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JOINT CONTINUITY OF GAUSSIAN LOCAL TIMES

BY JACK CUZICK AND JOHANNES P. DUPREEZ

University of Oxford and Plymouth Polytechnic

Sufficient conditions in terms of interpolation variances are given for a Gaussian process to have a jointly continuous local time. In the stationary case these conditions can be verified in terms of the spectral density and are seen to be within logarithmic factors of the best possible conditions. A bound for the modulus of continuity in the space variable is also obtained.

Let $X(t)$ be a jointly measurable Gaussian process on an interval of the real line with mean zero and bounded variance. When

$$\int_0^T \int_0^T \frac{ds\,dt}{\{E(X(t)-X(s))^2\}^{1/2}} < \infty, \tag{1}$$

Berman (1969) has established the existence of a square integrable density $\tilde{\alpha}(x, T)$ for the occupation measure of $X(t)$ on $[0, T]$ by means of the Fourier representation

$$\tilde{\alpha}(x, T) = \int_{-\infty}^{\infty} \int_0^T e^{i\lambda(X(t)-x)}dt\,d\lambda. \tag{2}$$

Here the equals sign is to be interpreted as almost sure L_2-equivalence. This paper is concerned with the regularity in x of a natural version of this occupation density or local time. In particular, joint continuity in (x, T) is established under conditions which are essentially weaker than those given by Berman (1973). However, our conditions are not strictly comparable to his because we use a different type of local nondeterminism which requires conditioning on an infinite instead of only a finite number of values. See Cuzick (1982) for more discussion of this point. Stationary Gaussian processes with spectral densities proportional to $(1+\lambda)^{-3}\log^{\beta}(e+\lambda)$, $\beta > 6$ provide examples which can be shown to have a jointly continuous local time by our methods, but not by those of Berman. The reader is referred to the review by Geman and Horowitz (1980) for general background on local times.

It is well known that

$$\alpha(x, T) = \lim_{\varepsilon\downarrow 0}\inf(2\varepsilon)^{-1}\int_0^T I_{\{|X(t)-x|\le\varepsilon\}}dt, \tag{3}$$

where I_A is the indicator function of the set A, defines a version of the local time when it exists, and that the limit in (3) actually exists for a.e. x a.s. Cuzick (1982) has also shown that when X is also stationary, the limit exists for any fixed x a.s. The version of the local time which we shall consider is the right continuous modification (in T) of (3).

Geman (1976) has shown that under (1), this version is continuous in T for a.e. x. However, to establish joint continuity, or even continuity in the space variable, stronger conditions are required. The best results to date are those of Berman (1973). He showed that when X is locally nondeterministic, i.e. for all $n > 0$, there exists $\varepsilon > 0$ and $K_n > 0$ such that for all $t_1 < \ldots < t_n$, with $t_n - t_1 \le \varepsilon$

$$\mathrm{Var}(X(t_n)\,|\,X(t_{n-1}), \cdots, X(t_1)) \ge K_n\,\mathrm{Var}(X(t_n) - X(t_{n-1})),$$

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a sufficient condition for joint continuity is that there exists $b(t)$ such that $b^2(t) \le E(X(s+t) - X(s))^2$ for all s and

$$\int_0^{\varepsilon} \frac{dt}{\{b(t)\}^{(1+\delta)}} < \infty$$

for some ε, $\delta > 0$. This method also yields a bound for the modulus of continuity in x in terms of δ (cf. Geman and Horowitz, 1980, page 46). We shall require the following generalisation of local nondeterminism:

DEFINITION. Let $\phi(x)$ be a continuous, non-decreasing function on $[0, 1]$ with $\phi(0) = 0$. A Gaussian process is said to be *strongly locally ϕ-nondeterministic* (*2-sided*) or *$SL\phi ND(2)$* if there exists $K > 0$, $\varepsilon > 0$ such that for all s

$$\operatorname{Var}\left(X(s) \,\middle|\, X(u), t \le |s-u| \le \varepsilon\right) \ge K\phi(t).$$

