingale $Z \in \mathscr{S}^2$ is a one-parameter semimartingale along the lines z_2 = constant and the lines z_1 = constant with the respective semimartingale representations

$$Z_z = \int_{R_z} Z_{W1}(z, s') \, dW_{s'} + \int_{R_z} Z_{\mu 1}(z, s') \, ds' \tag{2.2}$$

$$Z_z = \int_{R_z} Z_{W2}(z, s) \, dW_s + \int_{R_z} Z_{\mu 2}(z, s) \, ds \tag{2.3}$$

where

$$Z_{W1}(z, s') = q_{s'} + \int_{R_z} I(s \wedge s') r_{s,s'} \, dW_s + \int_{R_z} I(s \wedge s') \alpha_{s,s'} \, ds \tag{2.4}$$

$$Z_{\mu 1}(z, s') = b_{s'} + \int_{R_z} I(s \wedge s') \beta_{s,s'} \, dW_s \tag{2.5}$$

$$Z_{W2}(z, s) = q_s + \int_{R_z} I(s \wedge s') r_{s,s'} \, dW_{s'} + \int_{R_z} I(s \wedge s') \beta_{s,s'} \, ds' \tag{2.6}$$

$$Z_{\mu 2}(z, s) = b_s + \int_{R_z} I(s \wedge s') \alpha_{s,s'} \, dW_{s'}. \tag{2.7}$$

Equations (2.2) and (2.3) can conveniently be rewritten as

$$Z = Z_{Wi} \cdot W + Z_{\mu i} \cdot \mu \quad \text{for} \quad i = 1, 2.$$

REMARKS. (1) Equations (2.2) and (2.3) simply reflect the fact that the elementary stochastic integrals can be computed as iterated integrals. The stochastic integrals in (2.2) and (2.3) are special cases of stochastic integrals along "increasing curves." For example, the first term on the right of (2.2) is essentially a one-parameter integral along the lines z_2 = constant, and the result is a one-parameter local martingale along such lines.

(2) The restriction of a semimartingale $Z \in \mathscr{S}^2$ to a smooth, increasing curve yields a one-parameter semimartingale which is sample continuous and has a bounded variation component which is actually absolutely continuous. While the stochastic calculus of one-parameter processes extends to the full class of (possibly discontinuous) semimartingales, such an extension is still lacking in the two-parameter case.

(3) A version of $Z_{Wi}(z, s, \omega)$ which is jointly measurable in z, s, ω can and will be selected without further comment. See [2] for such considerations.

PROPOSITION 1. *Let $Z \in \mathscr{S}^p$ for $p \geq 2$. Then $(s, s'\ \omega) \rightsquigarrow Z_{Wi}(s \vee s', s', \omega)$ and $(s, s', \omega) \rightsquigarrow Z_{\mu i}(s \vee s', s', \omega)$ are in $\mathscr{L}_2^p$ for $i = 1, 2$.*

PROOF. It sufficies to prove that $Z_{W1} \in \mathscr{L}_2^p$, for example. Let $\rho_{s,s'} = \int_{R_{s \vee s'}} I(t \wedge s') r_{t,s'} \, dW_t$. By the approximation procedure of [11] applied to $(r^{(n)})^{p/2} \in \mathscr{L}_2^2$, there is a sequence of functions $r^{(n)} \in \mathscr{L}_2^p$ and an a.s. finite random variable N such that $r^{(n)}(s, s') = r(s, s')$ for all $s, s' \in R_{z_0}$ whenever $n \geq N(\omega)$ and such that $E[\int_{R_{z_0} R \otimes_{z_0}} | r^{(n)}(s, s')|^p \, ds \, ds'] \leq n$. Let $\rho_{s,s'}^{(n)} = \int_{R_{s' \times s}} I(t \wedge s') r_{t,s'}^{(n)} \, dW_t$. Then for fixed s', $\{\rho_{s,s'}^{(n)}, \mathscr{F}_{s \times s'}\}$ is one parameter martingale in s_2 (with $s = (s_1, s_2)$) so that

$$E[| \rho_{s,s'}^{(n)} |^p] \leq c'_p E\left[\left(\int_{R_{s' \times s}} I(t \wedge s') | r_{t,s'}^{(n)} |^2 \, dt\right)^{p/2}\right] \leq c_p \int_{R_{s' \times s}} I(t \wedge s') E[| r_{t,s'}^{(n)} |^p] \, dt \tag{2.8}$$

for a constant c'_p depending only on p by Burkholder's inequality [5]. The second inequality with $c_p = c'_p \, \mu \, (R_{z_0})(p - 2)/2$ results by Holder's inequality.
Hence

$$\begin{aligned} E\left[\int_{R_{z_0} \otimes R_{z_0}} | \rho_{s,s'}^{(n)} |^p \, ds \, ds'\right] &\leq c_p \int_{R_{z_0} \otimes R_{z_0}} ds \, ds' \int_{R_{s \times s'}} I(t \wedge s') E[| r_{t,s'}^{(n)} |^p] \, dt \\ &= c_p \int_{R_{z_0} \otimes R_{z_0}} h(t, s') E[| r_{t,s'}^{(n)} |^p] \, dt \, ds' \leq n c_p \mu(R_{z_0}) \end{aligned} \tag{2.9}$$

where

$$h(t, s') = \mu(R_{s' \times z_0} - R_{s' \times t}) \leq \mu(R_{z_0}).$$

Therefore $\rho^{(n)} \in \mathscr{L}_2^p$ for each n. Now $|\rho_{s,s'}^{(n)} - \rho_{s,s'}| I(N \leq n) = 0$ a.s. for each $(s, s') \in R_{z_0} \otimes R_{z_0}$ so by Fubini's lemma

$$\left(\int_{R_{z_0} \otimes R_{z_0}} | \rho_{s,s'}^{(n)} |^p \, ds \, ds' - \int_{R_{z_0} \otimes R_{z_0}} | \rho_{s,s'} |^p \, ds \, ds'\right) I(N \leq n) = 0 \quad \text{a.s.} \tag{2.10}$$

