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THE TWO-PARAMETER BROWNIAN BRIDGE: KOLMOGOROV INEQUALITIES AND UPPER AND LOWER BOUNDS FOR THE DISTRIBUTION OF THE MAXIMUM

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To Professor Rafael Laguardia (1906–1980)

In Memoriam

The aim of this paper is to give upper and lower bounds for the probability density at $(u-z)$ of the position at time (x, y) $(x, y, z, u \in R^+)$ of a standard Wiener process with two-dimensional parameter (x, y) with the requirement that it did not reach the barrier u in the "past" $\{(x', y')\colon 0 \le x' \le x, 0 \le y' \le y\}$. The fundamental tools are Kolmogorov forward inequalities for the density and certain bounds for the behaviour of p near the border.

1. Introduction. The aim of this paper is to give upper and lower estimates for the probability density

$$p(x, y, z, u) = \frac{\partial}{\partial z} P\{M(x, y) < u, \quad \beta(x, y) \ge u - z\} \tag{1}$$

at the position $u - z$ and epoch (x, y) $(x, y, z, u \in R^+)$ of a standard Wiener process β with two-dimensional parameter (x, y) with the requirement that

$$M(x, y) = \sup\{\beta(x', y')\colon \quad 0 \le x' \le x, \quad 0 \le y' \le y\}$$

is smaller than the positive constant u, that is, β does not reach the barrier u in the *past* $\{(x', y')\colon 0 \le x' \le x, 0 \le y' \le y\}$. ($\beta$ is a Gaussian process with continuous paths $\beta(\circ, \circ)$ and moments $E\beta(x, y) = 0$, $\mathrm{Cov}(\beta(x, y), \beta(x', y')) = (x \wedge x')(y \wedge y')$, $x, y, x', y' \in R^+$. We refer to [1, 10] for the construction and applications of such a process.)

The "Kolmogorov inequation" given by Theorem 2 and its consequence — Theorem 3 — correct [3] and Theorem 2 of [5] respectively.

The analytical procedure employed to obtain the estimates of p, requires the knowledge of a good upper bound of p near the border, and for this purpose we have adapted an argument given by V. Goodman in [6] to get lower bounds for the probability of crossing the barrier. Lower bounds for p lead naturally to upper bounds for the same probability.

Furthermore, from the densities one can compute the probability that the Brownian bridge with a two dimensional parameter — and condition $\beta(1, 1) = 0$ — crosses a fixed barrier, that appears in the two-dimensional generalization of the one-sided asymptotic Kolmogorov-Smirnov test. Here, we refer only to the case of independent coordinates, and our bounds permit estimates for the critical values of the test. This sort of application is described, for example, in [2]. At the end we add some tables with some of these lower and upper asymptotic critical values (see Table 2).

The function p depends on its four arguments in a relatively simpler way, as described in the following lemma:

LEMMA 1. (i) *p depends on x, y only through the product xy. We shall denote by the*

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same symbol p the function

$$p(xy, z, u) = p(x, y, z, u).$$

(ii) *furthermore*, $p(t, z, u) = t^{-1/2}p(1, zt^{-1/2}, ut^{-1/2})$ *for all* $t, z, u, \in R^+$.

PROOF. The discrete approach

$$\{M(x, y) < u, \beta(x, y) \geq u - z\} = \lim_{n\to\infty} \cap_{h,k=1}^{2^n} \{u^{-1}\beta(h2^{-n}x, k2^{-n}y) < 1,$$
$$|u - z|^{-1}\beta(x, y) \geq \operatorname{sgn}(u - z)\}$$

to the event in the right-hand side of (1) (with the unimportant restriction $z \neq u$) only involves at each step the finite set of Gaussian variables

$$|u - z|^{-1}\beta(x, y)$$

and $$u^{-1}\beta(h2^{-n}x, k2^{-n}y), \qquad h, k = 1, 2, \cdots, 2^n$$

whose joint distribution is invariant under the transformation $(x, y, z, u) \to (1, 1, z(xy)^{-1/2}, u(xy)^{-1/2})$. This implies that $P\{M(x, y) < u, \beta(x, y) \geq u - z\}$ is a function of $z(xy)^{-1/2}$, $u(xy)^{-1/2}$ only, and the remainder is plain.

REMARK. The preceding lemma shows that there is no loss of generality in restricting the domain of $p(t, z, u)$ to $[0, 1] \times R^+ \times R^+$, and this will be done from Section 3 on. Moreover, since we shall keep u fixed in most of the following, the abbreviation (t, z) will be used instead of (t, z, u) (for instance, $p(t, z) = p(t, z, u)$).

NOTATION. The standard Gaussian density will be denoted by

$$\varphi(t) = (2\pi)^{-1/2}e^{-t^2/2}$$

and the distribution function, by $\Phi(t) = \int_{-\infty}^{t} \varphi(t')\, dt'$.

2. An upper estimate for p.

THEOREM 1. *For every* $t, z, u > 0$, *and*

$$p_0(t, z) = t^{-1/2}\varphi(t^{-1/2}(u - z)) - t^{-1/2}\varphi(t^{-1/2}(u + z)),$$

the inequality

$$p(t, z) \leq p_0(t, z) - 2uzt^{-3/2}\varphi(t^{-1/2}(u + z)) \tag{2}$$

holds.

PROOF. Because of Lemma 1, it suffices to prove (2) for $t = 1$. Let C_0 be the Banach space of real valued continuous functions on $[0, 1]$ vanishing at 0. The real valued process β with parameter in $[0, 1] \times [0, 1]$ can be viewed as a C_0-valued process $\tilde{\beta}$ with parameter in $[0, 1]$, using the correspondence

$$\tilde{\beta}(y)(x) = \beta(x, y).$$

If U_u is the Borel subset of C_0 of functions bounded by $u > 0$, let us consider the defective probability measure $q(\cdot, u)$ defined on any Borel subset A of C_0 by

$$q(A, u) = P\{\tilde{\beta}(y) \in U_u \quad \text{for every} \quad y \in [0, 1], \tilde{\beta}(1) \in A\}.$$

In what follows, we shall denote by w any standard one-dimensional Wiener process, and by W the Wiener measure of A:

$$W(A) = P\{w \in A\}.$$

Since $\tilde{\beta}(1)$ is itself a standard Wiener process, $q(A, u)$ is not greater than $W(A)$, so that $q(\cdot, u)$ is absolutely continuous with respect to W.