This definition extends previous notions of local nondeterminism (Berman, 1973; Cuzick, 1978, 1982) to an interpolation set-up where information is available on both sides of the point of interest. It appears to be difficult to establish conditions under which general Gaussian processes possess the various forms of local nondeterminism. However, for *stationary* processes, the results of Cuzick (1978) are easily extended to the 2-sided case:

LEMMA 1. *Let $X(t)$ be a stationary Gaussian process with spectral distribution function $F(\lambda)$. Assume that the absolutely continuous part of $F(\lambda)$ is such that there exists a function $\phi(t)$ for which*

$$\frac{dF(\lambda/t)}{\phi(t)} \ge h(\lambda)\, d\lambda$$

for all $t < t_0$, where $h(\lambda)$ is non-increasing on $[0, \infty)$ *and*

$$\int_0^{\infty} \frac{\log(h(\lambda))}{1+\lambda^2}\, d\lambda > -\infty. \tag{4}$$

Then there exists a positive constant K such that

$$\operatorname{Var}\left(X(0) \,\middle|\, X(s), |s| \ge t\right) \ge K\phi(t),$$

i.e. $X(t)$ is $SL\phi ND(2)$.

PROOF. Let $Z^T(dF)$ be the span of the exponentials $\{e^{i\lambda t}, |t| \ge T\}$ in the Hilbert space of complex functions with inner product $\langle f, g\rangle = \int_0^\infty f(\lambda)\overline{g(\lambda)}\, dF(\lambda)$. Then

$$\operatorname{Var}(X(0) \mid X(s), |s| \ge t) = \inf_{g \in Z^t(dF)} \int_0^{\infty} |1 - g(\lambda)|^2\, dF(\lambda)$$

which after the change of variables $\lambda \to \lambda/t$ equals

$$\phi(t) \inf_{g \in Z^1(dF)} \int_0^{\infty} |1 - g(\lambda)|^2 \frac{dF(\lambda/t)}{\phi(t)}$$

which, for $t < t_0$, is greater than or equal to

$$\phi(t) \inf_{g \in Z^1(h)} \int_0^{\infty} |1 - g(\lambda)|^2 h(\lambda)\, d\lambda \tag{5}$$

since $Z^1(dF) \subseteq Z^1(h)$.

Now since $h(\lambda)$ satisfies (4) and is non-increasing it follows that the infinimum in (5) is positive (Dym-McKean, 1976, page 138).

With the aid of this concept, we can now state the main result:

THEOREM. *Let $X(t)$ be a mean zero Gaussian process on an interval of the real line whose variance is bounded on compact sets. If X is SLϕND with $\phi(t) = t^2 \log^\beta (e/t)$ for some $\beta > 6$, then the version of the local time given by* (3) *exists and is jointly continuous in (x, T). Furthermore if $0 < \delta < \frac{1}{4}(\beta - 6)$, there exists an* a.s. *finite (random) constant C depending only on δ (and the sample path), such that with probability one*

$$|\alpha(x+y, T) - \alpha(x, T)| \le C \log^{-\delta}\left(\frac{1}{|y|}\right)$$

for all x, all $|y| < e^{-1}$ and all $T \le T_0 < \infty$.

REMARK. When $\alpha(x, T)$ is continuous in x, the limit in (3) exists for all x. This can be seen by using the relation (Berman, 1969, page 270)

$$(6)\qquad \begin{aligned}(2\varepsilon)^{-1}\int_0^T I_{\{|X(t)-x|\le\varepsilon\}}dt &= (2\varepsilon)^{-1}\int_{-\infty}^{\infty} I_{\{|y-x|\le\varepsilon\}}\alpha(y, T)\, dy\\ &= (2\varepsilon)^{-1}\int_{|y-x|\le\varepsilon} \alpha(y, T)\, dy\end{aligned}$$

and noting that the limit in (3) defining $\alpha(y, T)$ exists for a.e. y so that when α is continuous in y, the limit as $\varepsilon \downarrow 0$ of (6) must exist for all x.