Since $P(N < +\infty) = 1$, this implies that $\rho \in \mathscr{L}_2^p$ as desired. Let

(2.11)
$$\eta_{s,s'} = \int_{R_{s\vee s'}} I(t \wedge s')\alpha_{t,s'}\,dt.$$

Then there exists $\alpha^{(n)} \in \mathscr{L}_2^\varrho$ and an a.s. finite random variable N such that $\alpha^{(n)}(s, s') = \alpha(s, s')$ whenever $n \geq N(\omega)$ and such that

$$E\left[\int_{R_{z_0}\otimes R_{z_0}} |\, \alpha^{(n)}(s, s')|^p \,ds\,ds'\right] \leq n.$$

Let $\eta^{(n)}$ be given by (2.11) with α replaced by $\alpha^{(n)}$. Then by Holder's inequality

$$E[|\eta^{(n)}(s, s')|^p] \leq \mu(R_{s'\vee s} - R_{s'})^{p-1} \int_{R_{s'\times s}} I(t \wedge s')E[|\, \alpha_{t,s'}^{(n)}\, |^p]\,dt$$

$$\leq c_p'' \int_{R_{s'\times s}} I(t \wedge s')E[|\, \alpha_{t,s'}^{(n)}\, |^p]\,dt$$

when $c_p'' = \mu(R_{z_0})^{p-1}$. This is the same as (2.8) so that $\eta \in \mathscr{L}_2^\varrho$ by the same argument used for ρ.

A similar but easier argument shows that ζ defined by $\zeta_{s,s'} = q_{s'} \in \mathscr{L}_2^\varrho$. Now $Z_{W1}(s \vee s', s') = \rho_{s,s'} + \eta_{s,s'} + \zeta_{s,s'}$ and $\rho + \eta + \zeta \in \mathscr{L}_2^\varrho$ as advertised. □

Certain binary operations on semimartingales will now be defined. Let Z be a semimartingale with representation (2.1) and let $\tilde{Z}$ be the semimartingale

$$\tilde{Z} = W\cdot\tilde{r}\cdot W + \tilde{q}\cdot W + \mu\cdot\tilde{\alpha}\cdot W + W\cdot\tilde{\beta}\cdot\mu + \tilde{b}\cdot\mu.$$

Suppose $\psi \in \mathscr{L}_1^\varrho$. Then define

(2.12)
$$[Z, \tilde{Z}] = (q\tilde{q})\cdot\mu + \mu\cdot(r\tilde{r})\cdot\mu,$$

(2.13)
$$\langle Z, \tilde{Z}\rangle_1(z) = \int_{R_z} Z_{W1}(z, s')\tilde{Z}_{W1}(z, s')\,ds',$$

(2.14)
$$\langle Z, \tilde{Z}\rangle_2(z) = \int_{R_z} Z_{W2}(z, s)\tilde{Z}_{W2}(z, s)\,ds,$$

(2.15)
$$\begin{aligned} Z*\tilde{Z} = {} & W\cdot(Z_{W2}(s \vee s', s)Z_{W1}(s \vee s', s'))\cdot W \\ & + \mu\cdot(Z_{\mu 2}(s \vee s', s)Z_{W1}(s \vee s', s'))\cdot W \\ & + W\cdot(Z_{W2}(s \vee s', s)Z_{\mu 1}(s \vee s', s'))\cdot\mu \\ & + \mu\cdot(Z_{\mu 2}(s \vee s', s)Z_{\mu 1}(s \vee s', s'))\cdot\mu, \end{aligned}$$

(2.16)
$$\begin{aligned} \psi\cdot Z = {} & (q\psi)\cdot W + W\cdot(r_{s,s'}\psi_{s\wedge s'})\cdot W + \mu\cdot(\alpha_{s,s'}\psi_{s\wedge s'})\cdot W \\ & + W\cdot(\beta_{s,s'}\psi_{s\wedge s'})\cdot\mu + (b\psi)\cdot\mu. \end{aligned}$$

An alternative expression for $\langle Z, \tilde{Z}\rangle_1$ *is*

(2.17)
$$\begin{aligned} \langle Z, \tilde{Z}\rangle_1 = {} & [Z, \tilde{Z}] + (W\cdot r + \mu\cdot\alpha)\tilde{Z}_{W1}(s' \vee s, s')\cdot\mu \\ & + (W\cdot\tilde{r} + \mu\cdot\tilde{\alpha})Z_{W1}(s' \vee s, s')\cdot\mu. \end{aligned}$$

To obtain (2.17), apply the one-parameter differential formula [5] to the integrand in (2.13) as a function of z_2 for fixed z_1 and use (2.4). Similarly,

(2.18)
$$\langle Z, \tilde{Z}\rangle_2 = [Z, \tilde{Z}] + \mu\cdot\tilde{Z}_{W2}(r\cdot W + \beta\cdot\mu) + \mu\cdot Z_{W2}(\tilde{r}\cdot W + \tilde{\beta}\cdot\mu).$$

PROPOSITION 2. *Let $2 \le r, s, t < +\infty$ such that $1/r + 1/s = 1/t$, and suppose $Z \in \mathscr{S}^r$, $\tilde{Z} \in \mathscr{S}^s$, and $\psi \in \mathscr{L}_1^s$. Then $[Z, \tilde{Z}]$, $\langle Z, \tilde{Z}\rangle_1$, $\langle Z, \tilde{Z}\rangle_2$, $Z*\tilde{Z}$ and $\psi\cdot Z$ are well-defined semimartingales in $\mathscr{S}^t$. If $s = +\infty$ (so $2 \le r = t < +\infty$) then $\psi\cdot Z$ is still a well-defined semimartingale in $\mathscr{S}^t$.*

PROOF. Proposition 2 is an easy consequence of Holder's inequality and Proposition 1.

