The Annals of Probability
1982, Vol. 10, No. 2, 435–450

AN EXTENSION OF THE STOCHASTIC INTEGRAL

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Two related extensions of the stochastic integral are discussed. These extensions allow the integrand to anticipate the Brownian motion, and arise in the study of linear stochastic integral equations. The development is based on the homogeneous chaos expansion of the integrand. Some properties of these extended integrals, and their commutativity with the classical integrals, are derived.

1. Introduction. In a number of recent papers the authors have examined linear Itô-Volterra operators

$$Tx(t) = \int_0^t \sigma(s, t)x(s)\, d\beta(s) + \int_0^t b(s, t)x(s)\, ds, \tag{1.1}$$

defined on the space of nonanticipating mean-square integrable stochastic processes. The formulation of a resolvent T_λ, satisfying $(I - \lambda T)^{-1} = I + \lambda T_\lambda$, entails the extension of the classical stochastic integral. This extension allows the integrand to anticipate the Brownian motion. In Berger and Mizel [4] the development is based on backward Riemann approximating sums. In Berger and Mizel [3] the assumption is made that σ and b are deterministic kernels, and the development is based on the orthogonal homogeneous chaos expansion. There note is made on the role of the β-derivative of the integrand in the definition and existence of the extended integral.

In the present work σ and b are allowed to be as general as possible. The development is again based on the orthogonal homogeneous chaos expansion. The role of the β-derivative is replaced by a more obscure operation, which reduces to the β-derivative in a special case. If $f(t)$ has the expansion

$$f(t) = \mathbb{E}f(t) + \textstyle\sum_{n=1}^{\infty} \displaystyle\int_{[0,T]^n} \phi_n(\tau_1, \cdots, \tau_n, t)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \tag{1.2}$$

then this obscure operation maps f to f', defined by

$$f'(t) = \phi_1(t, t) + \textstyle\sum_{n=2}^{\infty} n \displaystyle\int_{[0,T]^{n-1}} \phi_n(\tau_1, \cdots, \tau_{n-1}, t, t)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1}). \tag{1.3}$$

The integrand in (1.3) is taken as the right-hand limit. That is,

$$\begin{aligned} \phi_n(\tau_1, \cdots, \tau_{n-1}, t, t) &= \phi_n(\tau_1, \cdots, \tau_{n-1}, t+, t) \\ &= \lim_{\tau_n \downarrow t} \phi_n(\tau_1, \cdots, \tau_n, t). \end{aligned} \tag{1.4}$$

This operation has the remarkable property of measuring anticipation. If f is nonanticipating, then $f' = 0$. If f is purely anticipating, so that $f(t)$ is measurable relative to

Received August 7, 1979; revised February 25, 1981.

[1] Supported by National Science Foundation Grant MCS79-03550 and by the Lady Davis Fellowship Fund.

[2] Supported by National Science Foundation Grant MCS71-02776 A05, by a sabbatical grant from Carnegie-Mellon University, and by the Lady Davis Fellowship Fund.

AMS 1970 *subject classifications.* Primary 60H20, 45D05.

Key words and phrases. Brownian motion, integral equation, multiple Wiener integral, nonanticipating, resolvent, stochastic integral.

the sigma-algebra generated by $\{\beta(\tau) - \beta(t) : \tau \in [t, T]\}$, then $f' = -df/d\beta$. If $f(t) = F(\beta(t), \beta(T) - \beta(t))$ for some function $F(x, y)$, then $f'(t) = (\partial F/\partial y)\ (\beta(t), \beta(T) - \beta(t))$. Thus f' measures the β-derivative of the anticipating part of f. This is elaborated on later. Discussion of a related operator appears in Stroock [18]. In Section 2 a review of the orthogonal homogeneous chaos expansion is provided. In Section 3 two related extensions of the stochastic integral are discussed. In Section 4 the resolvent T_λ is constructed.

2. The Fundamental Expansion. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $\{\beta(t) : t \geq 0\}$ a Brownian motion on it. Let the sigma-algebras $\mathscr{F}(s, t)$ be defined for $s \leq t$ so as to satisfy: i) for $t_1 < t_2$, $\mathscr{F}(s, t_1) \subset \mathscr{F}(s, t_2)$; ii) $\beta(t) - \beta(s)$ is measurable with respect to $\mathscr{F}(s, t)$; iii) $\beta(t) - \beta(\tau)$ is independent of $\mathscr{F}(s, \tau)$ for $t \geq \tau$. In the discussion below, unless otherwise stated, $\mathscr{F}(s, t)$ is taken to be the sigma-algebra generated by $\beta(\tau) - \beta(s)$ for $\tau \in [s, t]$. For notational convenience $\mathscr{F}(0, s)$ is denoted by $\mathscr{F}(s)$. A stochastic process $f(s)$ is said to be nonanticipating if the random variable $f(s)$ is measurable with respect to $\mathscr{F}(s)$ for each s. Similarly, an n-parameter stochastic process $f(s_1, \cdots, s_n)$ is said to be nonanticipating in the kth parameter if the random variable $f(s_1, \cdots, s_n)$ is measurable with respect to $\mathscr{F}(s_k)$ for each $s_1, \cdots, s_n$. A stochastic process $f(s)$ is said to be mean-square integrable if $\int_0^t \mathbb{E}\,|f(s)|^2\,ds < \infty$ for each t.

Itô [10] has defined the classical stochastic integral $\int_0^t f(s)\,d\beta(s)$ for nonanticipating mean-square integrable processes $f(s)$. He has also defined in Itô [11] the multiple Weiner integral $\int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n)\,d\beta(\tau_1) \cdots d\beta(\tau_n)$ for deterministic functions $\phi \in L^2([0, t]^n)$. In fact, he considers a more general multiple integral than this, and in the present framework he shows that

$$\int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n)\,d\beta(\tau_1) \cdots d\beta(\tau_n) \tag{2.1}$$
$$= n! \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} \tilde{\phi}(\tau_1, \cdots, \tau_n)\,d\beta(\tau_1) \cdots d\beta(\tau_n),$$

where $\tilde{\phi}$ is the symmetric function $\phi(\tau_1, \cdots, \tau_n) = (1/n!) \sum_{\pi \in S_n} \phi(\tau_{\pi(1)}, \cdots, \tau_{\pi(n)})$, S_n being the permutation group on n letters. (There is a slight misprint in the statement of this Theorem in Itô [11], but the reader can easily verify the validity of (2.1) above.) Thus he relates the multiple Wiener integral to an iterated stochastic integral. Further, he provides an orthogonal expansion for random variables X measurable with respect to $\mathscr{F}(t)$ with finite second moments:

$$X = \mathbb{E}X + \sum_{n=1}^{\infty} \int_{[0,t]^n} \phi_n(\tau_1, \cdots, \tau_n)\,d\beta(\tau_1) \cdots d\beta(\tau_n), \tag{2.2}$$

where $\phi_n \in \tilde{L}^2(0, t]^n)$, the space of symmetric functions in $L^2([0, t]^n)$. These functions ϕ^n are also unique.

Example 2.A.

$$\beta^4(t) = 3t^2 + \int_{[0,t]^2} 6t\,d\beta(\tau_1)\,d\beta(\tau_2) + \int_{[0,t]^4} d\beta(\tau_1)\,d\beta(\tau_2)\,d\beta(\tau_3)\,d\beta(\tau_4).$$

Example 2.B. If $F(t, x)$ has the expansion $\sum_{n=0}^{\infty} a_n(t)H_n(t, x)$, where $H_n(t, x)$ is the Hermite polynomial of degree n, defined by

$$H_n(t, x) = (-t)^n e^{\frac{x^2}{2t}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2t}},$$

then:

$$F(t, \beta(t)) = \sum_{n=0}^{\infty} \int_{[0,t]^n} a_n(t)\, d\beta(\tau_1) \cdots d\beta(\tau_n).$$

Here, and throughout the discussion below, the first term in the sum on the right, corresponding to $n = 0$, is defined to be simply $a_0(t)$.