The proof of the theorem is based on the following basic result of Garsia:

LEMMA (*Garsia*, 1971). *Assume that $p(u)$ and $\psi(u)$ are symmetric, $p(u) \downarrow 0$ as $|u| \downarrow 0$, $\psi(u)$ is convex and $\psi(u) \uparrow \infty$ as $|u| \uparrow \infty$. Let I_0 denote the open unit hypercube in R^d and for every open hypercube I, let $e(I)$ denote the common length of its edges. If $f(\mathbf{x})$ is measurable in I_0 and*

$$\int_I\int_I \psi\left(\frac{f(\mathbf{x}) - f(\mathbf{y})}{p(e(I))}\right) d\mathbf{x}\, d\mathbf{y} \le B, \quad \text{for all } I \subseteq I_0$$

then for almost every $\mathbf{x}, \mathbf{y} \in I_0$

$$|f(\mathbf{x}) - f(\mathbf{y})| \le 8\int_0^{|\mathbf{x}-\mathbf{y}|} \psi^{-1}\left(\frac{B}{u^{2d}}\right) dP(u).$$

If in addition

$$f(\mathbf{x}) = \lim\nolimits_{\varepsilon\downarrow 0}(2\varepsilon)^{-d}\int_{\mathbf{y}\in\mathbf{x}+I_\varepsilon} f(\mathbf{y})\, d\mathbf{y}$$

for all $\mathbf{x} \in I_0$, where I_ε is the hypercube $(-\varepsilon, \varepsilon)^d$, then the result holds for all $\mathbf{x}, \mathbf{y} \in I_0$.

The basic estimates and relations required to prove the main theorem are contained in the following series of lemmas.

LEMMA 2. *Let $X_1, \cdots, X_N$ be the mean zero Gaussian variables which are linearly independent and assume that $\int_{-\infty}^{\infty} g(v)e^{-\varepsilon v^2}\, dv < \infty$ for all $\varepsilon > 0$. Then*

$$(7)\qquad \begin{aligned}&\int_{\mathbb{R}_N} g(v_1)\exp\{-\tfrac{1}{2}\operatorname{Var}(\textstyle\sum_{i=1}^N v_i X_i)\}\, dv_1 \cdots dv_N\\ &\qquad = (2\pi)^{(N-1)/2}(\det \operatorname{Cov}(X_1, \cdots, X_N))^{-1/2}\int_{-\infty}^{\infty} g\left(\frac{v}{\sigma}\right)e^{-v^2/2}\, dv\end{aligned}$$

where $\sigma^2 = \operatorname{Var}(X_1 \mid X_2, \cdots, X_N)$ *is the conditional variance of* X_1 *given* $X_2, \cdots, X_N$ *and* $\det \operatorname{Cov}(X_1, \cdots, X_N)$ *is the determinant of the covariance matrix of* $(X_1, \cdots, X_N)$.

PROOF. For any $N \times N$ positive definite matrix $\Sigma = (\Sigma_{ij})$

$$\frac{\{\det \sum\}^{1/2}}{(2\pi)^{N/2}} \int_{\mathbb{R}_{N-1}} \exp\left\{-\frac{1}{2} \mathbf{v}' \sum \mathbf{v}\right\} dv_2 \cdots dv_N = \frac{\exp\left\{-\frac{1}{2} v_1^2/(\sum^{-1})_{11}\right\}}{\sqrt{2\pi(\sum^{-1})_{11}}}$$

where $\mathbf{v} = (v_1, \cdots, v_N)'$. When $\sum_{ij} = EX_iX_j$, then $\sigma^2 \equiv (\sum^{-1})_{11} = \operatorname{Var}(X_1 \mid X_2, \cdots, X_N)$, and the lemma follows by applying this relation to (7) and then making the change of variables $v_1 = v\sigma$.

LEMMA 3. *Assume* $p(y)$ *is positive and non-decreasing on* $(0, \infty)$, $p(0) = 0$, $y^{2k-1}/p^{2k}(y)$ *is non-decreasing on* $[0, 1]$, *and* $\int_1^\infty dy/p^2(y) < \infty$. *Then there exists a constant* C *such that for all* $k \geq 1$

$$\int_0^\infty \frac{|e^{i\lambda y} - 1|^{2k-1}}{p^{2k}(y)} dy \leq C^k p_+^{-2k}\left(\frac{1}{\lambda}\right)$$

where $p_+(x) = \min(1, p(x))$, *so that* $p_+^{-2k}(x) = \max(1, p^{-2k}(x))$.

