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STRUCTURE OF A CLASS OF OPERATOR-SELFDECOMPOSABLE PROBABILITY MEASURES[1]

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In 1972, K. Urbanik introduced the notion of operator-selfdecomposable probability measures (originally they were called Lévy's measures). These measures are identified as limit distributions of partial sums of independent Banach space-valued random vectors normed by linear bounded operators. Recently, S. J. Wolfe has characterized the operator-selfdecomposable measures among the infinitely divisible ones. In this note we find examples of measures whose finite convolutions are a dense subset in a class of all operator-selfdecomposable ones.

1. Introduction. An important class of limit laws are those which serve as limiting probability distributions of normed sums of independent random variables. It seems to be quite natural that in multidimensional linear spaces the partial sums of sequence of random elements should be normed by linear operators or by affine transformations. In this setting we get the classes of operator-selfdecomposable and operator-stable probability measures. Jurek (1981a) gives a bibliography for those problems on Euclidean and Banach spaces. The aim of this note is to extend the results of Jurek (1980) and to give a structure characterization of the class of operator-selfdecomposable measures. This class was introduced by Urbanik (1978). His fundamental paper gives descriptions of operator-selfdecomposable μ in terms of properties of its decomposability semigroup $\mathbb{D}(\mu)$ (cf. Section 3) and its characteristic functional. This last one is proved via Choguet's Theorem on extreme points of compact convex sets. A more direct proof is given in Jurek (1981b). Recently, Wolfe (1980) has found another characterization of operator-selfdecomposable measures based on the Lévy spectral function associated with infinitely divisible measure. It seems that in order to carry out Wolfe's proof in detail we need the uniqueness of representation of each vector by a one-parameter group of operators and a subset of the unit sphere (cf. Proposition 2). The crucial step for our consideration is given in Proposition 5. Finally, we emphasize that originally, by Urbanik, the operator-selfdecomposable measures were called Lévy's probability measures.

2. Preliminaries and notations. Let E be a real separable Banach space with the dual E^* and a norm $\|\cdot\|$. By $\langle\cdot,\cdot\rangle$ we denote the duality pairing between E^* and E. Further, $\mathbb{B}(E)$ is the algebra of all continuous linear operators on E with the norm topology. For $A \in \mathbb{B}(E)$ and a strictly positive real number t, i.e. $t \in \mathbb{R}^+$, by t^A we mean the operator $\sum_{k=0}^{\infty} (B \log t)^k/k! \in \mathbb{B}(E)$. By $\mathscr{P}(E)$ we denote the topological semigroup of all Borel probability measures on E with the convolution $*$ and weak convergence. Given an operator $A \in \mathbb{B}(E)$ and a measure $\mu \in \mathscr{P}(E)$, we write $A\mu$ for the probability measure defined by $(A\mu)(F) := \mu(A^{-1}F)$ for all Borel subsets F of E. It is easy to check that

$$A(\mu * \nu) = A\mu * A\nu, \qquad A(B\mu) = (AB)\mu, \qquad (A\mu)\hat{\ }(y) = \hat{\mu}(A^*y)$$

where $A, B \in \mathbb{B}(E)$, $\mu, \nu \in \mathscr{P}(E)$, $\hat{\mu}$ is the characteristic functional of μ, A^* is the adjoint

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operator for A and $y \in E^*$. Moreover the mapping $h: \mathbb{B}(E) \times \mathscr{P}(E) \to \mathscr{P}(E)$, defined by $h(A, \mu) = A\mu$, is continuous i.e. if $A_n \to A$ in $\mathbb{B}(E)$ and $\mu_n \Rightarrow \mu$ in $\mathscr{P}(E)$ then $A_n\mu_n \Rightarrow A\mu$. By $\delta_x (x \in E)$ we denote the probability measure concentrated at the point x. We say that $\mu \in \mathscr{P}(E)$ is infinitely divisible if for each natural number $n \geq 2$ there exists a $\mu_n \in \mathscr{P}(E)$ such that $\mu_n^{*n} = \mu$. The class $ID(E)$ of all infinitely divisible probability measures on E is a closed subsemigroup of $\mathscr{P}(E)$. Moreover, it is known (see for example de Acosta, Araujo, Gine, 1978) that $\lambda \in ID(E)$ if and only if

$$\hat{\lambda}(y) = \hat{\rho}(y) \exp\left[i \langle y, x_0 \rangle + \int_E K(y, x) M(dx) \right], \qquad y \in E^*. \tag{2.1}$$