Goodman ([6], proof of Theorem 3) has shown that the Radon-Nikodym derivative

$$g(\eta, u) = \frac{dq(\eta, u)}{dW(\eta)}$$

satisfies the inequality

$$g(\eta, u) \leq 1 - e^{-2u\inf\{|u-\eta(r)|/r:0<r\leq 1\}} \tag{3}$$

for $\eta \in U_u$; the absolute value in (3), unnecessary when $\eta \in U_u$, is introduced in order that the inequality remains valid even when $\eta \notin U_u$.

We indicate in the following a proof of (3), adapting Goodman's formulation to our simpler requirements. Let us first notice that $\{\beta(r, y) - y\beta(r, 1): 0 \leq y \leq 1\}$ and $\{\beta(x, 1): 0 \leq x \leq 1\}$ are independent for each fixed r, as the direct computation of covariances show. Hence, for fixed $r \in [0, 1]$, the conditional distribution of

$$\beta(r, y) = (\beta(r, y) - y\beta(r, 1)) + y\beta(r, 1), \qquad 0 \leq y \leq 1$$

given $\tilde{\beta}(1)$ is the same as the one given $\beta(r, 1)$. In particular, for any $\eta \in U_u$,

$$\begin{aligned} &P\{\beta(r, y) < u \quad \text{for every} \quad y \in [0, 1]/\tilde{\beta}(1) = \eta\} \\ &= P\{\beta(r, y) < u \quad \text{for every} \quad y \in [0, 1]/\beta(r, 1) = \eta(1)\} \\ &= 1 - e^{-2u(u-\eta(r))/r}, \end{aligned} \tag{4}$$

the last equality been obtained from well known properties of Brownian bridge (see for instance [7]).

On the other hand, since $\tilde{\beta}(y) \in U_u$ implies trivially $\beta(r, y) < u$, (4) is an upper bound of $g(\eta, \mu)$, and then (3) follows by taking the infimum in r.

We proceed now to complete the proof of the theorem. Given any Borel set B in $(0, +\infty)$, we compute

$$\begin{aligned} \int_B p(1, z, u)\, dz &= P\{M(x, y) < u, u - \beta(1, 1) \in B\} \\ &= \int_{\{\eta: u-\eta(1)\in B\}} g(\eta, u)\, dW(\eta) = E\mathscr{I}_{\{u-w(1)\in B\}} g(w, u) \\ &= E(\mathscr{I}_{\{u-w(1)\in B\}} E(g(w, u)/w(1))) \\ &= \int_B \varphi(u - z) E(g(w, u)/w(1) = u - z)\, dz \end{aligned}$$

(where $\mathscr{I}$ denotes the indicator function and w is again a generic Wiener process), and apply (3) to obtain:

$$\begin{aligned} p(1, z) &\leq \varphi(u - z) E(1 - e^{-2u\inf_{0\leq r\leq 1}|u-w(r)|/r} \,|\, w(1) = u - z) \\ &= \varphi(u - z) \int_0^z (1 - e^{-2ux})(-dP\{\inf_{0<r\leq 1} |u - w(r)|/r > x \,|\, w(1) = u - z\}) \\ &= \varphi(u - z) \int_{0^-}^z e^{-2ux}\, dP\{w(r) < u - rx \quad \text{for every} \quad r \in [0, 1] \,|\, w(1) = u - z\} \\ &\quad + \varphi(u - z) P\{\inf_{0<r\leq 1} |u - w(r)|/r > 0 \,|\, w(1) = u - z\}. \end{aligned}$$

Now we apply again well known formulae about the barrier problem for Brownian bridge, for the computation of both probabilities and get

$$p(1, z, u) \leq \varphi(u - z) \int_0^z e^{-2ux}\, d(1 - e^{-2u(z-x)}) + \varphi(u - z) - \varphi(u + z);$$

hence the evaluation of the integral provides the required conclusion.

A Taylor expansion of the right hand-term in (3) leads readily to the following corollary.

COROLLARY 1. *For any fixed $u, Z > 0$, and a suitable constant C (depending on u, Z), the inequality*

$$p(t, z) \leq 2u^2z^2t^{-5/2}\varphi(t^{-1/2}u) + Cz^3t^{-3}\varphi(t^{-1/2}(u - z))$$

holds for $(t, z) \in (0, 1) \times (0, Z)$.

3. Two "Kolmogorov forward inequations" for p. In what follows, D denotes the differential heat operator

$$D = \frac{1}{2}\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}.$$

THEOREM 2. *The density p satisfies the inequalities*

(i) $Dp \geq 0$

and

(ii) $DtDp \geq 0$,

as a distribution.

For the proof, we shall use the following lemma.

LEMMA 2. *If ψ is a C^∞ function with compact support on $(0, 1) \times (0, 1) \times (0, +\infty)$ (or $(0, 1) \times (0, +\infty)$) then the integral*

$$\int_0^\infty p(xy, z)\psi(x, y, z)\, dz \qquad \left(\text{or} \int_0^\infty p(t, z)\psi(t, z)\, dz\right)$$

is a continuous function of (x, y) (respectively, t).