PROOF. When $\lambda \geq 1$

$$\int_0^\infty \frac{|e^{i\lambda y} - 1|^{2k-1}}{p^{2k}(y)} dy \leq \int_0^{1/\lambda} \frac{(\lambda y)^{2k-1}}{p^{2k}(y)} dy + 4^k\left[\int_{1/\lambda}^1 \frac{dy}{p^{2k}(y)} + \int_1^\infty \frac{dy}{p^{2k}(y)}\right] \tag{8}$$

$$\leq C^k p_+^{-2k}\left(\frac{1}{\lambda}\right), \quad \text{for some } C < \infty.$$

When $\lambda < 1$, the left hand side of (8) is bounded by

$$\int_0^1 \frac{(\lambda y)^{2k-1}}{p^{2k}(y)} dy + 4^k \int_1^\infty \frac{dy}{p^{2k}(y)} \leq \lambda^{2k-1} p^{-2k}(1) + 4^k p^{-2k}(1) \int_1^\infty \left(\frac{p(1)}{p(y)}\right)^2 dy$$

$$\leq C^k \text{ for some } C < \infty.$$

LEMMA 4. *As* $\alpha \to \infty$,

$$\int_1^\infty \log^\alpha(x) e^{-x^2/2} dx \leq \sqrt{\pi} (\log^\alpha \alpha) \tag{9}$$

PROOF. The left hand side of (9) is bounded by

$$\sup_{1 \leq x < \infty}[\log^\alpha(x) e^{-x^2/4}] \int_1^\infty e^{-y^2/4} dy \leq \sqrt{\pi} \sup_{1 \leq x < \infty} \log^\alpha(x) e^{-x^2/4}. \tag{10}$$

The supremum in the right hand side of (10) occurs when

$$x^2 \log x = 2\alpha \tag{11}$$

so that if x^* is the solution of (11), then $x^* \leq \sqrt{2\alpha}$ when $\alpha \geq ½e^2$, and it follows that for such α, (10) is less than

$$\sqrt{\pi} \log^\alpha(x^*) \leq \sqrt{\pi} (½ \log 2\alpha)^\alpha \leq \sqrt{\pi} (\log \alpha)^\alpha, \quad \text{since } \alpha \geq 2.$$

LEMMA 5. *Assume that* $X(t)$ *is* $SL\phi ND(2)$ *where* $\phi(t) = t^2 \log^\beta (e/t)$ *for some* $\beta > 6$.

Then for any γ satisfying $1 < \gamma < (\beta - 2)/4$, there exists a finite constant C, depending only on γ such that $X(t)$ has a local time α which satisfies

$$E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\alpha(x+y,T)-\alpha(x,T)}{p(y)}\right)^{2k} dx\,dy \le C^k(2k)!(\log k)^{k\beta} \tag{12}$$

where

$$p(y)=\begin{cases}0, & y=0\\ \log^{-\gamma}\left(\dfrac{e}{|y|}\right), & 0<|y|\le 1\\ \gamma|y|-\gamma+1, & |y|>1.\end{cases}$$

PROOF. It suffices to establish the result for all sufficiently large k. Let $R_{2k}(\mathbf{t}) = \det \operatorname{Cov}(X(t_0), \cdots, X(t_{2k-1}))$. Since (Cuzick, 1982, Lemma 3)

$$\int_{[0,T]} R_{2k}^{-1/2}(\mathbf{t})\,d\mathbf{t} = (2k)!\int_{0\le t_1\le\cdots\le t_{2k}\le T} R_{2k}^{-1/2}(\mathbf{t})\,d\mathbf{t} \le (2k)!\left(C\int_0^T \phi^{1/2}(t)\,dt\right)^{2k} < \infty$$