REMARK. Although the quantities in (2.12) through (2.16) are defined in terms of the stochastic integral representations of Z and $\tilde{Z}$, they have intrinsic meaning. Formally,

$$\langle Z, \tilde{Z}\rangle_1(dz_1, z_2) = Z(dz_1, z_2)\tilde{Z}(dz_1, z_2)$$

$$\langle Z, \tilde{Z}\rangle_2(z_1, dz_2) = Z(z_1, dz_2)\tilde{Z}(z_1, dz_2)$$

$$[Z, \tilde{Z}](dz_1, dz_2) = Z(dz_1, dz_2)\tilde{Z}(dz_1, dz_2)$$

$$Z*\tilde{Z}(dz_1, dz_2) = Z(z_1, dz_2)\tilde{Z}(dz_1, z_2)$$

$$\psi\cdot Z(dz_1, dz_2) = \psi(z)Z(dz_1, dz_2)$$

where dz_1 and dz_2 are forward increments from the point $z = (z_1, z_2) \in R_{z_2}$. $\langle Z, Z\rangle_1$ and $\langle Z, Z\rangle_2$ are versions of the Meyer compensator of Z^2 viewed as a one-parameter semimartingale along the lines $z_2 =$ constant and $z_1 =$ constant respectively. $[Z, Z]$ is the limit of a certain quadratic variation of Z [2] and $\psi\cdot Z$ is simply the stochastic integral of ψ with respect to Z. Finally, the non-symmetric operation "$*$" was introduced in [12], and its symmetrization can be seen to be "intrinsic" by applying the following differentiation formula to $Z\tilde{Z}$ (see (2.20).

THEOREM 2.3. (Differentiation Formula—Wong-Zakai-Ito Form). *Let $p \ge 2$ and suppose $Z \in \mathscr{S}^{4p}$ and $F \in C^4(\mathbb{R})$ are given. Define $F(Z)$ by $F(Z)_z = F(Z_z)$. Then $F(Z) \in \mathscr{S}^p$ and, with $F_k(x) = (d^k/dx^k)F(x)$,*

$$F(Z) = F(Z_0) + F_1(Z)\cdot Z + F_2(Z)\cdot(Z*Z) + \tfrac{1}{2} F_2(Z)\cdot(\langle Z, Z\rangle_1 + \langle Z, Z\rangle_2 - [Z, Z])$$

$$+ \tfrac{1}{2} F_3(Z)\cdot(Z*\langle Z, Z\rangle_1 + \langle Z, Z\rangle_2*Z + 2[Z, Z*Z]) + \tfrac{1}{4} F_4(Z)\cdot(\langle Z, Z\rangle_2*\langle Z, Z\rangle_1). \quad (2.19$$

PROOF. Since Z is sample continuous, $F_i(Z)$ is also and so $F_i(Z) \in \mathscr{L}_1^\infty$ for $i = 1, 2, 3, 4$. This fact and repeated application of Proposition 2.2 shows that the right hand side of (2.19) is in $\mathscr{S}^p$. The formula (2.19) is the same as that of [9] but with new notation. In [9], (2.19) is proved under the conditions that r, q, α, β, and b are bounded. The formula can be extended first to the case when r, α, β are $\mu \times \mu \times P$ $4p$th power integrable and q and b are $\mu \times P$ $4p$th power integrable. This extension can be accomplished using the fact that bounded functions are dense in the appropriate L^p-spaces and using estimates such as (2.9) and Doob's maximal L^p inequality. The general case follows by using the approximation technique of [11] and Fubini's theorem, as in the proof for Proposition 1. □

Let $\mathscr{S}^\omega = \bigcap_{2\le p<+\infty} \mathscr{S}^p$. Then $\mathscr{S}^\omega \subset \mathscr{S}^\infty\cdot\mathscr{S}^\omega$ is a natural collection of (two-parameter) semimartingales to use for a stochastic calculus. Indeed, if $f \in C^4(\mathbb{R})$ and if $X, Y \in \mathscr{S}^\omega$, then by Proposition 2.2 and Theorem 2.3, $\langle X, Y\rangle_i$, $[X, Y]$, $X\cdot Y$, $X*Y$ and $f(X) \in \mathscr{S}^\omega$. $\mathscr{S}^\omega$ is also closed under the operations introduced in the next section.

REMARK 1. If $Z = (Z_1, \cdots, Z_n)$ is a vector of n semimartingales, each in $\mathscr{S}^{4p}$, and if $F: \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives to fourth order, then $F(Z) \in \mathscr{S}^p$ and (2.19) still applies if the terms are interpreted appropriately. For example, identify

$$F_1(Z)\cdot Z = \textstyle\sum_i \frac{\partial F}{\partial z_i}\cdot Z_i$$

$$F_2(Z)\cdot(Z*Z) = \textstyle\sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j}\cdot(Z_i*Z_j)$$

$$F_3(Z)\cdot(Z^*\langle Z, Z\rangle_1) = \sum_{i,j,k} \frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k}\cdot(Z_i * \langle Z_j, Z_k\rangle_1).$$

For $n = 2$ and $F(z, \tilde{z}) = z\tilde{z}$, this yields

$$ Z\tilde{Z} = \tilde{Z}\cdot Z + \tilde{Z}\cdot Z + Z*\tilde{Z} + \tilde{Z}*Z + \langle Z, \tilde{Z}\rangle_1 + \langle Z, \tilde{Z}\rangle_2 - [Z, \tilde{Z}]. \tag{2.20}$$

REMARK 2. Another version of the differentiation formula is given in the next section.

3. Two parameter Stratonovich integrals and the related calculus. In this section stochastic integrals $Z \dot{-} \tilde{Z}$ and $Z \,\dot{*}\, \tilde{Z}$ are introduced which differ from the integrals $Z \cdot \tilde{Z}$ and $Z*\tilde{Z}$ by a correction term; see (3.6) and (3.7). The relationship of $Z \dot{-} \tilde{Z}$ and $Z \,\dot{*}\, \tilde{Z}$ to $Z \cdot \tilde{Z}$ and $Z*\tilde{Z}$ is the analog of the relationship of the Stratonovich integral to the Ito integral for one-parameter process. The Stratonovich integrals we introduce formally obey the rules of ordinary calculus and may be approximated by Riemann sums with integrands sampled in the center of the incremental rectangles. More importantly, using this Stratonovich type integral to interpret the partial differential equation (1.1) driven by white noise η, we arrive at an integral equation for which we can prove existence and uniqueness of solutions (Theorem 4.2). So far we cannot do the same using the Wong-Zakai-Ito interpretation.