Here ρ is a Gaussian measure on E (i.e. for all $y \in E^*$, the random variables $y\rho$ is Gaussian), $x_0 \in E$, K is the function on $E^* \times E$ given by

$$K(y, x) = \exp(i \langle y, x \rangle - 1 - i \langle y, x \rangle 1_B(x)) \tag{2.2}$$

(1_B denoting the indicator function of the unit ball $B = \{x \in E : \| x \| \leq 1\}$), and M is σ-finite measure on E which is finite on the complement of every neighbourhood of 0 and $M(\{0\}) = 0$. For a symmetric Gaussian measure ρ

$$\hat{\rho}(y) = \exp(-\tfrac{1}{2} \langle y, Ry \rangle), \qquad y \in E^*, \tag{2.3}$$

where R is called the *covariance operator*. R is a compact operator from E^* into E with the properties: $\langle y_1, Ry_2 \rangle = \langle y_2, Ry_1 \rangle$ for $y_1, y_2 \in E^*$ (symmetry) and $\langle y, Ry \rangle \geq 0$ (non-negativity). Since the representation (2.1) is unique we write $\lambda = [x_0, R, M]$, if $\hat{\lambda}$ is of the form (2.1) with (2.3). The measure M in (2.1) is called *generalized Poisson measure* (or Lévy measure) of λ (cf. de Acosta, Araujo, Gine, 1978, Theorem 1.6). Let us note that if $A \in \mathbb{B}(E)$ and $\lambda = [x_0, R, M] \in ID(E)$ then $A\lambda = [\tilde{x}_0, ARA^*, AM] \in ID(E)$, where $\tilde{x}_0 = Ax_0 + \int_B x(AM)(dx) - A \int_B xM(dx)$.

Let f and g be real-valued functions defined on an interval $[a, b]$ such that the right f'_+, g'_+ and the left derivatives f'_-, g'_- exist at every point of $[a, b)$ and $(a, b]$ respectively. In the sequel f' and g' denote either the right or left derivative possibly different one at different points.

PROPOSITION 1. *If $g' > 0$ and the function h ($h(x) = f'_+(x)/g'_+(x)$ or $h(x) = f'_-(x)/g'_-(x)$) does not increase in $[a, b]$ then for every $a < c < b$ such that $g(c) \neq g(a)$ and $g(b) \neq g(c)$ we have*

$$[f(c) - f(a)]/[g(c) - g(a)] \geq [f(b) - f(c)]/[g(b) - g(c)].$$

PROOF. For functions f and g having only the left and right derivatives the Cauchy mean value theorem has the following form (cf. Kubik, 1962); if $g(c) \neq g(a)$ then there exists $a < d < c$ such that

$$\{g'_+(d)[f(c) - f(a)]/[g(c) - g(a)] - f'_+(d)\} \cdot \{g'_-(d)[f(c) - f(a)]/[g(c) - g(a)] - f'_-(d)\} \leq 0.$$

The analogous inequality is obtained for the interval $[c, d]$ and some point $e \in (c, d)$. Since $g' > 0$ and h is not increasing we get

$$[f(c) - f(a)]/[g(c) - g(a)] \geq h(d) \geq h(e) \geq [f(b) - f(c)]/[g(b) - g(c)]$$

which completes the proof of the Proposition.

3. Operator-selfdecomposable measures. Let $\xi_1, \xi_2, \cdots$ be a sequence of independent E-valued random elements and let $A_1, A_2, \cdots$ be a sequence of operators from $\mathbb{B}(E)$ such that

(i) A_n is invertible ($n = 1, 2, \cdots$),

(ii) the closed multiplicative semigroup in $\mathbb{B}(E)$ generated by $\{A_m A_n^{-1}: n = 1, 2, \cdots, m; m = 1, 2, \cdots\}$ is compact in the norm topology,
(iii) the triangular array $A_n \xi_j (j = 1, 2, \cdots, n; n = 1, 2, \cdots)$ of random elements is infinitesimal (null array).