for some $C < \infty$, it follows that $E\int_{-\infty}^{\infty}\alpha^{2k}(x, T)\,dx < \infty$, $k \ge 1$ and $\alpha(x, T) \in L_p$ for all $p \ge 1$ (Geman and Horowitz, 1980, page 42). In particular, taking $k = 1$ shows that (1) is satisfied so that a square integrable local time exists and the representation (2) is valid. The relation (Berman, 1969, page 270)

$$\int_{-\infty}^{\infty}\alpha^{2k}(x,T)\,dx = \int_0^T \alpha^{2k-1}(X(t),T)\,dt$$

shows that

$$\int_{-\infty}^{\infty}\{\alpha(x+y,T)-\alpha(x,T)\}^{2k}\,dx$$

$$=\int_{-\infty}^{\infty}\{\alpha(x+y,T)-\alpha(x,T)\}^{2k-1}\{\alpha(x+y,T)-\alpha(x,T)\}\,dx$$

$$=\int_0^T\{\alpha(X(t),T)-\alpha(X(t)-y,T)\}^{2k-1}-\{\alpha(X(t)+y,T)-\alpha(X(t),T)\}^{2k-1}\,dt$$

so that the left hand side of (12) equals

$$E\int_0^T\int_{-\infty}^{\infty}\frac{\{\alpha(X(t),T)-\alpha(X(t)-y,T)\}^{2k-1}-\{\alpha(X(t)+y,T)-\alpha(X(t),T)\}^{2k-1}}{p^{2k}(y)}\,dy\,dt$$

which, upon using the Fourier representation (2), equals

$$\int_{[0,T]^{2k}} d\mathbf{t}\int_{\mathbb{R}^{2k-1}} d\boldsymbol{\lambda}\int_{\mathbb{R}^{dy}}\frac{\{\prod_{j=1}^{2k-1}(1-e^{-i\lambda_j y})-\prod_{j=1}^{2k-1}(e^{i\lambda_j y}-1)\}}{p^{2k}(y)}$$

$$\times E\exp\{i\textstyle\sum_{j=1}^{2k-1}\lambda_j(X(t_j)-X(t_0))\}$$

$$\le 2\int_{[0,T]^{2k}} d\mathbf{t}\int_{\mathbb{R}\times\mathbb{R}^{2k-1}} dy\,d\boldsymbol{\lambda}\,\{\textstyle\prod_{j=1}^{2k-1}|e^{i\lambda_j y}-1|\}p^{-2k}(y)$$

$$\times\exp\{-\tfrac{1}{2}\operatorname{Var}[\textstyle\sum_{j=1}^{2k-1}\lambda_j(X(t_j)-X(t_0))]\}.$$

Applying Hölder's inequality (Hardy, Littlewood and Polya, 1952, Theorem 11) to the integral $dt \times d\boldsymbol{\lambda}$, this expression is bounded by a constant times

$$\int_{[0,T]^{2k}} d\mathbf{t} \prod_{j=1}^{2k-1} \left[\int\int_{\mathbb{R}\times\mathbb{R}^{2k-1}} dy\, d\boldsymbol{\lambda} \frac{|e^{i\lambda_j y} - 1|^{2k-1}}{p^{2k}(y)} \times \exp\{-\tfrac{1}{2}\mathrm{Var}[\textstyle\sum_{j=1}^{2k-1} \lambda_j (X(t_j) - X(t_0))]\} \right]^{1/(2k-1)}$$

which by Lemma 2 equals

$$(13) \qquad (2\pi)^{k-1/2} \int_{[0,T]^{2k}} \frac{d\mathbf{t}}{(R^0_{2k-1}(\mathbf{t}))^{1/2}} \prod_{j=1}^{2k-1} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|e^{i\lambda y/\sigma_j} - 1|^{2k-1}}{p^{2k}(y)} e^{-\lambda^2/2}\, dy\, d\lambda \right]^{1/(2k-1)}$$

