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STABLE LAWS OF INDEX 2^{-n}

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The author expresses the distributions of stable laws of index 2^{-n} in terms of the normal and Chi squared distributions. This paper is an extension of an earlier result obtained by the author when he considered symmetric cases only.

1. Introduction and results. The logarithm of the characteristic function of a stable law as given in [1], has the representation

$$\log f(t) = i\gamma t - c\,|\,t\,|^{\alpha}\{1 - i\beta t\omega(t, \alpha)/|\,t\,|\} \tag{1.1}$$

where α, β, and γ are constants ($-1 \le \beta \le 1$, $0 < \alpha \le 2$, $c \ge 0$) and

$$\omega(t, \alpha) = \begin{cases} \tan(\Pi\alpha/2) & \text{if} \quad \alpha \ne 1 \\ (2/\Pi)\log|\,t\,| & \text{if} \quad \alpha = 1. \end{cases}$$

In a recent paper [3], the author has obtained the distributions that correspond to the symmetric stable laws of index 2^{-n}. In this article we shall still consider $\alpha = 2^{-n}$ but delete the restriction that $\beta = 0$. Since the characteristic function with $\gamma \ne 0$ and negative values of β will correspond to an appropriate linear transformation of a random variable with $\gamma = 0$ and positive values of β, we may assume without any loss of generality that $0 \le \beta \le 1$ and $\gamma = 0$.

We introduce in the following definitions two random variables U_n and W_n which will play fundamental roles in subsequent discussions.

DEFINITION 1.1. Let $X_1, X_2, \cdots, X_n$ be independent standard normal for $n \ge 2$. Define

$$U_n = X_1/V_n,$$

where

$$V_n = \begin{cases} X_2 & \text{for} \quad n = 2 \\ \exp_2[2^{n-2} - 1]X_2(X_3^2)(X_4^2)^2 \cdots (X_n^2)^{2^{n-3}} & \text{for} \quad n \ge 3, \end{cases}$$

where $\exp_2 b$ represents 2^b.

It is known [3] that the distribution of U_n is symmetric stable of index 2^{2-n}.

DEFINITION 1.2. Let $Y_1, Y_2, \cdots, Y_n$ be independent random variables, each being the reciprocal of a Chi squared random variable with 1 degree of freedom, i.e. the density of $Y_i (i = 1, 2, \cdots, n)$ is given by

$$f(y) = \begin{cases} (2\Pi)^{-1/2}\exp(-1/(2y))y^{-3/2} & y > 0, \\ 0 & y < 0. \end{cases}$$

Define

$$W_n = \begin{cases} Y_1 & \text{if} \quad n = 1 \\ Y_1(Y_2b_2^2)^2(Y_3b_3^2)^4 \cdots (Y_nb_n^2)^{2^{n-1}} & \text{if} \quad n \ge 2, \end{cases}$$

where

$$b_k = [2\sec(\Pi/[2^k])]^{-1/2}/\cos(\Pi/[2^{k+1}]) \quad k = 2, 3, \cdots n.$$

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Our main result is contained in the following.

THEOREM 1.1. *If U_{n+2} and W_n described respectively by Definitions* 1.1 *and* 1.2, *are independent then*

$$Y = [c(1-\beta)]^{2^n} U_{n+2} + (\beta c)^{2^n} W_n$$

has a stable distribution with index 2^{-n} and the given values of c and β.

2. Stable laws. The proof of Theorem 1.1 depends upon a few lemmas and corollaries.

LEMMA 2.1. *For an $a > 0$ and $b \geq 0$.*

$$(2.1) \quad \int_0^\infty \exp(itx - ax/2 - b^2/2x)x^{-1/2}\,dx$$
$$= (2\Pi)^{1/2}R^{-1}i\exp(-bR\sin\theta + i[bR\cos\theta - \theta]),$$

where

$$(2.2) \quad R = (a^2+4t^2)^{1/4};\ \sin\theta = [(R^2+a)/(2R^2)]^{1/2};\ \cos\theta = [(R^2-a)/(2R^2)]^{1/2}t/|t|.$$

PROOF. We denote the left side of (2.1) by *I*. From the properties of the normal distribution, we know

$$(2.3) \quad \exp(-b^2/2x) = x^{1/2}(2\Pi)^{-1/2}\int_{-\infty}^\infty \exp(ibu - xu^2/2)\,du.$$

Replacing $\exp(-b^2/2x)$ in (2.1) by the integral on the right side of (2.3), and interchanging the order of integration, we obtain

$$(2.4) \quad I = (2\Pi)^{-1/2}\int_{-\infty}^\infty\int_0^\infty \exp(ibu)\exp(itx - x(a+u^2)/2)\,dx\,du.$$

From the properties of the gamma distributions, the value of the inner integral turns out to be $2(a+u^2-2it)^{-1}$. Thus (2.4) yields

$$(2.5) \quad I = (2/\Pi)^{1/2}\int_{-\infty}^\infty \exp(ibu)(u^2+a-2it)^{-1}\,du.$$

Elementary calculations reveal that the only pole of the integrand in the upper half of the complex plane is at the point $Z_0 = R\exp(i\theta)$, and that it is a simple pole. The conclusion of Lemma 2.1 follows from a straightforward application of Cauchy's Residue Theorem and routine simplification.