By making use of (2.1), it follows that the expansion (2.2) gives rise to an alternate representation

$$X = \mathbb{E}X + \int_0^t k(s)\, d\beta(s), \tag{2.3}$$

where $k(s)$ is a nonanticipating mean-square integrable stochastic process. Further, if $f(s)$ is a nonanticipating mean-square integrable stochastic process, then it has the orthogonal expansion

$$f(s) = \mathbb{E}f(s) + \sum_{n=1}^{\infty} \int_{[0,s]^n} \phi_n(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \tag{2.4}$$

where $\phi(\cdot, s) \in \tilde{L}^2([0, s]^n)$ for each s, and $\phi \in L^2([0, t]_+^{n+1})$ for each t, where $[0, t]_+^m = \{(\tau_1, \cdots, \tau_m): 0 \le \tau_1, \cdots, \tau_{m-1} \le \tau_m \le t\}$. Here the alternate representation is

$$f(\tau) = \mathbb{E}f(\tau) + \int_0^\tau k(s, \tau)\, d\beta(s), \tag{2.5}$$

where $k(s, \tau)$ is nonanticipating in the first parameter, and $\int_0^t \int_0^\tau \mathbb{E}\,|k(s, \tau)|^2\, ds\, d\tau < \infty$ for each t. The reader can easily check that $\{f(\tau), \mathscr{F}(\tau): \tau \ge 0\}$ is a martingale if and only if $k(s, \tau)$ is independent of τ.

EXAMPLE 2.C. $\beta^4(\tau) = 3\tau^2 + \int_0^\tau 4[\beta^3(s) + 3(\tau - s)\beta(s)]\, d\beta(s)$.

EXAMPLE 2.D. $F(\tau, \beta(\tau)) = a_0(\tau) + \int_0^\tau [\sum_{n=1}^{\infty} n a_n(\tau) H_{n-1}(s, \beta(s))]\, d\beta(s)$.

Although a formula for ϕ_n in (2.2), based on the Fourier-Hermite series of Cameron and Martin [7], appears in Itô [11], in general, these functions are difficult to compute, and the authors know of no elementary algorithms for achieving this. (The reader can consult Lee and Schetzen [14] for a statistical approach used in systems theory.) Consider finding the expansion for $f(\tau) = \max_{s \in [0,\tau]} \beta(s)$.

Given the expansions (2.2) for X and Y, the expansion for $Z = XY$ can be obtained by using Itô's Formula to compute derivatives of products.

EXAMPLE 2.E.

$$\left[\int_{[0,t]} \phi(s)\, d\beta(s)\right]\left[\int_{[0,t]^n} \psi(\tau_1, \cdots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\right]$$
$$= \int_{[0,t]^{n+1}} \phi(s)\psi(\tau_1, \cdots, \tau_n)\, d\beta(s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$$
$$+ n \int_{[0,t]^{n+1}} \int_0^t \phi(s)\, \tilde{\psi}(\tau_1, \cdots, \tau_{n-1}, s)\, ds\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1}).$$

Thus, in a limited sense, one can develop an algebra of these series similar to that of the power series. However, the expansion for a general function $Z = F(X, Y)$ cannot be obtained directly, and thus the versatility of power series is lacking here.

3. Extended Stochastic Integral. In what follows it is necessary to deal with integrals $\int_0^t f(s)\, d\beta(s)$, where $f(s)$ is allowed to be measurable with respect to $\mathscr{F}(t)$ for each $s \leq t$. Here t is being held fixed. One approach is to rely on the orthogonal expansion

$$f(s) = \mathbb{E}f(s) + \sum_{n=1}^{\infty} \int_{[0,t]^n} \phi_n(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n) \tag{3.1}$$

based on (2.2), where $\phi_n(\cdot, s) \in \tilde{L}^2([0, t]^n)$ for each $s \leq t$. In this discussion f is assumed to be mean-square integrable, so that, in addition, $\phi_n \in L^2([0, t]^{n+1})$. Thus attention may be focused on the case $f(s) = \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$, where $\phi(\cdot, s) \in \tilde{L}^2([0, t]^n)$ and $\phi \in L^2([0, t]^{n+1})$. Then, as in Berger and Mizel [3, Section 2], the plan would be to extend the integral via (3.1) to the closure of these processes, $M(t)$, which consists of all mean-square integrable processes measurable with respect to $\mathscr{F}(t)$.

It would seem quite natural to define

$$\hat{\int_0^t} f(s)\, d\beta(s) = \int_{[0,t]^{n+1}} \phi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, d\beta(s), \tag{3.2}$$

and the implications of this definition are to be examined here. However, since Itô's multiple Wiener integral cannot properly be considered an iterated stochastic integral over $[0, t]$, as is apparent from Example 2.E above, it is advantageous to provide an alternate definition, denoted by $\bar{\int}$,

$$\begin{aligned} \bar{\int_0^t} f(s)\, d\beta(s) &= \hat{\int_0^t} f(s)\, d\beta(s) + n \int_0^t \int_{[0,t]^{n-1}} \phi(\tau_1, \cdots, t_{n-1}, s, s)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})\, ds \\ &= \hat{\int_0^t} f(s)\, d\beta(s) + n \int_{[0,t]^{n-1}} \int_0^t \phi(\tau_1, \cdots, \tau_{n-1}, s, s)\, ds\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1}), \end{aligned} \tag{3.3}$$

where $\phi(\cdot, s, s)$ denotes the right-hand limit $\phi(\cdot, s+, s)$.

In order to ensure the existence of $\bar{\int}$ it is assumed that the integrand ϕ has the following properties:

($H1$) The function $s \mapsto \phi(\cdot, s)$ is Riemann integrable (in this paragraph, this is taken to mean Bochner integrable, bounded and a.e. strongly L^2-continuous) over $[0, t]$;

($H2$) For each s, $\phi(\cdot, s)$ has a trace belonging to $\tilde{L}^2([0, t]^{n-1})$ on each $(n-1)$-dimensional hyperplane $\tau_k =$ const. Moreover, right continuity (in $\tilde{L}^2([0, t]^{n-1})$) holds uniformly in s and τ_k, $\phi(\cdot, \tau_k^m, \cdot, s) \to \phi(\cdot, \tau_k, \cdot, s)$ whenever $\tau_k^m \downarrow \tau_k$, and the function $(\tau_k, s) \mapsto \phi(\cdot, \tau_k, \cdot, s)$ is Riemann integrable over $[0, t]^2$;

($H3$) The function $s \mapsto \phi(\cdot, s, s)$ is Riemann integrable over $[0, t]$.

The class of processes in $M(t)$ for which each integrand ϕ_n in (3.1) satisfies ($H1$)–($H3$) and for which the formula (3.3) can be extended by closure in $L^2(\Omega)$ is properly contained in $M(t)$ and will be denoted by $\bar{M}(t)$. This is the class to which $\bar{\int}$ extends.

(Properties ($H1$)–($H3$) certainly hold if $\phi \in W^{2,2}([0, t]^{n+1})$, $n \geq 1$; and if $\sum_{n=1}^{\infty} n^2 \|\phi_n\|^2_{W^{2,2}([0,t]^{n+1})} < \infty$ then the process in (3.1) is in $\bar{M}(t)$. See, for example, Kufner, John and Fucik [13]. However, we are unable to present simple regularity conditions on a process f ensuring that each of its expansion coefficients in (3.1) possesses properties ($H1$)–($H3$).)

There are three important special cases to consider. The first is when $f(s)$ is a nonanticipating stochastic process. In this case $\phi(\cdot, s)$ is supported on $[0, s]^n$, as in (2.4). Because the diagonal has been taken as a right-hand limit it follows that $\bar{\int}$ coincides with $\hat{\int}$. Furthermore, the reader can check using (2.1) that in fact these integrals reduce to Ito's classical stochastic integral.

The second case is when $f(s)$ is measurable with respect to $\mathscr{F}(\tau)$, $\tau < t$, for each $s \leq \tau$, and $f(s) = 0$ for $s > \tau$. In this case ϕ is supported on $[0,\tau]^{n+1}$ and the integrals $\hat{\int}_0^t f(s)\, d\beta(s)$, $\bar{\int}_0^t f(s)\, d\beta(s)$ reduce to $\hat{\int}_0^\tau f(s)\, d\beta(s)$, $\bar{\int}_0^\tau f(s)\, d\beta(s)$, respectively.