where $\sigma_j^2 = \mathrm{Var}(X(t_j) - X(t_0) \,|\, X(t_i) - X(t_0), i \neq j)$ and $R^0_{2k-1}(\mathbf{t}) = \det \mathrm{Cov}(X(t_1) - X(t_0), \cdots, X(t_{2k-1}) - X(t_0))$. We now use Lemma 3 to bound the integral over dy and find that (13) is bounded by

$$(14) \qquad (\mathrm{Const})^k \int_{[0,T]^{2k}} \frac{d\mathbf{t}}{(R^0_{2k-1}(\mathbf{t}))^{1/2}} \prod_{j=1}^{2k-1} \left[\int_{-\infty}^{\infty} p_+^{-2k}\left(\frac{\sigma_j}{\lambda}\right) e^{-\lambda^2/2}\, d\lambda \right]^{1/(2k-1)}$$

As

$$p_+^{-2k}(x) = \begin{cases} \log^{2k\gamma}\left(\dfrac{e}{x}\right), & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

and $\log_+^\alpha(xy) \leq 2^\alpha[\log_+^\alpha x + \log_+^\alpha y]$ for $\alpha \geq 0$ where $\log_+(x) = \max(1, \log x)$, it follows that

$$\int_{-\infty}^{\infty} p_+^{-2k}\left(\frac{\sigma_j}{\lambda}\right) e^{-\lambda^2/2}\, d\lambda \leq (\mathrm{Const})^k \left[\int_{|\sigma_j/\lambda| \geq 1} e^{-\lambda^2/2}\, d\lambda + \int_{|\sigma_j/\lambda| < 1} \log_+^{2k\gamma}\left(\frac{e}{\sigma_j}\right) e^{-\lambda^2/2}\, d\lambda + \int_{|\sigma_j/\lambda|<1} \log_+^{2k\gamma}(\lambda) e^{-\lambda^2/2} d\lambda \right].$$

By Lemma 4, this is bounded by

$$(\mathrm{Const})^k \left[\log_+^{2k\gamma}\left(\frac{e}{\sigma_j}\right) + (\log 2k\gamma)^{2k\gamma} \right] \leq (\mathrm{Const})^k \left[\log_+^{2k\gamma}\left(\frac{e}{\sigma_j}\right) \right] [\log k]^{2k\gamma}.$$

It follows that (14) is bounded by

$$(15) \qquad (\mathrm{Const})^k (\log k)^{2k\gamma} \int_{[0,T]^{2k}} \frac{d\mathbf{t}}{(R^0_{2k-1}(\mathbf{t}))^{1/2}} \prod_{j=1}^{2k-1} \log_+^{\gamma^*}\left(\frac{e}{\sigma_j}\right)$$

where $\gamma^* = \gamma(1 + (2k-1)^{-1})$.

Now

$$\begin{aligned} R^0_{2k-1}(\mathbf{t}) &= \det \mathrm{Cov}(X(t_i) - X(t_0), i = 1, \cdots, 2k-1) \\ &= \prod_{j=1}^{2k-1} \mathrm{Var}(X(t_i) - X(t_0) \,|\, X(t_j) - X(t_0), 1 \leq j < i) \end{aligned}$$

and, as adding $X(t_0)$ to the conditioning set in each term reduces the conditional variance, this is greater than or equal to

$$\prod_{j=1}^{2k-1} \mathrm{Var}(X(t_i) \,|\, X(t_j), 0 \leq j < i) = \det \mathrm{Cov}(X(t_0), \cdots, X(t_{2k-1}))/\mathrm{Var}(X(t_0))$$

so that

$$(R^0_{2k}(t))^{-1/2} \leq (EX(t_0))^{1/2} R_{2k}^{-1/2}(\mathbf{t}).$$

Again, by adding $X(t_0)$ to the conditioning set

$$\sigma_j^2 = \mathrm{Var}(X(t_j) - X(t_0) \mid X(t_i) - X(t_0), i \neq j)$$
$$\geq \mathrm{Var}(X(t_j) \mid X(t_i), i \neq j)$$
$$\geq \mathrm{Var}\left(X(t_j) \,\middle|\, X(s), |s - t_j| \geq \min_{i\neq j}|t_i - t_j|\right)$$
$$\geq C\phi(\min_{i\neq j}|t_i - t_j|), \quad \text{for some } C > 0$$