A limit probability distribution μ of the sequence

$$A_n(\xi_1 + \xi_2 + \cdots + \xi_n) + x_n, \tag{3.1}$$

where $\{x_n\} \subseteq E$, is called *operator-selfdecomposable measure* (in Urbanik, 1978, these measures are called Lévy probability measure). Further, let us note that for *full* μ (i.e. the support of μ is not contained in any proper hyperplane of E) on finite dimensional spaces and on Banach spaces when A_n are multiples of identity operator I, the compactness condition (ii) can be omitted (cf. Urbanik, 1978). In the study of limit probability distribution of (3.1), Urbanik introduced the concept of *decomposability semigroup* $\mathbb{D}(\mu)$ of linear bounded operators associated with $\mu \in \mathscr{P}(E)$. Namely,

$$\mathbb{D}(\mu) := \{A \in \mathbb{B}(E): \exists (\nu \in \mathscr{P}(E))\ \mu = A\mu * \nu\}.$$

It is clear that $\mathbb{D}(\mu)$ is a semigroup under multiplication of operators and $\mathbb{D}(\mu)$ always contains the operators O and I.

THEOREM 1. (K. Urbanik, 1978). *A full measure $\mu \in \mathscr{P}(E)$ is operator-selfdecomposable if and only if there exists $Q \in \mathbb{B}(E)$ with the property $\lim_{t\to 0} t^Q = 0$ such that semigroup $\{t^Q: 0 < t < 1\}$ is contained in $\mathbb{D}(\mu)$.*

Let us note that for each one-parameter group $\{t^A: t \in \mathbb{R}^+\}$ satisfying the condition $\lim_{t\to 0} t^A = 0$, the orbits $r(x) = \{t^A x: t \in \mathbb{R}^+\}$ for $x \neq 0$ intersect the unit sphere in E at least once, but not necessarily exactly once.

EXAMPLE. Let $E = \mathbb{R}^2$ be the Euclidean space with the norm $\|\cdot\|$ and let $A = \begin{pmatrix} 1 & -k \\ 0 & 2 \end{pmatrix}$ for some $k \in \mathbb{R}$. Then, for $t > 0$, $t^A = t\begin{pmatrix} 1 & -k(t-1) \\ 0 & t \end{pmatrix}$ and for $x = (a, b) \in E$; $\|x\| = 1$, we get $\|t^A x\|^2 = t^2((a - bk(t-1))^2 + b^2t^2)$. Further, for f_x defined by $f_x(t) := \|t^A x\|^2 - 1$ we obtain $f_x(t) = (t-1)[b^2(k^2+1)t^3 + (b^2 - 2abk - b^2k^2)t^2 + t + 1]$. Hence, in the particular case, taking $x_0 = (\sqrt{15}/4, 1/4)$ and $k = \sqrt{15}$, we infer that $f_{x_0}(1) = f_{x_0}(2) = 0$, i.e. for $t = 1$ and $t = 2$ the orbit $r(x_0)$ intersects the unit sphere in E.

PROPOSITION 2. *Let $\{t^A: t \in \mathbb{R}^+\}$ be a one-parameter group in $\mathbb{B}(E)$ with the property $\lim_{t\to 0} t^A = 0$, and let $S_A := \{x \in E: (\|x\| = 1) \wedge \forall (t > 1) \|t^A x\| > 1\}$. Then the function $\Phi: S_A \times \mathbb{R}^+ \to E\backslash\{0\}$ defined by $\Phi(u, t) = t^A u$ is a Borel isomorphism between $S_A \times \mathbb{R}^+$ and $E\backslash\{0\}$.*

PROOF. By some simple arguments we get that S_A is a Borel subset of the unit sphere in E. Further, each orbit $r(x)$ intersects S_A exactly once ($x \neq 0$). Thus Φ is $1-1$, onto and continuous mapping. By Kuratowski's Theorem (see [6], Corollary 3.3, page 22) we get that Φ is a Borel isomorphism betwen the sets $S_A \times \mathbb{R}^+$ and $E\backslash\{0\}$, which completes the proof.