The third case is when $f(s)$ is measurable with respect to $\mathscr{F}(s, t)$ for each $s \leq t$. In this case (Berger and Mizel [3, Theorem 2.A]), $\phi(\cdot, s)$ is supported on $[s, t]^n$, and $f(s)$ has an alternate representation

$$f(s) = \int_{T_n(s,t)} \psi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \tag{3.4}$$

where $T_n(s, t) = \{(\tau_1, \cdots, \tau_n): s \leq \tau_1 \leq \cdots \leq \tau_n \leq t\}$. The definitions (3.2) and (3.3) become, respectively,

$$\hat{\int_0^t} f(s)\, d\beta(s) = \int_{T_{n+1}(0,t)} \psi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, d\beta(s), \tag{3.5}$$

$$\begin{aligned} \bar{\int_0^t} f(s)\, d\beta(s) &= \hat{\int_0^t} f(s)\, d\beta(s) + \int_0^t \int_{T_{n-1}(s,t)} \psi(s, \tau_1, \cdots, \tau_{n-1}, s)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})\, ds \\ &= \hat{\int_0^t} f(s)\, d\beta(s) + \int_{T_{n-1}(0,t)} \int_0^{\tau_1} \psi(s, \tau_1, \cdots, \tau_{n-1}, s)\, ds\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1}). \end{aligned} \tag{3.6}$$

Here there is a significant observation. The term

$$\int_{T_{n-1}(s,t)} \psi(s, \tau_1, \cdots, \tau_{n-1}, s)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})$$

is precisely $-\, df(s)/d\beta(s)$, the β-derivative of f (Isaacson [9], Ogawa [15]). Thus

$$\bar{\int_0^t} f(s)\, d\beta(s) = \hat{\int_0^t} f(s)\, d\beta(s) - \int_0^t \frac{df(s)}{d\beta(s)}\, ds. \tag{3.7}$$

It is shown below that this is independently related to Ogawa [16, Theorem 5].

Despite the fact that $\bar{\int}$ does not possess many of the usual probabilistic properties associated with the stochastic integral, whereas $\hat{\int}$ does, nonetheless there are a number of important features which distinguish it. The first is the ease of manipulating and evaluating it, without the necessity of resorting to the expansion (3.1).

THEOREM 3.A. *Let $f \in \bar{M}(t)$ be of the form $f(s) = g(s, \eta)$, where η is a random variable measurable with respect to $\mathscr{F}(t)$, g is jointly measurable in (s, η), Borel measurable in η, and, for each fixed x, $g(s, x)$ is a nonanticipating mean-square continuous stochastic process. Then $\bar{\int}_0^t f(s)\, d\beta(s) = h(\eta)$, where $h(x) = \int_0^t g(s, x)\, d\beta(s)$.*

A proof of this result can be constructed from the expansion (3.1). This approach is adopted in Berger and Mizel [3, Section 4]. However, it is also a corollary of Theorem 3.D below, which provides an alternate proof. Two examples illustrating Theorem 3.A follow.

EXAMPLE 3.B. Let $f(s) = \beta(t)$. Then $f(s) = \int_{[0,t]} d\beta(s)$, and it follows from (3.2) and (3.3) that

$$\hat{\int_0^t} f(s)\, d\beta(s) = \beta^2(t) - t,$$

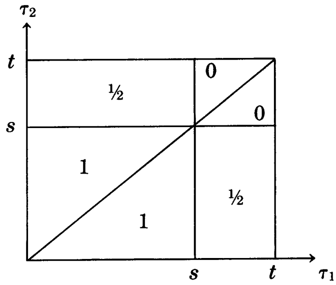
whereas

$$\bar{\int_0^t} f(s)\, d\beta(s) = \beta^2(t) = \beta(t) \int_0^t d\beta(s).$$

EXAMPLE 3.C. Let $f(s) = \beta(s)\beta(t)$. Then

$$f(s) = s + \int_{[0,t]^2} \phi(\tau_1, \tau_2, s)\, d\beta(\tau_1)\, d\beta(\tau_2),$$

where $\phi(\tau_1, \tau_2, s)$ is defined in the figure.



Thus,

$$\hat{\int_0^t} f(s)\, d\beta(s) = \frac{1}{2}\beta^3(t) - \frac{3}{2}t\beta(t) + \int_0^t s\, d\beta(s),$$

whereas

$$\bar{\int_0^t} f(s)\, d\beta(s) = \frac{1}{2}\beta^3(t) - \frac{1}{2}t\beta(t) = \beta(t)\int_0^t \beta(s)\, d\beta(s).$$

It should be noted that although $\bar{\int}_0^t$ is not a martingale in t, whereas $\hat{\int}_0^t$ is, (3.3) does provide its Doob-Meyer decomposition.

Another important feature of $\bar{\int}$ is its representation in terms of limits of Riemann sums. In fact it has the same backward approximating sums as the classical Itô integral.

THEOREM 3.D. *Let $f \in \bar{M}(t)$. Then in the sense of $L^2(\Omega)$-convergence*

$$\bar{\int_0^t} f(s)\, d\beta(s) = \lim_{\delta\downarrow 0} \sum_{k=0}^{m-1} f(t_k)[\beta(t_{k+1}) - \beta(t_k)],$$

where $0 = t_0 \le \cdots \le t_m = t$ *and* $\delta = \max_{0\le k\le m-1}(t_{k+1} - t_k)$.

The proof relies on the following two results, the first being combinatoric in nature, and the second analytic.

LEMMA 3.E. *The following identities hold for symmetric functions ϕ, ψ and deterministic functions h, k, θ.*

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(t_n)\right]\left[\int_0^t h(s)\, d\beta(s)\right] \\
(3.8)\qquad &\times \left[\int_{[0,t]^n} \psi(\tau_1, \cdots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\right]\left[\int_0^t k(s)\, d\beta(s)\right] \\
&= (n+1)! \int_{[0,t]^{n+1}} \widehat{\phi(\tau_1, \cdots, \tau_n)h(\tau_{n+1})}\,\widehat{\psi(\tau_1, \cdots, \tau_n)k(\tau_{n+1})}\, d\tau_1 \cdots d\tau_{n+1} \\
&+ nn! \int_{[0,t]^{n-1}} \left[\int_0^t \phi(\tau_1, \cdots, \tau_{n-1}, s)h(s)\, ds\right]\left[\int_0^t \psi(\tau_1, \cdots, \tau_{n-1}, s)k(s)\, ds\right] d\tau_1 \cdots d\tau_{n-1}.
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left[\int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\right]\left[\int_0^t h(s)d\beta(s)\right] \\
&\times \left[\int_{[0,t]^{n+1}} \theta(\tau_1, \cdots, \tau_{n+1})\, d\beta(\tau_1) \cdots d\beta(\tau_{n+1})\right] \\
&= (n+1)! \int_{[0,t]^{n+1}} \widetilde{\phi(\tau_1, \cdots, \tau_n)h(\tau_{n+1})}\tilde{\theta}(\tau_1, \cdots, \tau_{n+1})\, d\tau_1 \cdots d\tau_{n+1} \\
&= (n+1)! \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n)\left[\int_0^t \widetilde{\theta(\tau_1, \cdots, \tau_{n+1})h(\tau_{n+1})}\, d\tau_{n+1}\right] d\tau_1 \cdots d\tau_n.
\end{aligned} \tag{3.9}
$$

PROOF. These identities follow directly from those contained in Itô [11], in particular Theorem 2.2 there. (See also Example 2.E above.) Details are elaborated on in Trutzer [19]. □

LEMMA 3.F. *Let* $\Pi : 0 = t_0 \le \cdots \le t_m = t$ *be a partition of* $[0, t]$ *and let* $\delta = \text{mesh}(\Pi) = \max_{0\le k\le m-1}(t_{k+1} - t_k)$. *Assume that* ϕ *satisfies (H1)–(H3). Let* $\bar{t}_k \in [t_k, t_{k+1}]$, $k = 0, \cdots, m-1$ *and consider the Riemann approximation* $\phi_\Pi = \sum_{k=0}^{m-1} \phi(\cdot, s, \bar{t}_k) I_k(s)$, *where* I_k *denotes the indicator function of the set* (t_k, t_{k+1}). *Then* $\lim_{\delta\downarrow 0} \phi_\Pi = \phi(\cdot, s, s)$ *in the sense of* $L^2([0, t]^n)$.