by Lemma 1. From this it follows that (15) is bounded by

$$(16)\qquad (\mathrm{Const})^k(\log k)^{2k\gamma}\int_{[0,T]^{2k}} \frac{dt}{(R_{2k}(t))^{1/2}} \prod_{j=0}^{2k-1} \log_+^{\gamma^*}\left(\frac{1}{\phi_j}\right)$$

where $\phi_j = \min_{i\neq j}\phi(|t_i - t_j|)$. Note that the additional factor $\log_+^{\gamma^*}(1/\phi_0)$ now makes the integrand invariant under permutations of the t_j, so that if

$$T_{2k} = \{0 \leq t_0 \leq \cdots \leq t_{2k-1} \leq T\}, \qquad \text{then } \phi_j = \phi(\min(\Delta_j, \Delta_{j+1}))$$

where $\Delta_j = t_j - t_{j-1}, j = 1, \cdots, 2k - 1$, $\Delta_0 = \Delta_{2k} = T$ and (16) equals

$$(17)\qquad (\mathrm{Const})^k(2k)!(\log k)^{2k\gamma}\int_{T_{2k}} \frac{dt}{(R_{2k}(\mathbf{t}))^{1/2}} \prod_{j=0}^{2k-1} \log_+^{\gamma^*}(\max(\Delta_j^{-1}, \Delta_{j+1}^{-1})).$$

Now

$$R_{2k}(\mathbf{t}) = \prod_{j=0}^{2k-1} \mathrm{Var}(X(t_j) \mid X(t_i), i < j) \geq C^k\left(\prod_{i=1}^{2k-1} \phi(\Delta_i)\right) \mathrm{Var}\, X(t_0),$$

and

$$\log_+^{\gamma^*}(\max(\Delta_j^{-1}, \Delta_{j+1}^{-1})) \leq (\log_+^{\gamma^*} \Delta_j^{-1})(\log_+^{\gamma^*}(\Delta_{j+1}^{-1}))$$

so that after the change of variables

$$t_0 = t_0; \qquad \Delta_i = t_i - t_{i-1}, \qquad i = 1, \cdots, 2k - 1,$$

it follows that (17) is bounded by

$$(18)\qquad (\mathrm{Const})^k(2k)!(\log k)^{2k\gamma}T \log_+^{2\gamma^*}(T^{-1})\left[\int_0^T \frac{\log_+^{2\gamma^*}(\Delta^{-1})}{\phi^{1/2}(\Delta)}\, d\Delta\right]^{2k-1}.$$

As $\phi^{1/2}(\Delta) = \Delta \log^{\beta/2}\left(\frac{1}{\Delta}\right)$, and $\frac{1}{2}\beta > 1 + 2\gamma$, we can choose k so large that

$$\beta/2 - 2\gamma^* = \beta/2 - 2\gamma(1 + (2k - 1)^{-1}) > 1$$

and then (18) is bounded by $(\mathrm{Const})^k(2k)!(\log k)^{2k\gamma}$ and the lemma is proven since $2\gamma < \beta$.

Proof of Theorem. It follows from Lemma 5 and a power series expansion of e^x that when $\psi(z) = |z| \exp(|z|^\theta)$, $\theta < 1$

$$E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi\left(\frac{\alpha(x + y, T) - \alpha(x, T)}{p(y)}\right) dx\, dy < \infty$$

where $p(y)$ is as in Lemma 5. Thus by Garsia's lemma there are (random) constants B and C such that for almost every (x, y) with $|y| \leq e^{-1}$

$$|\alpha(x + y, T) - \alpha(x, T)| \leq \int_0^{|y|} \psi^{-1}\left(\frac{B}{u^2}\right) dp(u) \leq C\left(\log\left(\frac{1}{y}\right)\right)^{-(\gamma - 1/\theta)}.$$

This can then be extended to hold for all (x, y), $|y| \leq e^{-1}$ for the version of the local time given in (3) by using the second part of Garsia's lemma and the representation (6). Thus $\alpha(x, T)$ is a.s. continuous in x and given any $0 < \delta < \frac{1}{4}(\beta - 6)$ there exists a (random) constant C_δ such that for all x

$$|\alpha(x + y, T) - \alpha(x, T)| \leq C_\delta \log^{-\delta}\left(\frac{1}{|y|}\right), \quad |y| \leq e^{-1}$$

with probability one.