PROOF OF THE LEMMA. Because of its regularity, ψ may be uniformly approximated by finite combinations of the indicator functions of generalized intervals. Hence it suffices to prove that for each $z_1, z_2 (0 < z_1 < z_2 < \infty)$, $\int_{z_1}^{z_2} p(\cdot, z)\, dz$ is continuous on $(0, 1)$. Given any fixed $\tau > 0$ and $t, t + \delta$ such that $\tau \leq t < t + \delta \leq 1$,

$$\begin{aligned}
\int_{z_1}^{z_2} p(t + \delta, z)\, dz &= P\{M(t + \delta, 1) < u, u - z_2 < \beta(t + \delta, 1) < u - z_1\} \\
&\leq P\{M(t, 1) < u, u - z_2 < \beta(t + \delta, 1) < u - z_1\} \\
&\leq P\{M(t, 1) < u, u - z_2 - \sqrt[4]{\delta} < \beta(t, 1) < u - z_1 + \sqrt[4]{\delta}\} \\
&\quad + P\{|\beta(t + \delta, 1) - \beta(t, 1)| > \sqrt[4]{\delta}\} \\
&\leq \int_{z_1 - \sqrt[4]{\delta}}^{z_2 + \sqrt[4]{\delta}} p(t, z)\, dz + \sqrt[2]{\delta} = \int_{z_1}^{z_2} p(t, z)\, dz + o(1), \qquad \delta \downarrow 0
\end{aligned}$$

uniformly in t, because p is uniformly bounded (by the supremum of p_0 on $[\tau, 1] \times (0, \infty)$, for instance). A reversed inequality may be obtained in very much the same way.

PROOF OF THEOREM 2. Given any non-negative C^∞ function ψ with compact support on $(0, 1) \times (0, \infty)$, we define

$$S_x^{(n)} = \textstyle\sum_{i=1}^{2^n} \mathscr{I}_i^{(n)} \Delta_i^{(n)}\, \psi$$

where

$$\mathscr{I}_i^{(n)} = \mathscr{I}_{\{M((i-1)2^{-n},1)<u\}}$$

and

$$\Delta_i^{(n)}\psi = \psi(i2^{-n}, u - \beta(i2^{-n}, 1)) - \psi((i-1)2^{-n}, u - \beta((i-1)2^{-n}, 1)).$$

It is clear that $S_x^{(n)}$ is reduced for each ω to the value

$$\psi(X^{(n)}, u - \beta(X^{(n)}, 1))$$

where

$$X^{(n)} = X^{(n)}(\omega) = \max\{i2^{-n}: M((i-1)2^{-n}, 1) < u, \quad i = 1, 2, \cdots, 2^n\}.$$

On the other hand, after the Taylor expansion

$$\Delta_i^{(n)}\psi = \psi_t\delta - \psi_z\Delta_i\beta + \tfrac{1}{2}(\psi_{tt}\delta^2 - 2\psi_{tz}\delta\Delta_i\beta + \psi_{zz}(\Delta_i\beta)^2) + \cdots$$

with $\qquad \delta = 2^{-n}, \qquad \Delta_i\beta = \beta(i2^{-n}, 1) - \beta((i-1)2^{-n}, 1),$

the partial derivatives explicitly appearing computed in $((i-1)2^{-n}, u - \beta((i-1)2^{-n}, 1))$ and the ones in the remainder, of the third order, computed in an intermediate point, we notice that the expectation of this remainder is $o(\delta)$, since the derivatives are uniformly bounded and they appear multiplying $\delta^j(\Delta_i\beta)^{3-j}$ $(j = 0, \cdots, 3)$ whose absolute expectations are all $o(\delta)$.

We call $\mathscr{A}_{x,y}$ the σ-field generated by $\{\beta(x', y'): 0 \le x' \le x, 0 \le y' \le y\}$ and compute $ES_x^{(n)}$ using $E(\cdot) = E(E(\cdot/\mathscr{A}_{(i-1)2^{-n},1}))$ for the ith term $(i = 1, \cdots, 2^n)$. This leads to

$$ES_x^{(n)} = \sum_{i=1}^{2^n}\left[\delta E\mathscr{I}_i^{(n)}\left(\psi_t + \frac{1}{2}\psi_{zz}\right) + o(\delta)\right]$$

$$= \sum_{i=1}^{2^n} \delta \int_0^\infty p((i-1)2^{-n}, z)D^* \psi((i-1)2^{-n}, z)\, dz + o(1)$$

with

$$D^* = \frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial t},$$

in fact, a Riemann sum which tends to the integral in (0, 1) of

$$\int_0^\infty p(1, z)D^* \psi(\cdot, z)\, dz \tag{5}$$

as n goes to infinity, because of the continuity (Lemma 2) and boundedness of (5). Therefore

$$\iint_{(0,1)\times R^+} p(t, z)D^* \psi(t, z)\, dt\, dz = \lim_{n\to\infty} ES_x^{(n)} \ge 0 \tag{6}$$

(since $S_x^{(n)} \ge 0$) for any non-negative ψ, which ends the proof of (i) because D^* is the adjoint operator of D.

A completely analogous proof could have been made starting from

$$S_y^{(n)} = \psi(Y^{(n)}, u - \beta(1, Y^{(n)}))$$

with

$$Y^{(n)} = Y^{(n)}(\omega) = \max\{j2^{-n}: M(1, (j-1)2^{-n}) < u, \qquad j = 1, 2, \cdots, 2^n\}.$$

In order to prove (ii), we introduce the new sums

$$S^{(n)} = \sum_{i,j=1}^{2^n} \mathscr{I}_{i,j}^{(n)} \square_{i,j}^{(n)}\psi \tag{7}$$

with

$$\psi(x, y, z) = \psi(xy, z),$$
$$\mathscr{I}_{i,j}^{(n)} = \mathscr{I}_{\{M((i-1)2^{-n},(j-1)2^{-n})<u\}},$$

and $\square_{i,j}^{(n)} \psi$ equal to the double increment of

$$\psi(x, y, u - \beta(x, y)) \text{ on } R_{i,j}^{(n)} = ((i-1)2^{-n}, i2^{-n}] \times ((j-1)2^{-n}, j2^{-n}].$$

For each ω, the union of the squares appearing in $S^{(n)}$, that is, $\cup\{R_{i,j}^{(n)}: \mathscr{I}_{i,j}^{(n)} = 1\}$, has a border composed by $\{(x, y): xy = 0, 0 \le x \le 1, 0 \le y \le 1\}$ and a polygonal curve with edges in the directions of the coordinate axes, whose vertices are

$$(X^{(n)}, 1), \qquad (1, Y^{(n)})$$

and the points in

$$V^+ = V_n^+(\omega) = \{(i2^{-n}, j2^{-n}): i, j < 2^n, \mathscr{I}_{i,j}^{(n)} = 1, \mathscr{I}_{i+1,j}^{(n)} = \mathscr{I}_{i,j+1}^{(n)} = 0\}$$

and

$$V^- = V_n^-(\omega) = \{((i-1)2^{-n}, (j-1)2^{-n}): i, j \le 2^n, \quad \mathscr{I}_{i,j}^{(n)} = 0, \quad \mathscr{I}_{i-1,j}^{(n)} = \mathscr{I}_{i,j-1}^{(n)} = 1\}.$$