PROOF. This result follows from a "Duhamel-type" argument, as discussed in Olmstead [17]. Details are elaborated on in Trutzer [19]. □

PROOF OF THEOREM 3.D. It suffices to consider the case $f(s) = \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$, where $\phi(\cdot, s) \in \tilde{L}^2([0, t]^n)$ and $\phi \in L^2([0, t]^{n+1})$. Then it follows from Lemma 3.E and Lemma 3.F that

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \mathbb{E}\left|\textstyle\sum_{k=0}^{m-1} f(t_k)[\beta(t_{k+1}) - \beta(t_k)]\right|^2 \\
&= (n+1)! \int_{[0,t]^{n+1}} \tilde{\phi}^2(\tau_1, \cdots, \tau_{n+1})\, d\tau_1 \cdots d\tau_{n+1} \\
&\quad + nn! \int_{[0,t]^{n-1}} \left|\int_0^t \phi(\tau_1, \cdots, \tau_{n-1}, s, s)\, ds\right|^2 d\tau_1 \cdots d\tau_{n-1}.
\end{aligned} \tag{3.10}
$$

The details involved in this derivation are as follows. Let I_k denote the indicator function of the set (t_k, t_{k+1}). By virtue of (3.8)

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \mathbb{E}\left|\textstyle\sum_{k=0}^{m-1} f(t_k)[\beta(t_{k+1}) - \beta(t_k)]\right|^2 \\
&= (n+1)! \lim_{\delta\downarrow 0} \int_{[0,t]^{n+1}} \left|\widetilde{\textstyle\sum_{k=0}^{m-1} \phi(\tau_1, \cdots, \tau_n, t_k) I_k(\tau_{n+1})}\right|^2 d\tau_1 \cdots d\tau_{n+1} \\
&\quad + nn! \lim_{\delta\downarrow 0} \int_{[0,t]^{n-1}} \left|\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \phi(\tau_1, \cdots, \tau_{n-1}, s, t_k)\, ds\right|^2 d\tau_1 \cdots d\tau_{n-1}.
\end{aligned}
$$

Using the fact that symmetrization is a bounded operator (in fact it is an orthogonal projection), it follows from Lemma 3.F that

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \int_{[0,t]^{n+1}} \left|\widetilde{\textstyle\sum_{k=0}^{m-1} \phi(\tau_1, \cdots, \tau_n, t_k) I_k(\tau_{n+1})}\right|^2 d\tau_1 \cdots d\tau_{n+1} \\
&= \int_{[0,t]^{n+1}} \tilde{\phi}^2(\tau_1, \cdots, \tau_{n+1})\, d\tau_1 \cdots d\tau_{n+1}.
\end{aligned}
$$

Similarly it follows that

$$\lim_{\delta\downarrow 0}\int_{[0,t]^{n-1}}\left|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\phi(\tau_1,\cdots,\tau_{n-1},s,t_k)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}$$

$$=\int_{[0,t]^{n-1}}\left|\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}.$$

Using analogous steps the reader can also derive the following results.

$$\lim_{\delta\downarrow 0}\mathbb{E}\{\textstyle\sum_{k=0}^{m-1}f(t_k)[\beta(t_{k+1})-\beta(t_k)]\}\displaystyle\int_{[0,t]^{n+1}}\phi(\tau_1,\cdots,\tau_{n+1})\,d\beta(\tau_1)\cdots d\beta(\tau_{n+1})$$

$$=(n+1)!\int_{[0,t]^{n+1}}\tilde{\phi}^2(\tau_1,\cdots,\tau_{n+1})\,d\tau_1\cdots d\tau_{n+1},\tag{3.11}$$

$$\lim_{\delta\downarrow 0}\mathbb{E}\{\textstyle\sum_{k=0}^{m-1}(t_k)[\beta(t_{k+1})-\beta(t_k)]\}$$

$$\times\int_{[0,t]^{n-1}}\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\,d\beta(\tau_1)\cdots d\beta(\tau_{n-1})$$

$$=n!\int_{[0,t]^{n-1}}\left|\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}.\tag{3.12}$$

From (3.3), (3.10), (3.11), (3.12) it follows that

$$\lim_{\delta\downarrow 0}\mathbb{E}\left|\textstyle\sum_{k=0}^{m-1}f(t_k)[\beta(t_{k+1})-\beta(t_k)]-\displaystyle\bar{\int_0^t}f(s)\,d\beta(s)\right|^2$$

$$=(n+1)!\int_{[0,t]^{n+1}}\tilde{\phi}^2(\tau_1,\cdots,\tau_{n+1})\,d\tau_1\cdots d\tau_{n+1}$$

$$+nn!\int_{[0,t]^{n-1}}\left|\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}$$

$$-2(n+1)!\int_{[0,t]^{n+1}}\tilde{\phi}^2(\tau_1,\cdots,\tau_{n+1})\,d\tau_1\cdots d\tau_{n+1}$$

$$+(n+1)!\int_{[0,t]^{n+1}}\tilde{\phi}^2(\tau_1,\cdots,\tau_{n+1})\,d\tau_1\cdots d\tau_{n+1}$$

$$-2nn!\int_{[0,t]^{n-1}}\left|\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}$$

$$+nn!\int_{[0,t]^{n-1}}\left|\int_0^t\phi(\tau_1,\cdots,\tau_{n-1},s,s)\,ds\right|^2 d\tau_1\cdots d\tau_{n-1}$$

$$=0.$$ □

In a similar vein one can establish the following result concerning $\hat{\int}$.

THEOREM 3.G. *Let $f\in M(t)$, and let $f(s)$ be measurable with respect to $\mathscr{F}(s,t)$ for each $s\le t$. Then in the sense of $L^2(\Omega)$-convergence*

$$\hat{\int_0^t} f(s)\, d\beta(s) = \lim_{\delta\downarrow 0} \textstyle\sum_{k=0}^{m-1} f(t_{k+1})[\beta(t_{k+1}) - \beta(t_k)],$$

where $0 = t_0 \leq \cdots \leq t_m = t$ *and* $\delta = \max_{0\leq k\leq m-1}(t_{k+1} - t_k)$.

At this point the special case (3.7) becomes clearer. Define a Brownian motion β^* on $[0, t]$ by $\beta^*(s) = \beta(t) - \beta(t - s)$, and define $f^*(s) = f(t - s)$. Then f^* is a nonanticipating stochastic process with respect to β^*. Thus the integral $\int_0^t f^*(s)\, d\beta^*(s)$ is a classical Itô integral. It corresponds to $\hat\int_0^t f(s)\, d\beta(s)$ or $\bar\int_0^t f(s)\, d\beta(s)$ under the transformation $s \to t - s$. According to Theorem 3.G the evaluation of $\hat\int_0^t f(s)\, d\beta(s)$ carries over to the evaluation of $\int_0^t f^*(s)\, d\beta^*(s)$ by $\lim_{\delta\downarrow 0} \sum_{k=0}^{m-1} f^*(t'_k)[\beta^*(t'_{k+1}) - \beta^*(t'_k)]$, where $t'_k = t - t_k$. The basic observation here is that the time reversal $s \to t - s$ takes forward approximating sums into backward sums. Similarly, according to Theorem 3.D the evaluation of $\bar\int_0^t f(s)\, d\beta(s)$ carries over to the evaluation of $\int_0^t f^*(s)\, d\beta^*(s)$ by $\lim_{\delta\downarrow 0} \sum_{k=1}^{m-1} f^*(t'_{k+1})[\beta^*(t'_{k+1}) - \beta^*(t'_k)]$. By Theorem 5 in Ogawa [16], the difference $\bar\int_0^t f(s)\, d\beta(s) - \hat\int_0^t f(s)\, d\beta(s)$ is given by $\int_0^t (df^*(s)/d\beta^*(s))\, ds$. Furthermore, $(df^*(s)/d\beta^*(s)) = -df(t - s)/d\beta(t - s)$. (See Berger and Mizel [3, Section 6].)