A decomposition of $\langle Z, \tilde{Z}\rangle_i$ $i = 1, 2$ will be needed to define $Z \dot{-} \tilde{Z}$ and $Z \,\dot{*}\, \tilde{Z}$. Let Z and $\tilde{Z}$ have semimartingale representations

$$Z = W\cdot r\cdot W + q\cdot W + \mu\cdot\alpha\cdot W + W\cdot\beta\cdot\mu + b\cdot\mu \tag{3.1}$$

$$\tilde{Z} = W\cdot\tilde{r}\cdot W + \tilde{q}\cdot W + \mu\cdot\tilde{\alpha}\cdot W + W\cdot\tilde{\beta}\cdot\mu + \tilde{b}\cdot\mu. \tag{3.2}$$

Now, define

$$\prec\! Z, \tilde{Z}\rangle_1 = (W\cdot\tilde{r} + \mu\cdot\tilde{\alpha})Z_{W1}\cdot\mu$$

$$\prec\! Z, \tilde{Z}\rangle_2 = \mu\cdot Z_{W2}(\tilde{r}\cdot W + \tilde{\beta}\cdot\mu).$$

It follows from (2.17) and (2.18) that

$$\langle Z, \tilde{Z}\rangle_i = \prec\! Z, \tilde{Z}\rangle_i + \prec\! \tilde{Z}, Z\rangle_i + [Z, \tilde{Z}].$$

By Proposition 2.1 and Holder's inequality, it is clear that if $2 \le r, s, t < +\infty$ such that $1/r + 1/s = 1/t$, and if $Z \in \mathscr{S}^r$ and $\tilde{Z} \in \mathscr{S}^s$, then $\langle Z, \tilde{Z}\rangle_i \in \mathscr{S}^t$ for $i = 1, 2$.

The process $\langle Z, \tilde{Z}\rangle_i$ is intrinsically determined, as shown in the following proposition. Fix $z \in R_{z_0}$ and let $\sigma = \{z_{k,\ell}\}$ be a finite partition of R_z into congruent rectangles. Let $\|\sigma\| = |z_{k+1,\ell+1} - z_{k,\ell}|$. For any function f on R_{z_0} let $\Delta_{k,\ell}f = f(z_{k,\ell}, z_{k+1,\ell+1}]$.

PROPOSITION 3.1. *Let $Z, \tilde{Z}$ be given by* (3.1) *and* (3.2), *and suppose that all processes appearing as integrands in* (3.1) *and* (3.2) *are bounded. Then the following limits exist in $L^2(\Omega, \mathscr{F}, P)$:*

$$\langle Z, \tilde{Z}\rangle_{1,z} = \lim_{\|\sigma\|\to 0} \sum_{k,\ell,\ell'} (\Delta_{k,\ell}Z)(\Delta_{k,\ell'}\tilde{Z}) \tag{3.3}$$

$$[Z, \tilde{Z}]_z = \lim_{\|\sigma\|\to 0} \sum_{k,\ell=\ell'} (\Delta_{k,\ell}Z)(\Delta_{k,\ell'}\tilde{Z}) \tag{3.4}$$

$$\prec\! Z, \tilde{Z}\rangle_{1,z} = \lim_{\|\sigma\|\to 0} \sum_{k,\ell<\ell'} (\Delta_{k,\ell}Z)(\Delta_{k,\ell'}\tilde{Z}). \tag{3.5}$$

PROOF. Equation (3.3) is really a standard result for one-parameter semimartingales and may be proved by Burkholder's inequality [5]. Equation (3.4) follows by generalizing Burkholder's inequality to two parameter processes [3]. Equations (3.3) through (3.5) all may be proved by expressing the right hand sides in semimartingale form by use of the change of variable formula. Since we will not be using Propositon 3.1, details are omitted. □

Let X and Y be semimartingales. We define the Stratonovich (or "balanced" or "centered") integral of X with respect to Y by

$$X \dot{-} Y = X \cdot Y + \tfrac{1}{4}[X, Y] + \tfrac{1}{2}\langle X, Y\rangle_1 + \tfrac{1}{2}\langle X, Y\rangle_2 \tag{3.6}$$

and we define the Stratonovich version of $X * Y$ by

$$X \overset{*}{-} Y = X * Y + \tfrac{1}{4}[X, Y] + \tfrac{1}{2}\langle Y, X\rangle_1 + \tfrac{1}{2}\langle X, Y\rangle_2. \tag{3.7}$$

If $r, s, t \in [2, +\infty)$ with $1/r + 1/s = 1/t$, and if $X \in \mathscr{S}^r$, $Y \in \mathscr{S}^s$, then $X \dot{-} Y, X \overset{*}{-} Y \in \mathscr{S}^t$ by Proposition 2.1 and Holder's inequality.

PROPOSITION 3.2. (Differentiation formula in Stratonovich Form). *Let $F \in C^6(\mathbb{R})$ with $F(0) = 0$ and suppose $Z \in \mathscr{S}^{12}$. Then $F(Z), F'(Z), F''(Z), F'(Z) \dot{-} Z, Z \overset{*}{-} Z$ and $F''(Z) \dot{-} (Z \overset{*}{-} Z) \in \mathscr{S}^2$ and*

$$F(Z) = F'(Z) \dot{-} Z + F''(Z) \dot{-} (Z \overset{*}{-} Z). \tag{3.8}$$

If $X, Y, Z \in \mathscr{S}^\omega$, then

$$X \dot{-} (Y \dot{-} Z) = (XY) \dot{-} Z \tag{3.9}$$

$$\langle F(Y), X\rangle_i = F'(Y)\cdot\langle Y, X\rangle_i, \quad i = 1, 2 \tag{3.10}$$

$$\langle X, \langle Y, Z\rangle_2\rangle_1 = [Y * X, Z] \tag{3.11}$$

$$\langle X, \langle Y, Z\rangle_1\rangle_2 = [X * Y, Z] \tag{3.12}$$

$$\langle Y, Z\cdot X\rangle_i = Z\cdot\langle Y, X\rangle_i, \quad i = 1, 2 \tag{3.13}$$

$$\langle Y, X * Z\rangle_1 = X * \langle Y, Z\rangle_1 \tag{3.14}$$

$$\langle Y, Z * X\rangle_2 = \langle Y, Z\rangle_2 * X \tag{3.15}$$

PROOF. Equations (3.10) through (3.15) are easily proved by using the definitions of $\langle\ \rangle_i$, "$\dot{-}$" and "$\overset{*}{-}$", and passing to semimartingale representations. Using (3.10) through (3.15) with $X = Y = Z$, (3.8) is reduced to the differentiation formula (2.19) in Ito form. Similarly, (3.9) is proved using (3.10) through (3.15). □