Therefore, (7) may be written as

$$S^{(n)} = S_x^{(n)} + S_y^{(n)} + \textstyle\sum_{(x,y)\in V^+} \psi(x, y, u - \beta(x, y)) - \sum_{(x,y)\in V^-} \psi(x, y, u - \beta(x, y)).$$

Now let us prove

$$\lim_{n\to\infty} E \textstyle\sum_{(x,y)\in V^-} \psi(x, y, u - \beta(x, y)) = 0, \tag{8}$$

which implies

$$\liminf_{n\to\infty} E(S^{(n)} - S_x^{(n)} - S_y^{(n)}) \ge 0 \tag{9}$$

by Fatou Lemma applied to the nonnegative sum on V^+. For the proof of (8), let us notice that the points (x, y) in V^- verify the conditions

$$M(x, y) \ge u, \quad M(x - \delta, y) < u, \quad M(x, y - \delta) < u \ (\delta = 2^{-n}),$$

hence there exist δ_x, δ_y such that

$$0 < \delta_x \le \delta, \quad 0 < \delta_y \le \delta, \quad \beta(x - \delta_x, y - \delta_y) = u,$$

and therefore the point $(x - \delta_x, y - \delta_y, u - \beta(x - \delta_x, y - \delta_y))$ lies outside the support of ψ.

This last observation implies that the Taylor expansion of order $\alpha - 1$ of ψ around $(x - \delta_x, y - \delta_y, 0)$ reduces to the complementary term

$$\psi(x, y, u - \beta(x, y)) = \textstyle\sum_{i+j+k=\alpha} \dfrac{\alpha!}{i!j!k!} \dfrac{\partial^\alpha \psi}{\partial x^i \partial y^j \partial z^k} \delta_x^i \delta_y^j (\beta(x, y) - u)^k$$

where the partial derivatives are computed in a point of $(x - \delta_x, x) \times (y - \delta_y, y)$. Then, if M_α is a uniform absolute bound of all derivatives of order α, and μ_k is the absolute moment of order k of the standard normal distribution,

$$\begin{aligned} E \textstyle\sum_{(x,y)\in V^-} \psi(x, y, u - \beta(x, y)) &\le \textstyle\sum_{(x,y)\in V^-} \sum_{i+j+k=\alpha} \dfrac{\alpha!}{i!j!k!} M_\alpha \delta^{i+j} E\,|\beta(x, y) - u|^k \\ &\le \textstyle\sum_{i',j'=1}^{2^n} \sum_{i+j+k=\alpha} \dfrac{M_\alpha \delta^{i+j} \alpha!}{i!j!k!} 8\, \delta^{k/2} \mu_k \end{aligned} \tag{10}$$

because of the inequality

$$E \sup_{0<\delta_x<\delta, 0<\delta_y<\delta} |\beta(x, y) - \beta(x - \delta_x, y - \delta_y)|^k \le 8\delta^{k/2} \mu_k \tag{11}$$

which we prove later. The observation that (10) tends to zero as n goes to infinity, for $\alpha = 5$, leads to (8), and hence (9).

It remains to prove (11) (for $x, y \geq \delta$). In order to do that, we write

$$\beta(x, y) - \beta(x - \delta_x, y - \delta_y) = (\beta(x, y) - \beta(x - \delta_x, y)) + (\beta(x, y) - \beta(x, y - \delta_y)) - (\beta(x, y) - \beta(x - \delta_x, y) - \beta(x, y - \delta_y) + \beta(x - \delta_x, y - \delta_y)),$$

and use (for any $\lambda > 0$) the bounds given by the Reflection Principle and the trivial inequality $P\{M(1, 1) > u\} \leq 4\phi(-u)$ (see [4], for instance):

$$\begin{aligned} &P\{\sup\nolimits_{0<\delta_x<\delta, 0<\delta_y<\delta} |\beta(x, y) - \beta(x - \delta_x, y - \delta_y)| > \lambda\} \\ &\quad \leq 2P\{\sup\nolimits_{0<\delta_x<\delta}(\beta(x, y) - \beta(x - \delta_x, y)) > \lambda\} \\ &\qquad + 2P\{\sup\nolimits_{0<\delta_y<\delta}(\beta(x, y) - \beta(x, y - \delta_y)) > \lambda\} \\ &\qquad + 2P\{\sup\nolimits_{0<\delta_x<\delta, 0<\delta_y<\delta}(\beta(x, y) - \beta(x - \delta_x, y) \\ &\qquad - \beta(x, y - \delta_y) + \beta(x - \delta_x, y - \delta_y)) > \lambda\} \\ &\quad \leq 4P\{\beta(\delta, y) > \lambda\} + 4P\{\beta(x, \delta) > \lambda\} + 8P\{\beta(\delta, \delta) > \lambda\} \\ &\quad \leq 8P\{\beta(\delta, 1) > \lambda\} + 8P\{\beta(\delta, \delta) > \lambda\} < 8P\{|\beta(\delta, 1)| > \lambda\} \end{aligned}$$

(because $\delta < 1$). This implies for each positive k

$$E \sup\nolimits_{0<\delta_x<\delta, 0<\delta_y<\delta} |\beta(x, y) - \beta(x - \delta_x, y - \delta_y)|^k \leq 8E|\beta(\delta, 1)|^k$$

and (11) follows.