As outlined in Berger and Mizel [2], Theorem 3.D provides geometric insight for the investigation of iterated stochastic integration and results of Fubini-type. The following theorem, regarding different orders of integration over a triangular domain, is referred to as the Correction Formula.

THEOREM 3.H. *Let* $g(s, \tau)$ *be nonanticipating in the second parameter. Then*

$$\bar\int_0^t \int_s^t g(s, \tau)\, d\beta(\tau)\, d\beta(s) = \int_0^t \bar\int_0^\tau g(s, \tau)\, d\beta(s)\, d\beta(\tau) + \int_0^t g(s, s)\, ds,$$

whenever these integrals exist. Here $g(s, s)$ *is the* L^2 *limit* $g(s, s) = \lim_{\tau\downarrow s} g(s, \tau)$.

PROOF. Extend g to be zero for $s \geq \tau$. Then $\int_s^t g(s, \tau)\, d\beta(\tau) = \int_0^t g(s, \tau)\, d\beta(\tau)$ and, by the remark in Section 3, $\bar\int_0^\tau g(s, \tau)\, d\beta(s) = \bar\int_0^t g(s, \tau)\, d\beta(s)$. Furthermore, by the other remark there, $\bar\int$ is equal to $\int$ whenever the integrand is nonanticipating. Thus, in order to prove the theorem, it certainly suffices to establish the more general result that

$$\bar\int_0^t \bar\int_0^t h(s, \tau)\, d\beta(\tau)\, d\beta(s) + \int_0^t h^-(\tau, \tau)\, d\tau$$

(3.13)

$$= \bar\int_0^t \bar\int_0^t h(s, \tau)\, d\beta(s)\, d\beta(\tau) + \int_0^t h^+(s, s)\, ds,$$

for processes $h(s, \tau)$ which are measurable merely with respect to $\mathscr{F}(t)$ for each s, τ. Here $h^-(\tau, \tau)$ and $h^+(s, s)$ are the respective L^2 limits; $h^-(\tau, \tau) = \lim_{s\downarrow\tau} h(s, \tau)$, $h^+(s, s) = \lim_{\tau\downarrow s} h(s, \tau)$. Of course, in the theorem $g^- = 0$ since g is zero for $s \geq \tau$.

It suffices to consider $h(s, \tau) = \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$, $\phi(\cdot, s, \tau) \in \tilde{L}^2([0, t]^n)$ for each s, τ and $\phi \in L^2([0, t]^{n+2})$. Define ϕ_1 and ϕ_2 on $[0, t]^{n+2}$ by

(3.14)

$$\phi_1(\tau_1, \cdots, \tau_n, s, \tau) = \frac{1}{n} \textstyle\sum_{k=1}^n \phi(\tau_1, \cdots, \tau_{k-1}, s, \tau_{k+1}, \cdots, \tau_n, \tau_k, \tau),$$

$$\phi_2(\tau_1, \cdots, \tau_n, s, \tau) = \frac{1}{n} \textstyle\sum_{k=1}^n \phi(\tau_1, \cdots, \tau_{k-1}, \tau, \tau_{k+1}, \cdots, \tau_n, s, \tau_k).$$

Now proceed to evaluate the first double integral in (3.13).

$$\bar{\int}_0^t \bar{\int}_0^t h(s, \tau)\, d\beta(\tau)\, d\beta(s)$$

$$= \int_{[0,t]^{n+2}} \phi(\tau_1, \cdots, \tau_n, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, d\beta(\tau)\, d\beta(s)$$

$$+ n \int_0^t \int_{[0,t]^n} [\phi_2(\tau_1, \cdots, \tau_n, s, s) + \phi(\tau_1, \cdots, \tau_{n-1}, s, \tau_n, s)]\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, ds \tag{3.15}$$

$$+ n(n-1) \int_0^t \int_0^t \int_{[0,t]^{n-2}} \phi(\tau_1, \cdots, \tau_{n-2}, s, \tau, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-2})\, d\tau\, ds$$

$$+ \int_0^t h^+(s, s)\, ds.$$

Furthermore,

$$\int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_{n-1}, s, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$$

$$= \int_{[0,t]^n} \phi_1(\tau_1, \cdots, \tau_n, s, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n), \tag{3.16}$$

since the second integrand is but the symmetrization of the first in the variables of integration. Thus if the order of integration on the left-hand side of (3.15) is reversed, all of the terms on the right-hand side remain the same except the last, which becomes $\int_0^t h^-(\tau, \tau)\, d\tau$. □

Theorem 3.H suggests an alternate method of proving Theorem 3.D, as adopted in Berger and Mizel [4]. The proof proceeds by induction. If $f(s)$ is deterministic the result follows from the fact that $\bar{\int}$ coincides with the classical Itô integral. Suppose $f(s) = \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$. Then, by Theorem 3.H,

$$\bar{\int}_0^t f(s)\, d\beta(s) = \int_0^t \int_0^s g(s, \tau)\, d\beta(\tau)\, d\beta(s) + \bar{\int}_0^t \int_s^t g(s, \tau)\, d\beta(\tau)\, d\beta(s)$$

$$= \int_0^t \int_0^s g(s, \tau)\, d\beta(\tau)\, d\beta(s) + \int_0^t \bar{\int}_0^\tau g(s, \tau)\, d\beta(s)\, d\beta(\tau) + \int_0^t g(s, s)\, ds, \tag{3.17}$$

where $g(s, \tau) = n \int_{[0,\tau]^{n-1}} \phi(\tau_1, \cdots, \tau_{n-1}, \tau, s)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})$. Let $0 = t_0 \le \cdots \le t_m = t$. Then, since g is a multiple Wiener integral of order $n - 1$, it follows from the induction hypothesis that the $L^2(\Omega)$ approximating sums for the two double integrals on the right-hand side of (3.17) are

$$\sum_{k=0}^{m-1} \sum_{j=0}^{k-1} g(t_k, t_j)[\beta(t_{j+1}) - \beta(t_j)][\beta(t_{k+1}) - \beta(t_k)]$$
$$+ \sum_{k=0}^{m-1} \sum_{j=0}^{k-1} g(t_j, t_k)[\beta(t_{k+1}) - \beta(t_k)][\beta(t_{j+1}) - \beta(t_j)].$$

Furthermore, since $g(s, s)$ is a nonanticipating stochastic process, the diagonal sums $\sum_{k=0}^{m-1} g(t_k, t_k)[\beta(t_{k+1}) - \beta(t_k)]^2$ approximate $\int_0^t g(s, s)\, ds$ in $L^2(\Omega)$. Thus $\bar{\int}_0^t f(s)\, d\beta(s)$ is approximated in $L^2(\Omega)$ by

$$\sum_{k=0}^{m-1} \sum_{j=0}^{m-1} g(t_k, t_j)[\beta(t_{j+1}) - \beta(t_j)][\beta(t_{k+1}) - \beta(t_k)]. \tag{3.18}$$

On the other hand, $f(s)$ is approximated in $L^2(\Omega)$ by $\sum_{j=0}^{m-1} g(s, t_j)[\beta(t_{j+1}) - \beta(t_j)]$, and thus $\sum_{k=0}^{m-1} f(t_k)[\beta(t_{k+1}) - \beta(t_k)]$ is also approximated in $L^2(\Omega)$ by (3.18).

The following result is an analogue of Theorem 3.H for $\hat{\int}$. Its proof is a simpler version of the proof of Theorem 3.H, and is left to the reader.

THEOREM 3.I. *Let $g(s, \tau)$ be nonanticipating in the second parameter. Then*

$$\hat{\int_0^t} \int_s^t g(s, \tau)\, d\beta(\tau)\, d\beta(s) = \int_0^t \hat{\int_0^\tau} g(s, \tau)\, d\beta(s)\, d\beta(\tau),$$

whenever these integrals exist.