PROPOSITION 3.3. (Differentiation formula in Stratonovich Form—Vector Case). *Let $F: \mathbb{R}^n \to \mathbb{R}$ with $F(0) = 0$ have continuous derivatives through the sixth order, and let $Z_1, \cdots, Z_n \in \mathscr{S}^{12}$. Then*

$$F(Z_1, \cdots, Z_n) = \sum_i \frac{\partial F}{\partial z_i} \dot{-} Z_i + \sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j} \dot{-} (Z_i \overset{*}{-} Z_j) \tag{3.16}$$

and all terms in (3.16) *are semimartingales.*

REMARKS (1). The formula (3.16) shows that when the Stratanovich integral is used, semimartingales formally obey the ordinary rules of calculus. Indeed, let $g: R_{z_0} \to \mathbb{R}^n$ and $F: \mathbb{R}^n \to \mathbb{R}$ be smooth. Then by the chain rule of calculus

$$\frac{\partial^2}{\partial x_1 \partial x_2}(F \circ g) = \sum_i \frac{\partial F}{\partial g_i}\frac{\partial^2 g_i}{\partial x_1 \partial x_2} + \sum_{i,j} \frac{\partial^2 F}{\partial g_i \partial g_j}\frac{\partial g_i}{\partial x_1}\frac{\partial g_j}{\partial x_2},$$

which is (3.16) in differential form.

(2) As in the one parameter case, the multiple parameter Stratonovich integral may be approximated by Riemann sums in which the integrand is sampled in the center of rectangular increments. Let us describe a generalization.

Let $\sigma = \{z_{k,\ell}\}$ be a partition of R_z into congruent rectangles where $z_{k,\ell} = (t_k, s_\ell)$. Given $\lambda = (\lambda_1, \lambda_2)$, $0 \le \lambda_1, \lambda_2 \le 1$, let $t_k^\lambda = t_k + \lambda_1(t_{k+1} - t_k)$, let $s_k^\lambda = s_k + \lambda_2(s_{k+1} - s_k)$, and let $z_{k,\ell}^\lambda = (t_k^\lambda, s_\ell^\lambda)$. Suppose that X and Y are semimartingales with bounded integrand processes in their semimartingale representation. Then the following limits exist in $L^2(\Omega, P_0)$:

$$\lim_{\|\sigma\|\to 0} \sum_{k,\ell} X_{z_{k,\ell}^\lambda} \Delta_{k,\ell} Y = X \circ Y + \lambda_1\lambda_2[X, Y] + \lambda_2\langle X, Y\rangle_1 + \lambda_1\langle X, Y\rangle_{2|z}$$

and

$$\lim_{\|\sigma\|\to 0} \textstyle\sum_{k,\ell} (X_{(t_k^\lambda, s_{\ell+1})} - X_{(t_k^\lambda, s_\ell)})(Y_{(t_{k+1}, s_\ell^\lambda)} - Y_{(t_k, s_\ell^\lambda)})$$
$$= (X * Y) + \lambda_1 \langle Y, X\rangle_1 + \lambda_2 \langle X, Y\rangle_2 + \lambda_1\lambda_2 [X, Y]_{|z}.$$

As in the one-parameter case, these identities may be easily proven by expressing the left hand sides as the sum of quadratic variation expressions and a stochastic integral. The case $\lambda = (0, 0)$ yields the usual Ito integral, while $\lambda = (\frac{1}{2}, \frac{1}{2})$ yields the Stratonovich-like integrals we have defined.

EXAMPLE. For $Z, \tilde{Z} \in \mathcal{S}^\omega$, (3.16) implies that

$$Z\tilde{Z} = Z \dot{-} \tilde{Z} + \tilde{Z} \dot{-} Z + Z \underline{*} \tilde{Z} + \tilde{Z} \underline{*} Z. \tag{3.17}$$

To see (3.17) directly, fix $z \in R_{z_0}$ and, using the notation of the previous remark, write

$$Z_z\tilde{Z}_z = \textstyle\sum_{k,\ell} Z_z \Delta_{k,\ell}\tilde{Z} \tag{3.18}$$

and

$$Z_z = Z_{z_{k,\ell}^\lambda} + (Z_z - Z_{z\times z_{k,\ell}^\lambda} - Z_{z_{k,\ell}^\lambda \times z} + Z_{z_{k,\ell}^\lambda}) + (Z_{z_{k,\ell}^\lambda \times z} - Z_{z_{k,\ell}^\lambda}) + (Z_{z\times z_{k,\ell}^\lambda} - Z_{z_{k,\ell}^\lambda}). \tag{3.19}$$

Let $\lambda = (\frac{1}{2}, \frac{1}{2})$. Substitution of (3.19) into the right hand side of (3.18) yields an expression for $Z_z\tilde{Z}_z$ with four summations. If the integrand processes in the semimartingale representations (3.1) and (3.2) of Z and $\tilde{Z}$ are bounded then each sum converges in $L^2(\Omega, \mathcal{F}, P)$ to the corresponding term in (3.17).

This section will be completed with a lemma to be used in the next section.