To establish the joint continuity in (x, T) of the local time it follows by a slight modification of Lemma 5 that for $\psi(z) = |z| \exp(|z|^\theta)$, $\theta < 1$

$$E \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi\left(\frac{\alpha(x + y, t) - \alpha(x, s)}{p(y)}\right) dx\, dy\, ds\, dt < \infty$$

and since $p(y) \leq p(\max(|y|, |t - s|))$, it follows from Garsia's lemma that for a.e. x, y, s, t with $0 \leq s \leq t \leq 1$, $|y| \leq e^{-1}$

$$|\alpha(x + y, t) - \alpha(x, s)| \leq 8 \int_0^{\max(|y|, |t-s|)} \psi^{-1}\left(\frac{C}{u^4}\right) dp(u)$$

for random $C < \infty$. This result can then be extended to all x, y, s, t in the above set as before by noting that

$$(2\varepsilon)^{-2} \int_{|x-y| \leq \varepsilon, |s-T| \leq \varepsilon} \alpha(y, s)\, dy\, ds = (2\varepsilon)^{-2} \int_{T-\varepsilon}^{T+\varepsilon} \int_0^s I_{\{|X(u)-x| \leq \varepsilon\}}\, du\, ds \tag{19}$$
$$\leq (2\varepsilon)^{-1} \int_0^{T+\varepsilon} I_{\{|X(u)-x| \leq \varepsilon\}}\, du.$$

It follows that the limit in (19) as $\varepsilon \downarrow 0$ is bounded above by $\alpha(x, T + \delta)$ for any $\delta > 0$. Similarly a lower bound for (19) is $\alpha(x, T - \delta)$. As $\alpha(x, T)$ is continuous in T for any fixed x (Cuzick, 1982), it follows that the limit as $\varepsilon \downarrow 0$ of (19) equals $\alpha(x, T)$ and that

$$|\alpha(x + y, t + s) - \alpha(x, t)| \leq C_\delta \log^{-\delta}(\{\max(|y|, |s|)\}^{-1})$$

for all $|y| \leq e^{-1}$, $0 \leq t$, $s + t \leq 1$. In particular by taking $s = 0$, it follows that the modulus of continuity in the space variable given in the theorem is uniform in T.

REFERENCES

BERMAN, S. M. (1969). Harmonic analysis of local times and sample functions of Gaussian processes. *Trans. Amer. Math. Soc.* **137** 277–299.

BERMAN, S. M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69–94.

CUZICK, J. (1978). Local nondeterminism and the zeros of Gaussian processes. *Ann. Probability* **6** 72–84.

CUZICK, J. (1982). Continuity of Gaussian local times. *Ann. Probability* **10** 818–823.

DYM, H. and MCKEAN, H. P. (1976). *Gaussian Processes, Complex Function Theory and the Inverse Spectral Method.* Academic, New York.

GARSIA, A. M. (1971). Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. 6th Berk. Symp. Math. Statist. Probability* **II** 369–374.

GEMAN, D. (1976). A note on the continuity of local times. *Proc. Amer. Math. Soc.* **57** 321–326.

GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probability* **8** 1–67.

HARDY, G. H., LITTLEWOOD, J. E., and POLYA, G. (1952). *Inequalities.* Cambridge University Press, Cambridge.

MATHEMATICAL INSTITUTE,
24-29 ST. GILES,
UNIVERSITY OF OXFORD,
OXFORD OX1 3LB,
UNITED KINGDOM.

DEPARTMENT OF MATHEMATICS,
STATISTICS AND COMPUTING,
PLYMOUTH POLYTECHNIC,
PLYMOUGH, DEVON PL4 8AA,
UNITED KINGDOM.