Again, Theorem 3.I is motivated intuitively from the lattice point geometry suggested by Theorem 3.G. (See the discussions in Berger and Mizel [2], [4].) At the same time, Theorem 3.I suggests an alternate proof of Theorem 3.G, as follows. If $f(s)$ is deterministic the result is clear. Suppose, based on (3.4), that $f(s) = \int_{T_n(s,t)} \phi(\tau_1, \cdots, \tau_n, s)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$. Then, by Theorem 3.I,

$$\hat{\int_0^t} f(s)\, d\beta(s) = \hat{\int_0^t} \int_s^t g(s, \tau)\, d\beta(\tau)\, d\beta(s) = \int_0^t \hat{\int_0^\tau} g(s, \tau)\, d\beta(s)\, d\beta(\tau), \tag{3.19}$$

where $g(s, \tau) = \int_{T_n(s,\tau)} \phi(\tau_1, \cdots, \tau_{n-1}, \tau, s)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})$. Let $0 = t_0 \le \cdots \le t_m = t$. Then since g is an iterated stochastic integral of order $n - 1$, it follows from the induction hypothesis that the $L^2(\Omega)$ approximating sum for the right-hand side of (3.19) is

$$\sum_{j=0}^{m-1} \sum_{k=0}^{j-1} g(t_{k+1}, t_j)[\beta(t_{j+1}) - \beta(t_j)][\beta(t_{k+1}) - \beta(t_k)]. \tag{3.20}$$

On the other hand, $f(t_{k+1})$ is approximated in $L^2(\Omega)$ by $\sum_{j=k+1}^{m-1} g(t_{k+1}, t_j)[\beta(t_{j+1}) - \beta(t_j)]$, and thus $\sum_{k=0}^{m-1} f(t_{k+1})[\beta(t_{k+1}) - \beta(t_k)]$ is also approximated in $L^2(\Omega)$ by (3.20).

This section concludes with a lemma to be referred to later, also illustrating the usefulness of $\bar{\int}$.

LEMMA 3.J. *Let $g(s, \tau)$ be nonanticipating in the first parameter, let $f(s)$ be measurable with respect to $\mathscr{F}(t)$ for each s. Then*

$$\int_0^t f(\tau) \int_0^\tau g(s, \tau)\, d\beta(s)\, d\tau = \bar{\int_0^t} \int_s^t g(s, \tau) f(\tau)\, d\tau\, d\beta(s),$$

whenever these integrals exist.

PROOF. Extend $g(s, \tau)$ to be zero for $s \ge \tau$. Then, by Theorem 3.A, $f(\tau) \int_0^t g(s, \tau)\, d\beta(s) = \bar{\int_0^t} g(s, \tau) f(\tau)\, d\beta(s)$. Thus it suffices to establish the more general result that

$$\int_0^t \bar{\int_0^t} h(s, \tau)\, d\beta(s)\, d\tau = \bar{\int_0^t} \int_0^t h(s, \tau)\, d\tau\, d\beta(s), \tag{3.21}$$

where $h(s, \tau)$ is measurable with respect to $\mathscr{F}(t)$ for each s, τ. Following the argument in the proof of Theorem 3.G, let $h(s, \tau) = \int_{[0,t]^n} \phi(\tau_1, \cdots, \tau_n, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_n)$, where $\phi(\cdot, s, \tau) \in \tilde{L}^2([0, t]^n)$ for each s, τ and $\phi \in L^2([0, t]^{n+2})$. Then

$$\bar{\int_0^t} \int_0^t h(s, \tau)\, d\tau\, d\beta(s) = \int_{[0,t]^{n+1}} \int_0^t \phi(\tau_1, \cdots, \tau_n, s, \tau)\, d\tau\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, d\beta(s)$$

$$+ n \int_0^t \int_{[0,t]^{n-1}} \int_0^t \phi(\tau_1, \cdots, \tau_{n-1}, s, s, \tau)\, d\tau\, d\beta(\tau_1) \cdots d\beta(\tau_{nms1})\, ds$$

$$(3.22)\quad = \int_0^t \int_{[0,t]^{n+1}} \phi(\tau_1, \cdots, \tau_n, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_n)\, d\beta(s)\, d\tau$$

$$+ n \int_0^t \int_0^t \int_{[0,t]^{n-1}} \phi(\tau_1, \cdots, \tau_{n-1}, s, s, \tau)\, d\beta(\tau_1) \cdots d\beta(\tau_{n-1})\, ds\, d\tau$$

$$= \int_0^t \int_0^t h(s, \tau)\, d\beta(s)\, d\tau.$$

□

It seems from this lemma that $\bar{\int}$ arises in a very natural way from the classical Itô integral. This is because both integrals on the left-hand side of the equation in the lemma are classical integrals, and the only occurrence of any extended integral is on the right-hand side. In general, this lemma does not hold for $\hat{\int}$.

It is clear now that there are a number of results, concerning the commutativity of $\hat{\int}$ and $\bar{\int}$ with other integrals, that can be proven. A few are listed below. However, only those established above are necessary in connection with the material to follow. In a later paper, Berger and Mizel [6], these extended integrals will be more thoroughly discussed, with emphasis on their analytic and probabilistic properties, their β-derivatives, and their relationship to other integrals and to the different versions of the homogeneous chaos developed by Wiener [20] and Itô [11]. The above presentation is intended to be brief, merely to allow the discussion of stochastic integral equations to follow.

A few results concerning $\hat{\int}$ and $\bar{\int}$ are stated below. It is assumed that $g(s, \tau)$ is nonanticipating in the second parameter, and that $f(s)$ and $h(s, \tau)$ are measurable with respect to $\mathscr{F}(t)$ for each s, τ.

$$(3.23)\quad \bar{\int_0^t} \bar{\int_s^t} g(s, \tau) f(s)\, d\beta(\tau)\, d\beta(s) = \bar{\int_0^t} f(s) \bar{\int_0^\tau} g(s, \tau)\, d\beta(s)\, d\beta(\tau) + \int_0^t f(s) g(s, s)\, ds.$$

$$(3.24)\quad \hat{\int_0^t} \hat{\int_s^t} g(s, \tau) f(s)\, d\beta(\tau)\, d\beta(s) = \hat{\int_0^t} f(s) \hat{\int_0^\tau} g(s, \tau)\, d\beta(s)\, d\beta(\tau).$$

$$(3.25)\quad \hat{\int_0^t} \hat{\int_0^t} h(s, \tau)\, d\beta(\tau)\, d\beta(s) = \hat{\int_0^t} \hat{\int_0^t} h(s, \tau)\, d\beta(s)\, d\beta(\tau).$$

$$(3.26)\quad \int_0^t \bar{\int_s^t} h(s, \tau)\, d\beta(s)\, d\tau = \bar{\int_0^t} \int_0^\tau h(s, \tau)\, d\tau\, d\beta(s).$$

$$(3.27)\quad \int_0^t \hat{\int_s^t} h(s, \tau)\, d\beta(s)\, d\tau = \hat{\int_0^t} \int_0^\tau h(s, \tau)\, d\tau\, d\beta(s).$$

$$(3.28)\quad \hat{\int_0^t} \int_0^t h(s, \tau)\, d\tau\, d\beta(s) = \int_0^t \hat{\int_0^t} h(s, \tau)\, d\beta(s)\, d\tau.$$

$$(3.29)\quad \frac{d}{d\beta(t)} \bar{\int_0^t} g(s, t)\, d\beta(s) = g(t, t) + \bar{\int_0^t} \frac{\partial}{\partial\beta(t)} g(s, t)\, d\beta(s).$$

$$(3.30)\quad \frac{d}{d\beta(t)} \hat{\int_0^t} g(s, t)\, d\beta(s) = g(t, t) + \hat{\int_0^t} \frac{\partial}{\partial\beta(t)} g(s, t)\, d\beta(s).$$

4. Linear Stochastic Integral Equations. The basic equation to be considered is

$$(I - \lambda T)x(t) = f(t), \tag{4.1}$$

where

$$Tx(t) = \int_0^t \sigma(s, t)x(s)\, d\beta(s) + \int_0^t b(s, t)x(s)\, ds. \tag{4.2}$$