The next step of the proof follows the same lines as the proof of (i). We expand $\square_{i,j}^{(n)}$, omitting the index (n) and introducing the abbreviations

$$\sqsupset_{ij}\beta = \textstyle\sum_{i' \leq i} \square_{ij}\beta, \qquad \sqcap_{ij}\beta = \textstyle\sum_{j' \leq j} \square_{ij}\beta, \qquad \urcorner_{ij} = \sqsupset_{ij}\beta + \sqcap_{ij}\beta - \square_{ij}\beta$$

as follows:

$$\begin{aligned} \square_{ij}\psi = &- \psi_z \square_{ij}\beta + \tfrac{1}{2}[2\psi_{xy}\delta^2 - 2\psi_{xy}\delta \sqsupset_{ij}\beta - 2\psi_{yz}\delta \sqcap_{ij}\beta + \psi_{zz}\square_{ij}(\beta^2)] \\ &+ \tfrac{1}{6}[3(\psi_{xxy} + \psi_{xyy})\delta^3 - 3\psi_{xxz}\delta^2 \sqsupset_{ij}\beta - 6\psi_{xyz}\delta^2 \urcorner_{ij}\beta \\ &- 3\psi_{yyz}\delta^2 \sqcap_{ij}\beta + 3\psi_{xzz}\delta \sqsupset_{ij}(\beta^2) + 3\psi_{yzz}\delta \sqcap_{ij}(\beta^2) - \psi_{zzz}\square_{ij}(\beta^3)] \\ &+ \tfrac{1}{24}[(4\psi_{xxxy} + 6\psi_{xxyy} + 4\psi_{xyyy})\delta^4 - 4\psi_{xxxz}\delta^3 \sqsupset_{ij}\beta - 12\psi_{xxyz}\delta^3 \urcorner_{ij}\beta \\ &- 12\,\psi_{xyyz}\delta^3 \urcorner \beta_{ij} - 4\psi_{yyyz}\delta^3 \sqcap_{ij}\beta + 6\psi_{xxzz}\delta^2 \sqsupset_{ij}(\beta^2) + 12\psi_{xyzz}\delta^2(\urcorner_{ij}\beta^2) \\ &+ 6\psi_{yyzz}\delta^2 \sqcap_{ij}(\beta^2) - 4\psi_{xzzz}\delta \sqsupset_{ij}(\beta^3) - 4\psi_{yzzz}\delta \sqcap_{ij}(\beta^3) \\ &+ \psi_{zzzz}\square_{ij}(\beta^4)] + \cdots. \end{aligned} \tag{12}$$

The derivatives appearing explicitly in (12) are computed in $((i-1)2^{-n}, (j-1)2^{-n}, u - \beta((i-1)2^{-n}, (j-1)2^{-n}))$ and the omitted ones, in the complementary term, correspond to the fifth order and the expectation of this term is $o(\delta^2)$ as is easily verified. As in the proof of (i), we compute expectations in (7), now applying

$$E(\cdot) = E(E(\cdot / \mathscr{A}_{(i-1)2^{-n}, (j-1)2^{-n}}))$$

in each term $\mathscr{I}_{i,j}^{(n)} \square_{i,j}^{(n)} \psi$, and obtain

$$\begin{aligned} ES^{(n)} &= \textstyle\sum_{ij}\delta^2 E\{\mathscr{I}_{ij}^{(n)}(\psi_{xy} + \tfrac{1}{2}\psi_{zz} + \tfrac{1}{2}(i-1)2^{-n}\psi_{xzz} + \tfrac{1}{2}(j-1)2^{-n}\psi_{yzz} \\ &\quad + \tfrac{1}{4}(i-1)(j-1)2^{-2n}\psi_{zzzz})\} + o(1) \\ &= \textstyle\sum_{ij}\delta^2 \displaystyle\int_0^\infty p((i-1)(j-1)2^{-2n}, z)\left(\frac{x}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial y}\right)\left(\frac{y}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial x}\right) \\ &\quad \cdot \psi((i-1)2^{-n}, (j-1)2^{-n}, z)\, dz + o(1). \end{aligned}$$

We use again Lemma 2 to obtain from the continuity of the integral that the limit of the sums is

$$\lim ES^{(n)} = \int_0^1 dx \int_0^1 dy \int_0^1 p(xy, z) \left(\frac{x}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial y}\right)\left(\frac{y}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial x}\right)\psi(x, y, z)\, dz$$

$$= \int_0^1 dt \int_0^\infty (-\log t) p(t, z) D^* t D^* \psi(t, z)\, dz,$$

and from (6) and (9) we conclude

$$\iint p(t, z)((-\log t) D^* t D^* - 2D^*)\psi(t, z)\, dt\, dz \geq 0.$$

We finally notice that

$$(-\log t) D^* t D^* - 2D^* = D^* t D^* (-\log t),$$

and, since any nonnegative C^∞ function with compact support on $(0, 1) \times (0, \infty)$ can be written in the form $(-\log t)\psi(t, z)$, with ψ of the same class of functions, (ii) follows.

4. A Lower Bound for p. Theorem 2 shows that in fact, the distributions

$$h = DtDp \geq 0$$

and

$$w = tDp \geq 0$$

are formal densities of Borel measures.

A lower bound $\tilde{p}$ of p will be obtained as a solution of the equation

$$D\tilde{p} = \frac{\tilde{w}}{t}$$

where $\tilde{p}$ satisfies initial and border conditions that are not greater than those corresponding to p, and the cooling term $\tilde{w}/t$ is greater than or equal to w/t in the interior of $(0, 1) \times R^+$.

We shall find such a function $\tilde{w}$ as a solution of the homogeneous heat equation

$$D\tilde{w} = 0,$$

with the same initial and border conditions as w. Hence, $\tilde{w} \geq w$ will follow from $Dw = h \geq 0$.

So, our next task is to get initial and border conditions for $\tilde{w}$ and $\tilde{p}$. For that purpose, let Ψ and χ be fixed, non-negative, C^∞-functions of one real variable, support contained in $(-1, 0)$ and

$$\int_{-1}^0 \Psi(t)\, dt = \int_{-1}^0 \chi(z)\, dz = 1;$$

Ψ and χ will be chosen afterward.