By taking $b > 0$, differentiating both sides of (2.1) with respect to *b*, dividing both sides by *b* and performing standard algebraic operations, we obtain the following.

COROLLARY 2.1. *For $a > 0$, $b > 0$*

$$\int_0^\infty \exp(itx - ax/2 - b^2/2x)x^{-3/2}\,dx = (2\Pi)^{1/2}b^{-1}\exp(-bR\sin\theta + ibR\cos\theta).$$

Taking $b = 1$ and $a \to 0$ in Corollary 2.1, we observe from (2.2) that $R \to (2|t|)^{1/2}$, $\sin\theta \to 2^{-1/2}$ and $\cos\theta \to t/|t|2^{-1/2}$. Thus Corollary 2.1 yields the following.

COROLLARY 2.2. $(2\Pi)^{-1/2}\int_0^\infty \exp(itx - \tfrac{1}{2}x)x^{-3/2}\,dx = \exp[-|t|^{1/2}(1 - it/|t|)] = \exp[-|t|^{1/2}(1 - it\omega(t, \tfrac{1}{2})/|t|)]$.

Corollary 2.2 affirms that the reciprocal of the Chi squared distribution with 1 degree of freedom has a stable distribution of index ½, $\beta = 1$ and $c = 1$. This result has already been derived by Paul Levy [2].

We need one more theorem before we can prove our main result.

THEOREM 2.1. *The distribution of the random variable W_n, described in Definition 1.2, is stable with index 2^{-n}, $c = 1$, and $\beta = 1$.*

PROOF. The theorem will be proved by induction. Corollary 2.2 affirms the validity of the theorem for $n = 1$. Assume next that the theorem is true for $n = k$. Let $\alpha = 2^{-k}$. By the induction hypothesis, we have

$$E[\exp(itW_k)] = \exp[-|t|^{\alpha}(1 - it\omega(t, \alpha)/|t|)]. \tag{2.6}$$

Observe that

$$W_{k+1} = (Y_{k+1}b_{k+1}^2)^{2^k} W_k. \tag{2.7}$$

We have

$$\begin{aligned} E(\exp(itW_{k+1})] &= E[\exp(it(Y_{k+1}b_{k+1}^2)^{2^k} W_k)] \\ &= \int_0^{\infty} (2\Pi)^{-1/2}\exp[-|t|^{\alpha}y_{k+1}b_{k+1}^2(1 - it\omega(t, \alpha)/|t|)] \exp(-1/(2y_{k+1}))(y_{k+1})^{-3/2}\, dy_{k+1}. \end{aligned} \tag{2.8}$$

Replacing a by $2|t|^{\alpha}b_{k+1}^2$ and t by $|t|^{\alpha}b_{k+1}^2[t\omega(t, \alpha)/|t|]$ in (2.2) we obtain

$$R = [4|t|^{2\alpha}b_{k+1}^4 + 4|t|^{2\alpha}b_{k+1}^4\omega^2(t, \alpha)]^{1/4} = 2^{1/2}|t|^{\alpha/2}b_{k+1}[\sec(\Pi\alpha/2)]^{1/2}, \tag{2.9}$$

$$\sin\theta = \left[\frac{1 + \cos(\Pi\alpha/2)}{2}\right]^{1/2} = \cos(\Pi\alpha/4);\ \cos\theta = t\sin(\Pi\alpha/4)/|t|. \tag{2.10}$$

From Corollary 2.1, (2.8)–(2.10) and the fact that $\alpha = 2^{-k}$,

$$b_{k+1} = [2\sec(\Pi/[2^{k+1}])]^{-1/2}/\cos(\Pi/[2^{k+2}]).$$

we obtain the desired result for the characteristic function of W_{k+1}.

Thus the theorem is proved by induction.

PROOF OF THEOREM 1.1. Since U_{n+2} and W_n are independent, the characteristic function of Y equals the product of the characteristic functions of $[c(1-\beta)]^{2^n}U_{n+2}$ and $(\beta c)^{2^n}W_n$.

With the results of Theorem 1 in [3] and Theorem 2.1 of this article we obtain with $\alpha = 2^{-n}$

$$\begin{aligned} E[\exp[itY)] &= \exp[-c(1-\beta)|t|^{2^{-n}}]\exp[-\beta c|t|^{2^{-n}}(1 - it\omega(t, \alpha)/|t|)] \\ &= \exp[-c|t|^{2^{-n}}(1 - it\beta\omega(t, \alpha)/|t|)]. \end{aligned}$$

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