Here $\sigma(s, t)$ and $b(s, t)$ are nonanticipating in the first and second parameters, respectively, and $f(t)$ is nonanticipating. A discussion of this equation regarding existence, uniqueness and the convergence of the successive approximants appears in Berger and Mizel [4, Section 3], [5, Section 9] and [3, Section 5]. The successive approximants are defined by $x_n(t) = \sum_{k=0}^n \lambda^k T^k f(t)$. Define the resolvent approximants iteratively by $\sigma_1(s, t) = \sigma(s, t)$, $b_1(s, t) = b(s, t)$, and

$$\begin{aligned} \sigma_{n+1}(s, t) &= \int_s^t \sigma_n(s, \tau)\sigma(\tau, t)\, d\beta(\tau) + \int_s^t \sigma_n(s, \tau)b(\tau, t)\, d\tau, \\ b_{n+1}(s, t) &= \int_s^t b_n(s, \tau)\sigma(\tau, t)\, d\beta(\tau) + \int_s^t b_n(s, \tau)b(\tau, t)\, d\tau. \end{aligned} \tag{4.3}$$

Then it follows from Theorem 3.I that

$$x_n(t) = f(t) + \hat{\int_0^t} [\textstyle\sum_{k=1}^n \lambda^k \sigma_k(s, t)]f(s)\, d\beta(s) + \int_0^t [\textstyle\sum_{k=1}^n \lambda^k b_k(s, t)]f(s)\, ds. \tag{4.4}$$

Thus if $\sigma_\lambda(s, t) = \sum_{k=1}^\infty \lambda^{k-1}\sigma_k(s, t)$ and $b_\lambda(s,t) = \sum_{k-1}^\infty \lambda^{k-1} b_k(s, t)$,

$$x(t) = (I - \lambda T)^{-1} f(t) = (I + \lambda \hat{T}_\lambda) f(t), \tag{4.5}$$

where

$$\hat{T}_\lambda f(t) = \hat{\int_0^t} \sigma_\lambda(s, t)f(s)\, d\beta(s) + \int_0^t b_\lambda(s, t)f(s)\, ds. \tag{4.6}$$

Suppose σ and f are deterministic, and that $b(s, t)$ is measurable with respect to $\mathscr{F}(s, t)$ for each s, t. Then $\sigma_\lambda(s, t)$ is measurable with respect to $\mathscr{F}(s, t)$ for each $s, \bar{t}$. Furthermore, $d\sigma_{n+1}(s, t)/d\beta(s) = -\sigma(s, s)\sigma_n(s, t)$, so that $d\sigma_\lambda(s, t)/d\beta(s) = -\lambda\sigma(s, s)\sigma_\lambda(s, t)$. Thus it follows from (3.7) that

$$\hat{T}_\lambda f(t) = \bar{T}_\lambda f(t) - \lambda R_\lambda f(t), \tag{4.7}$$

where

$$\bar{T}_\lambda f(t) = \bar{\int_0^t} \sigma_\lambda(s, t)f(s)\, d\beta(s) + \int_0^t b_\lambda(s, t)f(s)\, ds, \tag{4.8}$$

$$R_\lambda f(t) = \int_0^t \sigma(s, s)\sigma_\lambda(s, t)f(s)\, ds. \tag{4.9}$$

In fact, (4.7) holds in general. This follows from Theorem 3.H applied to (4.4). According to Theorem 3.A, this facilitates the evaluation of $x(t)$, as demonstrated in the example below.

EXAMPLE 4.A. Suppose $\sigma(s, t) = \theta(s)\eta(t)$ and $b = 0$. Then

$$\sigma_n(s, t) = \frac{1}{(n-1)!}\sigma(s, t)H_{n-1}\left(\int_s^t \sigma(\tau, \tau)\, d\beta(\tau), \int_s^t |\sigma(\tau, \tau)|^2\, d\tau\right),$$

where $H_n(s, t)$ is the Hermite polynomial of degree n. Thus

$$\sigma_\lambda(s, t) = \sigma(s, t)\exp[\lambda \int_s^t \sigma(\tau,\tau)\, d\beta(\tau) - 1/2\lambda^2 \int_s^t |\sigma(\tau,\tau)|^2\, d\tau.]$$

Let $a(t) = \exp[\lambda \int_0^t \sigma(\tau,\tau)\, d\beta(\tau) - 1/2\lambda^2 \int_0^t |\sigma(\tau,\tau)|^2\, d\tau]$. Then it follows from (4.7) that

$$x(t) = f(t) + \lambda a(t) \int_0^t \frac{\sigma(s, t)}{a(s)} f(s)\, d\beta(s) - \lambda^2 \int_0^t \sigma(s, s)\sigma_\lambda(s, t) f(s)\, ds.$$

This can be checked directly by differentiating (4.1) and converting it into a stochastic differential equation, as in Berger [1, Section 4].

It is clear from Section 3 above that (4.1) can be generalized to allow $\sigma(s, t)$ to be nonanticipating in the second parameter. If the stochastic integral in (4.2) is $\hat{\int}$, then (4.6) holds. If the integral is $\bar{\int}$, then (4.7) holds. In general, (4.6) and (4.7) do not both hold.

There are two interesting transformations to apply to (4.1). Decompose T into $T_1 + T_2$, where $T_1 x(t) = \int_0^t \sigma(s, t) x(s)\, d\beta(s)$ and $T_2 x(t) = \int_0^t b(s, t) x(s)\, ds$. Let $b_\lambda^*(s, t)$ be the resolvent kernel for T_2, defined by

(4.10)
$$b_\lambda^*(s, t) = \sum_{n=1}^\infty \lambda^{n-1} b_n^*(s, t),$$
$$b_1^*(s, t) = b(s, t), \qquad b_{n+1}^*(s, t) = \int_s^t b_n^*(s,\tau) b(\tau, t)\, d\tau,$$

or, equivalently,

(4.11)
$$b_\lambda^*(s, t) = b(s, t) + \lambda \int_s^t b_\lambda^*(s, \tau) b(\tau, t)\, d\tau.$$

Then $(I - \lambda T_2)^{-1} y(t) = (I + \lambda T_{2,\lambda}) y(t)$, where $T_{2,\lambda} y(t) = \int_0^t b_\lambda^*(s, t) y(s)\, ds$. Define

(4.12)
$$\bar{A}(\lambda) x(t) = (I - \lambda T_2)^{-1} T_1 x(t) = \bar{\int_0^t} \tilde{\sigma}_\lambda(s, t) x(s)\, d\beta(s),$$
$$\tilde{\sigma}_\lambda(s, t) = \sigma(s, t) + \lambda \int_s^t \sigma(s, \tau) b_\lambda^*(\tau, t)\, d\tau.$$

By applying $(I - \lambda T_2)^{-1}$ to (4.1) and making use of Lemma 3.J it follows that

(4.13)
$$[I - \lambda \bar{A}(\lambda)] x(t) = g_\lambda(t),$$

where $g_\lambda(t) = (I - \lambda T_2)^{-1} f(t)$. Next let $\tilde{\sigma}_\lambda^\#(s, t)$ be the resolvent kernel for $\bar{A}(\lambda)$, defined by

(4.14)
$$\tilde{\sigma}_\lambda^\#(s, t) = \sum_{n=1}^\infty \lambda^{n-1} \tilde{\sigma}_{\lambda,n}^\#(s, t),$$
$$\tilde{\sigma}_{\lambda,1}^\#(s, t) = \tilde{\sigma}_\lambda(s, t), \qquad \tilde{\sigma}_{\lambda,n+1}^\#(s, t) = \int_s^t \tilde{\sigma}_{\lambda,n}^\#(s, \tau) \tilde{\sigma}_\lambda(\tau, t)\, d\beta(\tau).$$

Then

(4.15)
$$x(t) = [I - \lambda \bar{A}(\lambda)]^{-1} g_\lambda(t) = (I + \lambda \bar{A}_\lambda - \lambda^2 \tilde{R}_\lambda) g_\lambda(t),$$

where $\bar{A}_\lambda g_\lambda(t) = \bar{\int_0^t} \tilde{\sigma}_\lambda^\#(s, t) g_\lambda(s)\, d\beta(s)$ and $R_\lambda g_\lambda(t) = \int_0^t \tilde{\sigma}(s, s)\, \tilde{\sigma}_\lambda^\#(s, t) g_\lambda(s)\, ds$.