The existence of the set S_A, for which the statement in Proposition 2 holds true, in Euclidean spaces was proved in Jurek (1979). The above construction of S_A is essentially due to Hudson and Mason (1981). It is worth noticing that there is an analogue of Proposition 2 for strongly continuous one-parameter group as well, but with S_A replaced by a Borel subset of $E\backslash\{0\}$, cf. Jurek (1981b), Proposition 2.1.

For multiplicative semigroup $\{t^A: 0 < t \le 1\}$ in $\mathbb{B}(E)$ satisfying the condition $\lim_{t\to 0} t^A = 0$ there exist positive numbers β and γ such that

$$\|t^A\| \le \beta t^\gamma \text{ for each } 0 < t \le 1, \tag{3.2}$$

(cf. Yosida, 1965, page 232). Hence, for $x \in E$ and $t \in [a, b] \subseteq \mathbb{R}^+$ we have

$$\| t^A x \| \le \beta (t/b)^\gamma \| b^A x \| \quad \text{and} \quad \beta^{-1}(t/a)^\gamma \| a^A x \| \le \| t^A x \|,$$

and consequently for $x \neq 0$ we get

$$g(a, b; x) := \sup_{t\in[a,b]} \| t^A x \| / \inf_{t\in[a,b]} \| t^A x \| \le \beta^2 \| b^A x \| / \| a^A x \| \le \beta^3 (b/a)^\gamma. \tag{3.3}$$

Let be given the one-parameter group $\{t^A : t \in \mathbb{R}^+\}$ and the measure $\mu = [x_0, R, M] \in ID(E)$. For a Borel subset F of S_A and a Borel subset I of $\mathbb{R}^+$ we put $[F, I] := \Phi(F \times I)$ i.e.

$$[F, I] = \{t^A x \in E : x \in F,\, t \in I\}.$$

It is easy to check that $s^A([F, I]) = [F, sI]$. Finally, A-*Lévy spectral function* $L^A(F, r)$ of μ we define by

$$L^A(F, r) := -M(\Phi(F \times (r, \infty))), \qquad r \in \mathbb{R}^+. \tag{3.4}$$

In view of Proposition 2 we have that $-L^A(., r)$ is finite Borel measure on S_A, $L^A(F, .)$ is non-decreasing function, right-continuous with $\lim_{r\to\infty} L^A(F, r) = 0$. Now we can formulate Wolfe's result as follows.

THEOREM 2. (Wolfe, 1980). *A full measure $\mu \in \mathscr{P}(E)$ is operator-selfdecomposable if and only if $\mu = [x_0, R, M]$ and there is a $Q \in \mathbb{B}(E)$ with the property* $\lim_{t\to 0} t^Q = 0$ *such that Q-Lévy spectral function $L^Q(F, r)$ of μ, for each Borel subset F of S_Q, has right and left derivatives at each value $r \in \mathbb{R}^+$, the function $r[\partial L^Q(F, r)/\partial r]$ is non-increasing on $\mathbb{R}^+$ for each F and in addition $QR + RQ^*$ is non-negative linear operator on E^*. [Here $\partial L^Q(E, r)/\partial r$ denotes the right or left derivatives, possible different ones at different points.]*

4. Generators of a set $\mathscr{K}_Q$. Let $Q \in \mathbb{B}(E)$ with the property $\lim_{t\to 0} t^Q = 0$ be fixed. We define the following class of measures

$$\mathscr{K}_Q := \{\mu \in \mathscr{P}(E) : s^Q \in \mathbb{D}(\mu) \quad \text{for each} \quad 0 < s < 1\},$$

where $\mathbb{D}(\mu)$ denotes the decomposability semigroup of μ. For the set $\mathscr{K}_Q$ we have the following.