LEMMA 3. *Let $f: \mathbb{R} \to \mathbb{R}$ be six times continuously differentiable and let $Z \in \mathcal{S}^\omega$. Then*

$$f(Z) \underline{*} f(Z) = f'(Z)^2 \dot{-} (Z \underline{*} Z). \tag{3.20}$$

PROOF. Let $Y_t = \frac{1}{2} f(Z_t)^2$. Applying the differentiation formula (3.8) to Y_t as a function of Z_t yields that

$$Y = (f(Z)f'(Z)) \dot{-} Z + (f(Z)f''(Z) + f'(Z)^2) \dot{-} (Z \underline{*} Z). \tag{3.21}$$

The same formula, applied to Y_t as a function of $f(Z_t)$ yields

$$Y = f(Z) \dot{-} f(Z) + f(Z) \underline{*} f(Z).$$

Or, since $f(Z) = f'(Z) \dot{-} Z + f''(Z) \dot{-} (Z \underline{*} Z)$,

$$Y = (f(Z)f'(Z)) \dot{-} Z + (f(Z)f''(Z)) \dot{-} (Z \underline{*} Z) + f(Z) \underline{*} f(Z). \tag{3.22}$$

Comparison of (3.21) and (3.22) yields (3.20). □

4. Existence, uniqueness, and Markov property of solution to PDE.

The proof of our main result (Theorem 4.2) will depend on the following theorem.

THEOREM 4.1. *Let θ and σ satisfy $|\theta(x) - \theta(x')| \le L_\theta |x - x'|$ and $|\sigma(x) - \sigma(x')| \le L_\sigma |x - x'|$ for all $x, x' \in \mathbb{R}$. Then there is a unique adapted, sample continuous random process $Z = \{Z_z : z \in R_{z_0}\}$ such that*

$$Z = \theta(Z)\cdot\mu + \sigma(Z)\cdot W. \tag{4.1}$$

The solution z is in $\mathcal{S}^\infty$ and is Markov relative to $\{\mathcal{F}_z : z \in R_{z_0}\}$.

REMARK. Equation (4.1) is the integrated version (using Ito-type integrals) of the equation

$$\frac{\partial^2 Z}{\partial t_1 \partial t_2} = \theta(Z_t) + \sigma(Z_t)\eta_t, \qquad t \in R_{z_0}$$

$$Z_{t_1 t_2} = 0 \text{ if } t_1 t_2 = 0$$

where $\eta_t = \frac{\partial^2}{\partial t_1 \partial t_2} W$ is Gaussian white noise. This equation has been studied in [1] and [8].

PROOF. The existence of a solution will be demonstrated by a Picard iteration argument. The right hand side of equation (4.1) is in $\mathscr{S}^\infty$ for any adapted, sample continuous process Z. Let $Z^{(0)} = 0$ and for $k > 0$ define

$$Z^{(k)} = \theta(Z^{(k-1)}) \cdot \mu + \sigma(Z^{(k-1)}) \cdot W$$

and

$$\varphi_z^{(k)} = \sup_{0<s<z} | Z_s^{(k+1)} - Z_s^{(k)} |.$$

First, note that

$$\varphi_{z_0}^{(0)} = \sup_{0<z<z_0} | \theta(0)\mu(R_z) + \sigma(0) W_z |$$

so using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and Doob's inequality extended to two parameter processes [2] respectively yields that

$$\begin{aligned} E[(\phi_{z_0}^{(0)})^2] &\leq 2\theta(0)^2 \mu(R_{z_0})^2 + 2\sigma(0)^2 E[\sup_{0<s<z_0} W_s^2] \\ &\leq 2\theta(0)^2 \mu(R_{z_0})^2 + 32\sigma(0)^2 \mu(R_{z_0}) < +\infty. \end{aligned}$$

Using the same inequalities and the Schwarz inequality also yields, for $k > 0$

$$\begin{aligned} E[(\varphi_z^{(k)})^2] &\leq 2E[\sup_{0<s<z} (\theta(Z^{(k)}) - \theta(Z^{(k-1)})) \cdot \mu_s^2] \\ &\quad + 2E[\sup_{0<s<z} (\sigma(Z^{(k)}) - \sigma(Z^{(k-1)})) \cdot W_s^2] \\ &\leq 2\mu(R_z) E[(\theta(Z^{(k)}) - \theta(Z^{(k-1)}))^2 \cdot \mu_z] \\ &\quad + 2 \times 16 E[(\sigma(Z^{(k)}) - \sigma(Z^{(k-1)}))^2 \cdot \mu_z] \\ &\leq cE[(\varphi^{(k-1)})^2] \cdot \mu_z \end{aligned}$$

where $c = 2\mu(R_{z_0}) L_\theta + 32 L_\sigma$. By iteration, this implies that

$$E[(\varphi_z^{(k)})^2] \leq \frac{\mu(R_z)^k c^k}{k!} E[(\varphi_{z_0}^{(0)})^2].$$

Thus, $\sum_{k=0}^\infty P(\varphi_{z_0}^{(k)} \geq 1/k^2) < +\infty$ so $Z^{(k)}$ converges uniformly a.s. by the Borel-Cantelli lemma to a sample continuous, adapted process Z. Since $Z^{(k)}$ converges to Z in $L^2(\Omega \times R_{z_0}, P \times \mu)$ as well, Z must be a solution to (4.1).

Suppose that Z_1 and Z_2 are each solutions to (4.1). Then $Z_1, Z_2 \in \mathscr{S}^\infty$. For $k > 0$ let $I_k(\cdot, \omega)$ be the indicator function of the random set

$$A_k(\omega) = \{z \in R_{z_0} : | Z_i(s) | \leq k \text{ for } 0 < s < z \text{ and } i = 1, 2\}.$$

Then $\lim_{k\to\infty} \inf_{z \in R_{z_0}} I_k(z, \omega) = 1$ a.s. and I_k is non-increasing in the sense that if $I_k(z, w) = 1$ then $I_k(s, \omega) = 1$ for all $s < z$. For $i = 1, 2$, define

$$\tilde{Z}_i = (\theta(Z_i) I_k) \cdot \mu + (\sigma(Z_i) I_k) \cdot W.$$

Then $(Z_i - \tilde{Z}_i) I_k = 0$ a.s. and $E[\tilde{Z}_i(s)^2]$ is bounded for $i = 1, 2$. Furthermore, $\tilde{Z}_1$ and $\tilde{Z}_2$ are each solutions to the equation

$$\tilde{Z} = (\theta(\tilde{Z}) I_k) \cdot \mu + (\sigma(\tilde{Z}) I_k) \cdot W.$$

A now trivial Picard iteration argument establishes that $\tilde{Z}_1 = \tilde{Z}_2$ a.s. so that $Z_1(s) = Z_2(s)$

for $s \in A_k(\omega)$ a.s. for all k. Thus $Z_1(s) = Z_2(s)$ for all s a.s., so the solution Z to (4.1) is unique.