Similarly, let $\sigma_\lambda^\#(s, t)$ be the resolvent kernel for T_1 (defined analogously to $\tilde{\sigma}_\lambda^\#(s, t)$ above). Then $(I - \lambda T_1)^{-1} y(t) = (I + \lambda \bar{T}_{1,\lambda} - \lambda^2 \tilde{R}'_\lambda) y(t)$, where $\bar{T}_{1,\lambda} y(t) = \bar{\int_0^t} \sigma_\lambda^\#(s, t) y(s)\, d\beta(s)$ and $\tilde{R}'_\lambda y(t) = \int_0^t \sigma(s, s) \sigma_\lambda^\#(s, t) y(s)\, ds$. Define

(4.16)
$$\bar{B}(\lambda)x(t) = (I - \lambda T_1)^{-1}T_2 x(t) = \int_0^t \tilde{b}_\lambda(s, t)x(s)\, ds,$$
$$\tilde{b}_\lambda(s, t) = \bar{b}(s, t) + \lambda \int_s^t b(s, \tau)\sigma_\lambda^{\#}(\tau, t)\, d\beta(\tau) - \lambda^2 \int_s^t b(s, \tau)\sigma(\tau, \tau)\sigma_\lambda^{\#}(\tau, t)\, d\tau.$$

By applying $(I - \lambda T_1)^{-1}$ to (4.1) and making use of Lemma 3.J it follows that

$$[I - \lambda\bar{B}(\lambda)]x(t) = h_\lambda(t), \tag{4.17}$$

where $h_\lambda(t) = (I - \lambda T_1)^{-1}f(t)$. Next let $\tilde{b}_\lambda^*(s, t)$ be the resolvent kernel for $\bar{B}(\lambda)$ (defined analogously to $b_\lambda^*(s, t)$ above). Then

$$x(t) = [I - \lambda\bar{B}(\lambda)]^{-1}h_\lambda(t) = (I + \lambda\bar{B}_\lambda)h_\lambda(t), \tag{4.18}$$

where $\bar{B}_\lambda h_\lambda(t) = \int_0^t \tilde{b}_\lambda^*(s, t)h_\lambda(s)\, ds$.

It is also of interest to examine (4.1) with respect to norms. Suppose that σ, b and f are all deterministic. Then if $m(t) = \mathbb{E}\,|x(t)|^2$ and $A_2(\lambda)m(t) = \int_0^t |\tilde{\sigma}_\lambda(s, t)|^2 m(s)\, ds$, it follows that

$$[I - \lambda^2 A_2(\lambda)]m(t) = |g_\lambda(t)|^2 = |\,\mathbb{E}\,x(t)\,|^2. \tag{4.19}$$

Equivalently,

$$\mathbb{E}\,|[I - \lambda\bar{A}(\lambda)]^{-1}g_\lambda(t)|^2 = [I - \lambda^2 A_2(\lambda)]^{-1}|g_\lambda(t)|^2. \tag{4.20}$$

Furthermore, in this case the expansion $\tilde{\sigma}_\lambda^{\#}(s, t) = \sum_{n=1}^\infty \lambda^{n-1}\tilde{\sigma}_{\lambda,n}^{\#}(s, t)$, where, as above, $\tilde{\sigma}_{\lambda,1}^{\#}(s, t) = \tilde{\sigma}_\lambda(s, t)$, $\tilde{\sigma}_{\lambda,n+1}^{\#}(s, t) = \int_s^t \tilde{\sigma}_{\lambda,n}^{\#}(s, \tau)\,\tilde{\sigma}_\lambda(\tau, t)\,d\beta(\tau)$, is orthogonal. Let $m_{\lambda,n}(s, t) = \mathbb{E}\,|\,\tilde{\sigma}_{\lambda,n}^{\#}(s, t)|^2$. Then $m_{\lambda,1}(s, t) = |\tilde{\sigma}_\lambda(s, t)|^2$, $m_{\lambda,n+1}(s, t) = \int_s^t m_{\lambda,n}(s, \tau)\,|\,\tilde{\sigma}_\lambda(\tau, t)|^2\, d\tau$, and these are precisely the iterates for the resolvent of $A_2(\lambda)$, $m_\lambda^*(s, t)$. Thus $m_\lambda^*(s, t) = \sum_{n=1}^\infty \lambda^{n-1} m_{\lambda,n}(s, t)$. Therefore, the existence of $m_\lambda^*(s, t)$ ensures the existence of $\tilde{\sigma}_\lambda^{\#}(s, t)$.

Further, it is to be noted that in this case the expansion $\tilde{\sigma}_\lambda^{\#}(s, t) = \sum_{n=1}^\infty \lambda^{n-1}\sigma_{\lambda,n}^{\#}(s, t)$ is precisely the expansion (2.4), which generates the corresponding expansion for $x(t)$. Consider, now, that any nonanticipating mean-square integrable process $x(t)$ can be represented as in (2.5). When $k(s, t) = \sigma(s, t)x(s)$, where σ is deterministic, $x(t)$ is said to be generated by $\sigma(s, t)$, since its expansion (2.5) can be developed at once from $\sigma(s, t)$. It is clear that no non-trivial mean zero processes have generators, and that all deterministic functions have zero as generator. Any process that arises from a linear differential equation

$$\mathscr{L}x(t) = f_1(t) + f_2(t)x(t)\xi(t), \tag{4.21}$$

where $f_1(t)$ and $f_2(t)$ are deterministic, and $\xi(t)$ is a white noise, has a generator, by virtue of the Green's function representation.

EXAMPLE 4.B. The process

$$x(t) = \exp\Big[\int_0^t [a(s) - 1/2|b(s)|^2]ds + \int_0^t b(s)\,d\beta(s)\Big] + \int_0^t f(s)\exp\Big[\int_s^t [a(\tau) - 1/2|b(\tau)|^2]d\tau + \int_s^t b(\tau)\,d\beta(\tau)\Big]\, ds$$

has generator $\sigma(s, t) = b(s)\exp[\int_s^t a(\tau)\,d\tau]$, since it satisfies $(d/dt)x(t) - ax(t) = f(t) + b(t)x(t)\xi(t)$. Here $\xi(t)$ is formally $(d/dt)\beta(t)$.

Suppose $y_0(t) = x(t) - \mathbb{E}\,x(t)$ can be successively stochastically differentiated infinitely many times, and define

$$y_{n+1}(t) = \frac{d}{dt}\left[y_n(t) - \int_0^t \frac{dy_n(s)}{d\beta(s)} d\beta(s) \right]. \tag{4.22}$$

Then if $x(t)$ has a generator $\sigma(s, t)$, $y_n(t) = \int_0^t =(\partial^n/\partial t^n)\ \sigma(s, t)x(s)\,d\beta(s)$, and

$$\frac{1}{x(t)}\frac{dy_n(t)}{d\beta(t)} = \frac{\partial^n}{\partial t^n}\sigma(t, t). \tag{4.23}$$

Thus a necessary and sufficient condition that $x(t)$ have a generator is that the functions $z_n(t) = (1/x(t))(dy_n(t)/d\beta(t))$ are deterministic and satisfy $\sum_{n=0}^{\infty} (t^n/n!)\ |z_n(s)| < \infty$ for $s \le t$. If this is the case, then $\sigma(s, t) = \sum_{n=0}^{\infty} ((t-s)^n/n!)\ z_n(s)$.

EXAMPLE 4.C. $F(t, \beta(t))$ has a generator if and only if $F(t, x) = a(t)e^{bx}$ in which case the generator is $\sigma(s, t) = b(a(t)/a(s))\exp[1/2|b|^2(t-s)]$.

Acknowledgment. The authors gratefully acknowledge the help of V. Trutzer who supplied the proof of Theorem 3.D adopted below. The referee was most helpful in correcting the original version and citing the related works Hitsuda [8] and Itô [12].

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