PROPOSITION 3. (i) *If $AQ = QA$, $\mu \in \mathscr{K}_Q$ and $x \in E$ then $A\mu * \delta_x \in \mathscr{K}_Q$.*

(ii) *For each $\alpha > 0$, $\mathscr{K}_Q = \mathscr{K}_{\alpha Q}$.*
(iii) *$\mathscr{K}_Q$ is closed subsemigroup of $ID(E)$.*

PROOF. The properties (i), (ii) and semigroup structure of $\mathscr{K}_Q$ are simple consequences of the definition of $\mathscr{K}_Q$. Lemma 4.4 in Urbanik (1978) implies that $\mathscr{K}_Q \subseteq ID(E)$. Finally, if $\mu_n = s^Q \mu_n * \nu_{s,n}$ for every $0 < s < 1$ and $\mu_n \Rightarrow \mu$ then the sequence $\{\nu_{s,n}\}_{n=1}^{\infty}$ is conditionally compact (cf. Parthasarathy, 1967, page 58) and $\lim_{n\to\infty} \hat{\nu}_{s,n}(y)$ exists. Hence $\mu = s^Q \mu * \nu_s$ for some $\nu_s \in \mathscr{P}(E)$, which proves the closedness of $\mathscr{K}_Q$, and Proposition 3 is completely proved.

Let m be a finite Borel measure on S_Q (cf. Proposition 2), and λ_α ($\alpha \in \mathbb{R}^+$) be a Borel measure on $\mathbb{R}^+$ with density $1_{(0,\alpha]}(t)t^{-1}$. By $M_{\alpha,m}$ we denote the Borel measure on $E \setminus \{0\}$ defined as follows:

$$M^Q_{\alpha,m} := \Phi(m \times \lambda_\alpha) \quad \text{i.e.} \quad M^Q_{\alpha,m}(F) = \int_{S_Q} \int_0^\alpha 1_F(s^Q x) s^{-1}\, ds\, m(dx). \tag{4.1}$$

Further, by Proposition 2 and the inequality (3.2) we get

$$\int_E (1 \wedge \|x\|) M_{\alpha,m}(dx) \le M_{\alpha,m}(\|x\| > 1) + \int_{S_Q} \int_0^{\alpha \wedge 1} \|s^Q x\| s^{-1} \, ds m(dx)$$

$$\le M_{\alpha,m}(\|x\| > 1) + m(S_Q)\gamma^{-1}\beta(\alpha \wedge 1)^{\gamma} < \infty,$$

where $a \wedge b$ denotes the minimum of real numbers. Therefore Araujo and Gine (1978) implies that $M_{\alpha,m}$ is generalized Poisson exponent, i.e. $[x, 0, M_{\alpha,m}] \in ID(E)$. Let us note that if we replace the operator Q by $Q_1 = aQ (a \in \mathbb{R}^+)$ then $S_{Q_1} = S_Q$ and

(4.2) $$M^{Q_1}_{\alpha,m} = M^{Q}_{\alpha_1,m_1}, \quad \text{where} \quad \alpha_1 = \alpha^a, \qquad m_1 := a^{-1} m.$$

Hence the general shape of the measures $M^Q_{\alpha,m}$ is invariant under the transformation $Q \to aQ (a \in \mathbb{R}^+)$.

Proposition 4. *The infinitely divisible measures* $\mu = [x, 0, M_{\alpha,m}]$, *where* $M_{\alpha,m}$ *is of the form* (4.1), *and* $\mu = [x, R, 0]$ *such that the operator* $QR + RQ^*$ *is non-negative, belong to the class* K_Q.

Proof. Of course, it is sufficient to verify that in the first case

(4.3) $$M_{\alpha,m} \ge s^Q M_{\alpha,m} \quad \text{for each} \quad 0 < s < 1,$$

and in the second case that

(4.4) $$R - s^Q R S^{Q^*} \ge 0 \quad \text{for each} \quad 0 < s < 1.$$

Taking into account (4.1) we have

$$s^Q M_{\alpha,M}([F, A]) = m(F) \int_A 1_{(0,s\alpha]}(t) t^{-1} \, dt \le M_{\alpha,m}([F, A]),$$

i.e. (4.3) is fulfilled on the sets $[F, A]$, and by Proposition 2 we infer that (4.3) is true for arbitrary Borel subset of $E \backslash \{0\}$.

Given $y \in E^*$, we put

$$f_y(s) := \langle y, (R - s^Q R s^{Q^*})y \rangle \quad \text{for} \quad 0 < s \le 1.$$

By a simple computation we get

$$df_y(s)/ds = -s^{-1} \langle s^{Q^*} y, (QR + RQ^*) s^{Q^*} y \rangle$$

which implies $df(s)/ds \le 0$. On the other hand, $f_y(1) = 0$ and $\lim_{s \to 0} f_y(s) = \langle y, Ry \rangle \ge 0$. Thus the operator $R - s^Q R s^{Q^*}$ is non-negative for each $0 < s < 1$, which completes the proof of Proposition 4.