The proof that Z is Markov will be based on the following claim: for each $a \in \mathscr{A}$, $\mathscr{O}_{dW}(a^c)$ is independent of $\mathscr{F}_a$. There exist $0 \times z_0 = t_0, t_1, \cdots, t_n = z_0 \times 0$ in R_{z_0} such that $t_i \wedge t_j$ if $i < j$ and such that $a = R_{t_1} \cup \cdots \cup R_{t_n}$. Let $\mathscr{D}_i = \mathscr{O}_{dW}(a^c \cap (R_{z_0 \times t_{i-1}} - R_{z_0 \times t_i}))$ for $1 \le i \le n$. In the first place $\mathscr{O}_{dW}(a^c) = \bigvee_{i=1}^n \mathscr{D}_i$. In the second place, $\mathscr{D}_i$ is independent of $\mathscr{D}_{i+1} \vee \cdots \vee \mathscr{D}_n \vee \mathscr{F}_a$ for $i = 1, \cdots, n$ by the definition of $\mathscr{F}.$-Wiener process. These two facts imply that $\mathscr{O}_{dW}(a^c)$ is independent of $\mathscr{F}_a$ as claimed.

Let $a \in \mathscr{A}$ be given by $a = R_{t_1} \cup \cdots \cup R_{t_n}$ with $t_0, \cdots, t_n$ given as before, and let $z_i = t_{i-1} \times t_i$. Then $\{Z_z : z \in \bar{a}^c\}$ satisfies the equations

$$Z(z_i, z] = \int_{(z_i, z]} \theta(Z_s)\, ds + \int_{(z_i, z]} \sigma(Z_s)\, dW_s, \qquad i = 1, \cdots, n. \tag{4.2}$$

Moreover, the equations (4.2) show that $\{Z_z : z \in a^c\}$ is uniquely determined by $\{dW_z : z \in a\}$ and $\{Z_z : z \in \partial a\}$ given as initial conditions. Indeed, this can be shown by Picard iteration as used in the first part of the proof. The initial boundary data $\{Z_z : z \in \partial a\}$ for this problem is a.s. continuous rather than zero as implicit in (4.1), but the previous proof of existence and uniqueness is routinely modified to cover this case. Given $\{Z_z : z \in \partial a\}$, the "future" $\{Z_z : z \in a^c\}$ depends only on the future white noise $\{dW_z : z \in a^c\}$ and so is conditionally independent of $\mathscr{F}_a$. Thus, Z is Markov. □

The stochastic equation

$$Y - a(Y) \dot{-} (Y \overset{*}{-} Y) - b(Y) \cdot \mu - c(Y) \dot{-} W = 0 \tag{4.3}$$

will be considered now. Equation (4.3) is the Stratonovich-type stochastic integral version of the differential equation (1.1) driven by Gaussian white noise $\eta = \partial^2/(\partial t_1 \partial t_2)\, W$.

THEOREM 4.2. *Let a, b, c have four bounded, continuous derivatives. Then there is a unique solution of* (4.3) *contained in $\mathscr{S}^\omega$. The solution Y is actually in $\mathscr{S}^\infty$ and is Markov relative to $\{\mathscr{F}_z : z \in R_{z_0}\}$. If $h \in C^6(\mathbb{R})$, $h(0) = 0$ and h' is strictly positive and bounded, then the Markov process $X = h(Y)$ again satisfies an equation of the form* (4.3).

PROOF. Suppose $Y \in \mathscr{S}^\omega$ is a solution to (4.3). Define $F \in C^4(\mathbb{R})$ by

$$f(0) = 0 \text{ and } f'(x) = \exp\left(-\int_0^x a(s)\, ds\right).$$

By the differentiation formula (3.8) and (3.9),

$$\begin{aligned} f(Y) &= f'(Y) \dot{-} Y + f''(Y) \dot{-} (Y \overset{*}{-} Y) = f'(Y) \dot{-} \left\{ Y + \left(\frac{f''(Y)}{f'(Y)}\right) \dot{-} (Y \overset{*}{-} Y) \right\} \\ &= f'(Y) \dot{-} \{Y - a(Y) \dot{-} (Y \overset{*}{-} Y)\}. \end{aligned} \tag{4.4}$$

Using (4.3) this becomes

$$f(Y) = f'(Y) \dot{-} \{b(Y) \cdot \mu + c(Y) \dot{-} W\}. \tag{4.5}$$

The inverse f^{-1} of f is well defined on $f(\mathbb{R})$ since $f'(x) > 0$. Let $\sigma = (f'c) \circ f^{-1}$ and $\rho = (f'b) \circ f^{-1}$ (here "$\circ$" denotes composition of functions). Let $Z_t = f(Y_t)$. Then (4.5) may be rewritten as

$$Z = \rho(Z) \cdot \mu + \sigma(Z) \dot{-} W. \tag{4.6}$$

Using the definition of "$\dot{-}$", the differentiation formula (2.19) on $\sigma(Z)$, and (4.6) respectively yields that

$$\begin{aligned} \sigma(Z) \dot{-} W &= \sigma(Z) \cdot W + \tfrac{1}{4}[\sigma(Z), W] + \tfrac{1}{2}\langle \sigma(Z), W \rangle_1 + \tfrac{1}{2}\langle \sigma(Z), W \rangle_2 \\ &= \sigma(Z) \cdot W + \tfrac{1}{4}\sigma'(Z) \cdot [Z, W] + 0 + 0 = \sigma(Z) \cdot W + \tfrac{1}{4}\sigma'(Z)\sigma(Z) \cdot [W, W]. \end{aligned} \tag{4.7}$$