For each $\varepsilon > 0$, denote:

$$\Psi_\varepsilon(t) = \varepsilon^{-1}\Psi(t/\varepsilon); \qquad \chi_\varepsilon(z) = \varepsilon^{-1}\chi(z/\varepsilon).$$

If $q(t, z)$ is a formal density, we shall denote by

$$q_\varepsilon(t, z) = \iint \Psi_\varepsilon(t - t')\chi_\varepsilon(z - z') q(t', z')\, dt'\, dz'$$

its convolution with the kernel $\Psi_\varepsilon(t)\chi_\varepsilon(z)$. The C^∞-function q_ε is defined for $(t, z) \in (0, 1 - \varepsilon) \times R^+$ and $q_\varepsilon(t, z)\, dt\, dz$ converges weakly (as a measure) to $q(t, z)\, dt\, dz$ as $\varepsilon \to 0$.

LEMMA 3. (*Initial conditions*).

(i) $\limsup_{t\downarrow 0} \int_0^{+\infty} [p_0(t, z) - p(t, z)]\, dz = 0$

(ii) $\lim_{\varepsilon\downarrow 0}\limsup_{t\downarrow 0} \int_0^{+\infty} w_\varepsilon(t, z)\, dz = 0.$

PROOF. The standard reflection principle for the (one-dimensional) Wiener process says that:

$$\int_0^{+\infty} p_0(t, z)\, dz = 1 - 2\Phi(- u\cdot t^{-1/2}).$$

This, plus the trivial bound

$$P(M(1, 1) > u) = 1 - \int_0^{+\infty} p(t, z)\, dz \le 4\Phi(- u\cdot t^{-1/2})$$

(see for example [4]), implies:

$$\int_0^{+\infty} [p_0(t, z) - p(t, z)]\, dz \le 2\, \Phi(- u\cdot t^{-1/2}), \tag{12}$$

and (i) follows.

To prove (ii) compute:

$$\int_0^{+\infty} w_\varepsilon(t, z)\, dz = \int_0^{+\infty} dz \iint \Big[\Psi_\varepsilon(t - t')\chi_\varepsilon(z - z') + \frac{t'}{2}\Psi_\varepsilon(t - t')\chi_\varepsilon''(z - z') - t'\Psi_\varepsilon'(t - t')\chi_\varepsilon(z - z') \Big]\cdot[p(t', z') - p_0(t', z')]\, dt'\, dz'$$

where we have used the fact that $w = t\, Dp = t\, D(p - p_0)$. We separate the right hand member into two terms, and use (12):

$$\begin{aligned} &\int_0^{+\infty} dz \iint |\Psi_\varepsilon(t - t')\chi_\varepsilon(z - z') - t'\Psi_\varepsilon'(t - t')\chi_\varepsilon(z - z')| \\ &\qquad \cdot(p_0(t', z') - p(t', z'))\, dt'\, dz' \\ &\le \int |\Psi_\varepsilon(t - t') - t'\Psi_\varepsilon'(t - t')|\cdot 2\Phi(- u\cdot t'^{-1/2})\, dt' \\ &\le (1 + \varepsilon^{-1}\|\Psi'\|_\infty)2\Phi(- u(t + \varepsilon)^{-1/2}), \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_0^{+\infty} dz \iint \frac{t'}{2}\Psi_\varepsilon(t - t')|\chi_\varepsilon''(z - z')|(p_0(t', z') - p(t', z'))\, dt'\, dz' \\ &\le \frac{1}{2}\int \Psi_\varepsilon(t - t')\, dt'\cdot\varepsilon^{-3}\|\chi''\|_\infty \int_0^{+\infty} (p_0(t', z') - p(t', z'))\, dz' \\ &\le \tfrac{1}{2}\, \varepsilon^{-3}\|\chi''\|_\infty\cdot 2\Phi(- u(t + \varepsilon)^{-1/2}). \end{aligned} \tag{14}$$

Letting $t \downarrow 0$ and $\varepsilon \downarrow 0$ in (13) and (14), we obtain part (ii) of the Lemma.

LEMMA 4. (*Border conditions*). *Put* $w_\varepsilon(t, 0) = \limsup_{z\downarrow 0} w_\varepsilon(t, z)$. *Then, for each positive number* θ, *the functions* Ψ, χ *above can be chosen in such a way that if* $\varepsilon > 0$ *is* *small enough, one has*

$$w_\varepsilon(t, 0) \le G(t, u) + \theta \quad \textit{for all} \quad t \in (0, 1 - \epsilon),$$

where the function $G(t, u)$ *is defined by*:

$$G(t, u) = 2u^2 t^{-3/2} \varphi(u \cdot t^{-1/2}).$$

PROOF. From Corollary 1, if z is small enough (say $0 < z < u/2$), one has:

$$p(t, z) \le z^2(G(t, u)/t) + Cz^3 \tag{15}$$

uniformly on t, where C is a constant. So:

$$\begin{aligned} w_\varepsilon(t, z) \le \iint \bigg| & \Psi_\varepsilon(t - t')\chi_\varepsilon(z - z') + \frac{t}{2}\Psi_\varepsilon(t - t')\chi_\varepsilon''(z - z') \\ & - t'\Psi_\varepsilon'(t - t')\chi_\varepsilon(z - z') \bigg| \, p(t', z')\, dt'\, dz' \\ \le \iint \bigg| & \Psi(-\tau)\chi(-\zeta) + \frac{1}{2}\varepsilon^{-2}(t + \varepsilon\tau)\Psi(-\tau)\chi''(-\zeta) \\ & - \varepsilon^{-1}(t + \varepsilon\tau)\Psi'(-\tau)\chi(-\zeta) \bigg| \\ & \cdot [(t + \varepsilon\tau)^{-1} G(t + \varepsilon\tau, u)(z + \varepsilon\zeta)^2 + C(z + \varepsilon\zeta)^3]\, d\tau\, d\zeta. \end{aligned}$$