Proposition 5. *For each* $\mu = [0, 0, M] \in \mathscr{K}_Q$ *there exist subsequences* $\{k_n\}$ *of positive integers, positive real numbers* α_{nj} *and positive finite Borel measures* m_{nj} *on* $S_Q (j = 1, 2, \cdots, k_n; n = 1, 2, \cdots)$ *such that*

(a) $$N_n := \textstyle\sum_{j=1}^{k_n} M_{\alpha_{nj}, m_{nj}} \Rightarrow M,$$

outside every neighbourhood of zero in E. Moreover, if for some $p > 0$, $\int_E (1 \wedge \|x\|^p) M(dx) < \infty$ *then*

(b) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{\|x\| \le \varepsilon} \|x\|^p N_n(dx) = 0.$$

Proof. Let L^Q be Q-Lévy spectral function of μ (cf. (3.4)). By Proposition 2, $-L^Q(\cdot, t)$ is a finite Borel measure on S_Q for each $t > 0$. For Borel subset F of S_Q let us put $b_{nk}(F) = L^Q(F, k/2^n)(k = 1, 2, \cdots, n)$ and

(4.5) $$a_{nk}(F) = [b_{nk}(F) - b_{nk-1}(F)]/\log(k/(k-1)), \quad \text{for} \quad k = 2, 3, \cdots, n2^n,$$

and $a_{n1}(F) = a_{n2}(F)$, $a_{n,n2^n+1}(F) = b_{n,n2^n+1}(F) \equiv 0$ for all Borel subset F of S_Q. Taking into account Theorem 2 and applying Proposition 1 for the function $r\partial L^Q(F, r)/\partial r$ we infer that

$$a_{n,k+1}(F) \le a_{nk}(F), \quad \text{for} \quad k = 1, 2, \cdots, n2^n.$$

Hence

(4.6) $$m_{nk} = a_{nk} - a_{n,k+1}, \quad \text{for} \quad k = 1, 2, \cdots, n2^n,$$

are non-negative finite Borel measures on S_Q. Further, let us put

(4.7) $$\alpha_{nk} = k/2^n \quad \text{for} \quad k = 1, 2, \cdots, n2^n \quad \text{and} \quad k_n = n2^n.$$

We start by proving the condition (a). It is enough to verify it for the sets of the form $[F, (t, \infty)]$. Let H_n^Q be the Q-Lévy spectral function of $[0, 0, N_n]$ i.e. $H_n^Q(F, r) = -N_n([F, (r, \infty)])$. Taking into account (4.1), (4.5) through (4.7), we infer that if $(i-1)/2^n \le r \le i/2^n$ for some $i \in \{2, 3, \cdots, k_n\}$ then

$$\begin{aligned} H_n^Q(F, r) &= \textstyle\sum_{k=2}^{k_n} (a_{nk}(F) - a_{n,k+1}(F))\log(1 \wedge (2^n r)/k) \\ &= a_{ni}(F)\log(2^n r/i) - \textstyle\sum_{k=i}^{k_n} [a_{nk}(F) - a_{n,k+1}(F)]\log k \\ &= a_{ni}(F)\log(2^n r/i) + b_{ni}(F) - b_{n,k_n}(F). \end{aligned}$$

For fixed $i \ge 2$, we define the functions $L_{ni}(F, r)$ as follows

$$L_{ni}(F, r) := a_{ni}(F)\log(2^n r/i) + b_{ni}(F), \quad \text{for} \quad (i-1)/2^n \le r \le i/2^n.$$

Hence we get

$$|L_{ni}(F, r) - L^Q(F, r)| \le L^Q(F, i/2^n) - L^Q(F, (i-1)/2^n),$$

because of $L_{ni}(F, i/2^n) = L^Q(F, i/2^n)$ and $L_{ni}(F, (i-1)/2_n) = L^Q(F, (i-1)/2^n)$. Since for $\varepsilon > 0$ there exists a positive integer n_0 such that for $n \ge n_0$ and $k = [2^n t] + 1, \cdots, k_n$ we have

$$L^Q(F, k/2^n) - L^Q(F, (k-1)/2^n) \le \varepsilon, \qquad -L^Q(F, n) \le \varepsilon,$$

therefore $\lim_{n\to\infty} H_n^Q(F, t) = L^Q(F, t)$, which proves the condition (a).