Substituting this into (4.6) yields that

$$Z = (\rho(Z) + \tfrac{1}{4}\sigma(Z)\sigma'(Z))\cdot\mu + \sigma(Z)\cdot W. \tag{4.8}$$

Note that

$$\sigma'(x) = ((f'c)\circ f^{-1})'(x) = -\frac{(f'c)'}{f'}\circ f^{-1}(x)$$
$$= \left(-\frac{f''}{f'}c - c'\right)\circ f^{-1}(x) = (ac - c')\circ f^{-1}(x).$$

Similarly,

$$\rho'(x) = (ab - b')\circ f^{-1}(x)$$

and

$$(\sigma\sigma')'(x) = (f'c(ac - c'))\circ f^{-1}(x) = \{ac(ac - c') - (c(ac - c'))'\}\circ f^{-1}(x).$$

Hence σ, ρ and $\sigma\sigma'$ have bounded, continuous first derivatives, and are thus Lipschitz continuous. The equation (4.8) is hence the same as (4.1) with $\theta = \rho + \tfrac{1}{4}\sigma\sigma'$, and the conditions of Theorem 4.1 are satisfied. Hence, there is a unique sample continuous, adapted solution to (4.8). This implies the uniqueness assertion of Theorem 4.2.

To prove the existence of a solution Y to (4.3), let $Z \in \mathscr{S}^\infty$ be the unique sample continuous adapted process satisfying (4.8) and let $Y_t = f^{-1}(Z_t)$. Now (4.8) (instead of (4.6)) implies that $[Z, W] = \sigma(Z)\cdot[W, W]$ so that (4.7) is true as before. Hence, Z satisfies (4.6) which is equivalent to

$$f(Y) = f'(Y) \mathbin{\dot-} \{b(Y)\cdot\mu + c(Y)\cdot W\}. \tag{4.9}$$

On the other hand, as in (4.4), the change of variable formula and the definition of f yield that

$$f(Y) = f'(Y) \mathbin{\dot-} \{Y - a(Y) \mathbin{\dot-} (Y \overset{*}{-} Y)\}. \tag{4.10}$$

Now $f' > 0$ so that $(f'(Y))^{-1} \in \mathscr{L}_1^\infty$. By (4.9) and (4.10),

$$(f'(Y))^{-1} \mathbin{\dot-} f(Y) = b(Y)\cdot\mu + c(Y) \mathbin{\dot-} W = Y \,\&\, (Y) \mathbin{\dot-} (Y \underset{*}{\pm} Y)$$

which demonstrates that Y satisfies (4.3) as desired.

Z is Markov relative to $\{\mathscr{F}_z : z \in R_{z_0}\}$ so the solution $Y = f^{-1}(Z)$ to (4.3) is also Markov since the Markov property is preserved under one-to-one transformation of the state space $\mathbb{R}$.

Note that $Z \in \mathscr{S}^\infty$ and Z has no double integrals in its representation, so that Z_{Wi} and $Z_{\mu i}$ are a.s. bounded. Hence, examining the differentiation formula (2.19) applied to $Y = f^{-1}(Z)$ reveals that $Y \in \mathscr{S}^\infty$.

Let $g = h^{-1}$. Then $g \in C^6(\mathbb{R})$, $g(0) = 0$, and $g'(x) \geq \epsilon > 0$. Application of the differentiation formula (3.8) to $Y = g(X)$ and Lemma 3.3 imply that (4.3) may be rewritten as

$$g'(X) \mathbin{\dot-} X + g''(X) \mathbin{\dot-} (X \underset{*}{\pm} X) - ((a\circ g(X))g'(X)^2) \mathbin{\dot-} (X \underset{*}{\pm} X)$$
$$- (b\circ g(X))\cdot\mu - (c\circ g(X))\cdot W = 0.$$

Integrating $\dfrac{1}{g'(Z)} \in \mathscr{L}_1^\infty$ with respect to this yields that

$$X - \left(\frac{(a\circ g)(g')^2 - g''}{g'}(X)\right) \mathbin{\dot-} (X \underset{*}{\pm} X) - \left(\frac{b\circ g(X)}{g'(X)}\right)\cdot\mu - \left(\frac{c\circ g(X)}{g'(X)}\right) \mathbin{\dot-} W = 0$$

which has the same form as (4.3). □

REMARKS.

(1) It is not difficult to prove (see [1]) that there exists a transition semigroup corresponding to the Markov process solutions to equation (4.1). The semigroup might, for

example, act on continuous functions over sets of the form ∂a, for a $\in \mathscr{A}$. Whether or not there exists such a semigroup for the solutions to (4.3), however, is unknown. The problem is related to the initial value problem of (1.1) with non-zero initial data. The appearance of first derivatives in (1.1) causes the solution to be strongly affected by perturbations of the input data. Hence, some smoothness should be required of the initial data. For example, in the stochastic case, we can prove an existence and uniqueness theorem for (1.1) driven by white Gaussian noise when continuous one-parameter semi-martingales $\{Y_{(t_1,0)}:\mathscr{F}_{(t_1,0)}\}$ and $\{Y_{(0,t_2)}:\mathscr{F}_{(0,t_2)}\}$ are given as initial conditions. But it is not clear how to treat the case of arbitrary continuous (even deterministic) initial data.

(2) It is hoped that the Stratonovich formulation of (1.1) ensures certain stability properties of the solution when the driving term W is approximated by smoother random processes. See [15], [7] and references therein for discussion of the one parameter case.

(3) The σ-fields $\{\mathscr{O}_z(Y):z \in R_{z_0}\}$ generated by the solution Y to (4.1) satisfy condition (F4) of Section II. It is, moreover, clear that in order to represent a continuous process on R_{z_0} which is Markov with respect to its own σ-fields, either condition (F4) must be assumed or else there is no hope of representing the process as the solution of a stochastic differential equation of the form (4.2) driven by a white "innovations" process.

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