Let A be a bound of $t^{-1}G(t, u)$ for $t \in (0, 1)$. Then:

$$\begin{aligned} w_\varepsilon(t, 0) \le & \iint \left| \varepsilon^2\Psi(-\tau)\chi(-\zeta) + \frac{1}{2}(t + \varepsilon\tau)\Psi(-\tau)\chi''(-\zeta) - \varepsilon(t + \varepsilon\tau)\Psi'(-\tau)\chi(-\zeta) \right| \\ & \cdot [(t + \varepsilon\tau)^{-1} G(t + \varepsilon\tau, u) + C\,\varepsilon\,\zeta]\zeta^2\, d\tau\, d\zeta \\ \le & (A + \varepsilon C)(\varepsilon^2 + \varepsilon(1 + \varepsilon)\|\Psi'\|_\infty) \\ & + \left[\int_0^1 \Psi(-\tau)(G(t + \varepsilon\tau, u) + C\varepsilon(t + \varepsilon\tau))\, d\tau\right] \int_0^1 \frac{1}{2}\zeta^2 |\chi''(-\zeta)|\, d\zeta. \end{aligned}$$

Now (ii) follows easily if the function χ can be chosen with the additional requirement that

$$\int_0^1 \frac{1}{2}\zeta^2 |\chi''(-\zeta)|\, d\zeta < 1 + \theta,$$

and this can be shown by means of an elementary computation.

THEOREM 3.
(i) *The function*

$$\tilde{w}(t, z, u) = \int_0^t z(t - s)^{-3/2} \varphi(z \cdot (t - s)^{-1/2}) G(s, u)\, ds$$

is the density of a measure that is greater than or equal to the measure with formal density $w(t, z, u)$.
(ii) *The function*

$$\tilde{p}(t, z, u) = p_0(t, z, u) - \int_0^t (\log t - \log s)(t - s)^{-3/2} z\varphi(z(t - s)^{-1/2}) G(s, u)\, ds$$

is a lower bound for $p(t, z, u)$.

PROOF. The solution of the equation $Dq = 0$ (in $(t, z) \in (0, 1) \times R^+$) with zero initial condition and border condition $q(t, 0) = Q(t)$ is given by

$$q(t, z) = \int_0^t z(t-s)^{-3/2}\varphi(z(t-s)^{-1/2})Q(s)\,ds;$$

see for example [11]. So, the function $\tilde{w}$ satisfies $D\tilde{w} = 0$, with zero initial condition and border value $G(t, u)$.

Let now $\theta > 0$ be given. Using Lemma 3(ii), for ε small enough, one has

$$\limsup_{t\downarrow 0} \int_0^{+\infty} [\tilde{w}(t, z) - w_\varepsilon(t, z)]\,dz > -\theta. \tag{16}$$

The functions Ψ, χ can be chosen according to Lemma 4 so that

$$w_\varepsilon(t, 0) \le G(t, u) + \theta/2$$

for ε small enough, and as a consequence, for z small enough:

$$[\tilde{w}(t, z) - w_\varepsilon(t, z)] \ge -\theta \tag{17}$$

uniformly on t (c.f. the proof of Lemma 4).
Observe also that $D(\tilde{w} - w_\varepsilon) = -h_\varepsilon \le 0$.

Combining now the maximum principle for the heat operator with the fact that the solution of $Dq = 0$ with zero border condition and initial total mass bounded by θ is bounded by

$$q(t, z) \le \theta(2\pi t)^{-1/2},$$

it follows from (16) and (17) that

$$\tilde{w}(t, z) - w_\varepsilon(t, z) > -\theta - \theta(2\pi t)^{-1/2}$$

in the distributions sense. Since $\theta > 0$ is arbitrary, we get (i). Consider now the function

$$\begin{aligned}\tilde{v}(t, z) &= p_0(t, z) - \tilde{p}(t, z) - (\log t)\tilde{w}(t, z)\\ &= \int_0^t z(t-s)^{-3/2}\varphi(z(t-s)^{-1/2})(-\log s)G(s, u)\,ds\end{aligned}$$

which is the solution of $D\tilde{v} = 0$ with zero initial condition and border condition $\tilde{v}(t, 0) = (-\log t)G(t, u)$.

Clearly,

$$\tilde{p}(t, z) = p_0(t, z) - \tilde{v}(t, z) - (\log t)\tilde{w}(t, z)$$

satisfies $D\tilde{p} = \tilde{w}/t$ and $(p_0 - \tilde{p})$ has zero initial and border values. But $D(p - \tilde{p}) = (w - \tilde{w})/t \le 0$ and by Corollary 1, and Lemma 3 (i), $p - \tilde{p} = (p_0 - \tilde{p}) - (p_0 - p)$ has zero initial and border values. A new application of the maximum principle for the heat operator implies that $p - \tilde{p} \ge 0$, and finishes the proof.

5. Numerical results. The computation of tables for the function $\tilde{p}$ can be done from Theorem 3 (ii). The same holds for $\tilde{P}(t, u)$, the upper bound of $P(M(x, y) > u)$ where $xy = t$, given by:

$$\tilde{P}(t, u) = 1 - \int_0^{+\infty} \tilde{p}(t, z, u)\,dz. \tag{18}$$

Lemma 1 shows that it is sufficient to make the computations for $t = 1$.

Moreover, the bound is obviously improved if $\tilde{p}$ is replaced by $\tilde{p} \vee 0$ in the integrand in (18).

The conditional probability

$$P(M(1,1) > u/\beta(1,1) = 0) = 1 - \frac{p(1,u,u)}{\varphi(0)}$$

is bounded above by $1 - (\tilde{p}(1, u, u))/\varphi(0)$, and appears in the two-dimensional one-sided asymptotic Kolmogorov-Smirnov test in the case of independent coordinates.

Similar lower bounds are obtained from p_G. The following tables contain some results as well as some other known ones for comparison purposes.