Now we proceed to a proof of the condition (b). At first, note that for every $\varepsilon > 0$ there exists a Borel subset A_ε of $\mathbb{R}^+$ such that $\{x \in E : 0 < \|x\| \le \varepsilon\} = [S_q, A_\varepsilon]$ (cf. Proposition 2), and $\eta := \sup A_\varepsilon \to 0$ if $\varepsilon \to 0$. In view of the formulae (4.5) through (4.7) and summing by parts, we get

$$\int_{\|x\|\le\varepsilon} \|x\|^p N_n\,(dx) \le \textstyle\sum_{k=1}^{k_n} \displaystyle\int_{S_Q}\int_0^{n\wedge\alpha_{nk}} \|t^Q x\|^p t^{-1}\,dt\, m_{nk}\,(dx)$$

(4.8) $$= \int_{S_Q}\int_0^{2^{-n}} \|t^Q x\|^p t^{-1}\,dt\, a_{n2}\,(dx) + \textstyle\sum_{k=2}^{[2^n\eta]} \displaystyle\int_{S_Q}\int_{(k-1)/2^n}^{k/2^n} \|t^Q x\|^p t^{-1}\,dt\, a_{nk}\,(dx)$$

$$+ \int_{S_Q}\int_{[2^n\eta]/2^n}^{\eta} \|t^Q x\|^p t^{-1}\,dt\, a_{n,[2^n\eta]+1}\,(dx),$$

where $[x]$ denotes the integral part of a real number x. Further, by the First Mean Value Theorem and the inequality (3.3), for $x \in S_Q$ and the interval $I_{nk} := ((k-1)/2^n, k/2^n]$, in $\mathbb{R}^+ (k = 2, 3, \cdots, [2^n\eta])$, there exists $t_{nk} \in I_{nk} \cup \{(k-1)/2^n\}$ such that

$$\int_{I_{nk}} \|t^Q x\|^p d \log t = \|t_{nk}^Q x\|^p \log(k/k-1)$$
$$\le g^p((k-1)/2^n, k/2^n; x)\log(k/k-1)\|s^Q x\|^p$$
$$\le C \log(k/k-1)\|s^Q x\|^p, \ (C = C(\beta, \gamma, p) = \beta^{3p}2^{\gamma p}),$$

for arbitrary $s \in I_{nk}$. This, together with (4.5), gives us

$$\int_{S_Q}\int_{I_{nk}} \|t^Q x\|^p t^{-1}\, dt\, a_{nk}\,(dx) = \int_{S_Q}\int_{I_{nk}} \|t_{nk}^Q x\|^p M(\Phi\,(dx, ds))$$
$$\le C \int_{S_Q}\int_{I_{nk}} \|s^Q x\|^p M(\Phi\,(dx, ds)) = C\int_{[S_Q, I_{nk}]} \|z\|^p M\,(dz).$$

Hence the second component in (4.8) is not greater than $C\int_{[S_Q,(0,\eta]]}\|z\|^p M\,(dz)$, therefore it converges to zero if $n \to \infty$ and then $\varepsilon \to 0$. By the same arguments we obtain that the third component converges to zero too. Since for the first component in (4.8) we have

$$\int_{S_Q}\int_0^{2^{-n}} \|t^Q x\|^p t^{-1}\, dt\, a_{n2}\,(dx) = \int_{S_Q}\int_{2^{-n}}^{2^{-n+1}} \|(t-2^{-n})^Q x\|^p (t-2^{-n})^{-1}\, dt\, a_{n2}\,(dx)$$
$$\le \beta^p \int_{S_Q}\int_{2^{-n}}^{2^{-n+1}} ((t-2^{-n})/t)^{\gamma p-1}\|t^Q x\|^p t^{-1}\, dt\, a_{n2}\,(dx)$$
$$\le C\beta^p \int_{[S_Q, I_{n2}]} \|z\|^p M\,(dx),$$

it converges to zero as $n \to \infty$. In the last inequality we could assume that $\gamma p \ge 1$ because of (4.2) or Proposition 3 (ii). Thus Proposition 5 is completely proved.

Let us denote by $\mathscr{G}_Q$ the class of all measures of the form $[x_0, 0, M_{\alpha,m}]$ or $[x_0, R, 0]$, where $x_0 \in E$, the operator $QR + RQ^*$ is non-negative and $M_{\alpha,m}$ is defined by (4.1). By Proposition 4 we have $\mathscr{G}_Q \subseteq \mathscr{K}_Q$. Further, let $\tilde{\mathscr{K}}_Q$ denotes those measures $[x_0, R, M]$ in $\mathscr{K}_Q$ for which $\int_E (1 \wedge \|x\|)M\,(dx) < \infty$. Note that $\mathscr{G}_Q \subseteq \tilde{\mathscr{K}}_Q$ and $\tilde{\mathscr{K}}_Q$ is a proper subset of $\mathscr{K}_Q$.

As a simple consequence of Proposition 5 with $p = 2$ and Theorem 5.5, Chapter VI in Parthasarathy (1967), we get the following structural characterization of the class $\mathscr{K}_Q$.

THEOREM 3. *Let E be a real separable Hilbert space. The class $\mathscr{K}_Q$ is the smallest closed subsemigroup in $ID(E)$ generated by the set $\mathscr{G}_Q$.*

For an arbitrary Banach space we are able to prove only the following partial characterization.

THEOREM 4. *Each measure from $\tilde{\mathscr{K}}_Q$ is a weak limit of finite convolutions of measures from the set $\mathscr{G}_Q$.*

PROOF. Let $[0, 0, M] \in \tilde{\mathscr{K}}_Q$ and M be symmetric generalized Poisson exponent. The Proposition 5 with $p = 1$ and Theorem V.10.5 in Woyczynski (1978) gives us that $[0, 0, N_n] \Rightarrow [0, 0, M]$, where $[0, 0, N_n]$ is the finite convolution of elements from $\mathscr{G}_Q$. If M is not a symmetric measure then we may take its symmetrization i.e. $M + M^-(M^-(F) := M(-F)$, F an arbitrary Borel subset of $E\setminus\{0\}$). Therefore

$$[0, 0, N_n] * [0, 0, N_n^-] \Rightarrow [0, 0, M + M^-].$$

Hence and by Acosta, Araujo, Gine (1978) (Corollary 1.5 and Theorem 1.10) we get $[0, 0, N_n] \Rightarrow [0, 0, M]$, which completes the proof of Theorem 4.

5. A structure of operator-selfdecomposable measures. In this section we shall summarize our information about the structure of the class of all operator-selfdecomposable measures. Namely, combining Theorem 1 with Theorem 3 and 4 we obtain the following.

COROLLARY 1. *Let H be a real separable Hilbert space. A full probability measure μ is operator-selfdecomposable if and only if there exists $Q \in \mathbb{B}(H)$ with the property* $\lim_{t\to 0} t^Q = 0$, *such that μ is a weak limit of finite convolutions of measures from the set $\mathcal{G}_Q$.*

COROLLARY 2. *Let E be a real separable Banach space. If an infinitely divisible measure $\mu = [x_0, R, M]$ satisfying the condition $\int_E (1 \wedge \|x\|)M(dx) < \infty$, is a full operator-selfdecomposable measure then there exists $Q \in \mathbb{B}(E)$ with the property* $\lim_{t\to 0} t^Q = 0$, *such that μ is a weak limit of finite convolutions of measures from the set $\mathcal{G}_Q$. Conversely, each weak limit of finite convolutions of elements from $\mathcal{G}_Q$ is operator-selfdecomposable.*

In case Q is the identity operator, the above statements characterize the set $\mathcal{L}$ of all self-decomposable probability measures (cf. Jurek, 1980). Moreover, it seems to be true that in Corollary 2 (or in Theorem 4) the integrability conditions may be omitted, i.e. $\mathcal{G}_Q$ generates the whole class of operator-selfdecomposable measures on arbitrary Banach space.

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