TABLE 1.
Bounds for $P(M(1, 1) > u)$

Lower bound:

$$P_G(u) = 1 - \int_0^{+\infty} p_G(1, z, u)\, dz$$

Upper bounds:

$$\tilde{P}(u) = \{1 - \int_0^{+\infty} [0 \vee \tilde{p}(1, z, u)]dz\} \wedge 1;$$

$$P_T(u) = 1 \wedge \{4\,\Phi(-u)\}.$$

u	P_G	$\tilde{P}$	P_T
.1	.991	1.	1.
.2	.964	1.	1.
.3	.924	1.	1.
.4	.874	.993	1.
.5	.812	.961	1.
.6	.751	.901	1.
.7	.684	.825	.968
.8	.616	.740	.847
.9	.549	.655	.736
1.0	.484	.573	.635
1.1	.422	.495	.543
1.2	.365	.424	.460
1.3	.312	.360	.387
1.4	.264	.302	.323
1.5	.222	.251	.267
1.6	.184	.207	.219
1.7	.151	.169	.178
1.8	.123	.137	.144
1.9	.0994	.1099	.1149
2.0	.0795	.0873	.0910
2.1	.0629	.0688	.0714
2.2	.0493	.0537	.0556
2.3	.0383	.0415	.0429
2.4	.0295	.0318	.0328
2.5	.0224	.0241	.0248
2.6	.0169	.0181	.0186
2.7	.0127	.0135	.0139
2.8	.0094	.0100	.0109
2.9	.0069	.0072	.0075
3.0	.0050	.0053	.0054

TABLE 2.
Bounds for the critical values for the two-dimensional asymptotic Kolmogorov-Smirnov test in the independence case.
Lower bound: U_G; Upper bound: $\tilde{U}$

$$\alpha = 1 - (2\pi)^{1/2} p_G(1, U_G, U_G) = 1 - (2\pi)^{1/2}\tilde{p}(1, \tilde{U}, \tilde{U})$$

Level α	U_G	$\tilde{U}$
.10	1.39	1.47
.05	1.54	1.60
.025	1.67	1.73
.01	1.82	1.87

TABLE 3

Values of $\tilde{p}(1, z, u), p_G(1, z, u), p_0(1, z, u)$

	Lower bound	Upper bounds	
z	$\tilde{p}$	p_G	p_0
u = 0,75			
0.2	0.0000	0.0127	0.0889
0.4	0.0000	0.0457	0.1693
0.6	0.0000	0.0898	0.2341
0.8	0.0332	0.1344	0.2784
1.0	0.0732	0.1710	0.3004
1.2	0.1073	0.1937	0.3009
1.4	0.1296	0.2004	0.2834
1.6	0.1380	0.1922	0.2528
1.8	0.1335	0.1727	0.2144
2.0	0.1195	0.1463	0.1736
2.2	0.1000	0.1173	0.1343
2.4	0.0788	0.0894	0.0995
2.6	0.0587	0.0649	0.0706
2.8	0.0415	0.0450	0.0481
3.0	0.0280	0.0298	0.0314
u = 1			
0.2	0.0000	0.0178	0.0955
0.4	0.0124	0.0637	0.1835
0.6	0.0601	0.1242	0.2573
0.8	0.1186	0.1858	0.3121
1.0	0.1743	0.2370	0.3450
1.2	0.2172	0.2704	0.3556
1.4	0.2412	0.2832	0.3459
1.6	0.2453	0.2762	0.3196
1.8	0.2319	0.2533	0.2818
2.0	0.2058	0.2198	0.2375
2.2	0.1727	0.1813	0.1918
2.4	0.1375	0.1426	0.1485
2.6	0.1043	0.1071	0.1103
2.8	0.0766	0.0770	0.0787
3.0	0.0523	0.0531	0.0539
u = 1,25			
0.2	0.0013	0.0207	0.0905
0.4	0.0398	0.0735	0.1757
0.6	0.1020	0.1428	0.2509
0.8	0.1728	0.2142	0.3117
1.0	0.2385	0.2756	0.3549
1.2	0.2888	0.3191	0.3786
1.4	0.3180	0.3409	0.3826
1.6	0.3247	0.3409	0.3684
1.8	0.3113	0.3220	0.3391
2,0	0.2823	0.2890	0.2991
2.2	0.2433	0.2473	0.2530
2.4	0.2001	0.2024	0.2054
2.6	0.1574	0.1586	0.1601
2.8	0.1185	0.1191	0.1199
3.0	0.0856	0.0859	0.0862

	Lower bound	Upper bounds	
z	$\tilde{p}$	p_G	p_0
u = 1,5			
0.2	0.0086	0.0209	0.0773
0.4	0.0527	0.0735	0.1522
0.6	0.1185	0.1429	0.2221
0.8	0.1921	0.2159	0.2839
1.0	0.2614	0.2820	0.3345
1.2	0.3174	0.3335	0.3710
1.4	0.3544	0.3660	0.3910
1.6	0.3702	0.3780	0.3937
1.8	0.3654	0.3704	0.3797
2.0	0.3430	0.3460	0.3512
2.2	0.3073	0.3090	0.3138
2.4	0.2636	0.2645	0.2659
2.6	0.2166	0.2171	0.2178
2.8	0.1708	0.1710	0.1713
3.0	0.1293	0.1294	0.1295
u = 1,75			
0.2	0.0114	0.0187	0.0604
0.4	0.0534	0.0655	0.1208
0.6	0.1141	0.1278	0.1807
0.8	0.1825	0.1954	0.2386
1.0	0.2496	0.2602	0.2920
1.2	0.3081	0.3162	0.3378
1.4	0.3532	0.3588	0.3724
1.6	0.3813	0.3849	0.3930
1.8	0.3909	0.3931	0.3977
2.0	0.3826	0.3838	0.3863
2.2	0.3584	0.3891	0.3604
2.4	0.3220	0.3223	0.3229
2.6	0.2775	0.2777	0.2780
2.8	0.2297	0.2297	0.2299
3.0	0.1826	0.1826	0.1826
u = 2			
0.2	0.0110	0.0151	0.0435
0.4	0.0461	0.0527	0.0885
0.6	0.0963	0.1035	0.1361
0.8	0.1544	0.1609	0.1863
1.0	0.2147	0.2198	0.2375
1.2	0.2722	0.2759	0.2873
1.4	0.3226	0.3251	0.3320
1.6	0.3622	0.3637	0.3677
1.8	0.3878	0.3886	0.3908
2.0	0.3973	0.3977	0.3988
2.2	0.3902	0.3905	0.3910
2.4	0.3679	0.3680	0.3682
2.6	0.3331	0.3331	0.3332
2.8	0.2896	0.2896	0.2897
3.0	0.2419	0.2420	0.2420

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