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SOME RESULTS ON DISTRIBUTIONS ARISING FROM COIN TOSSING

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Let $\{X_n\}$ be a sequence of independent identically distributed random variables which take the values ± 1 with probability ½. Let $X = \sum_{n=1}^{\infty} a_n X_n$ where $\sum a_n^2 < \infty$. We show that if

$$n^{-\alpha} \le |a_n| \le n^{-\beta}$$

for some $\alpha > ½$ and $0 \le \alpha - \beta < ½$ then the distribution of $X = \sum a_n X_n$ is absolutely continuous with respect to Lebesgue measure. We then prove similar results for more general independent sequences.

We also show that if

$$\liminf 2^N \sqrt{\textstyle\sum_{n=N+1}^{\infty} a_n^2} = 0$$

then the distribution of $X = \sum a_n X_n$ is singular with respect to Lebesgue measure.

1. Introduction. Let $\{X_n\}$ be a sequence of independent, random variables with mean 0 and uniformly bounded variance; let $\{a_n\}$ be a square summable sequence and $X = \sum a_n X_n$. We consider the problem: what can be said about the distribution of X, from knowing $\{a_n\}$ and the distributions of the X_n's? The answer in general is, very little. See, e.g., [1, page 49]. However, if the X_n's are distributed on a countable set, then the Jessen-Wintner law of pure types [1, page 49] says that X has a distribution of pure type, i.e., the distribution of X is either discrete or continuous but singular with respect to Lebesgue measure (denoted by singular dx) or absolutely continuous with respect to Lebesgue measure (denoted by a.c. dx). It should be noted that the Jessen-Wintner law of pure types gives no clues to which type of distribution it is. Also observe the following well known fact: if we divide the positive integers into two disjoint sets, say A and B, and let

$$Y = \textstyle\sum_{n \in A} a_n X_n \quad \text{and} \quad Z = \sum_{n \in B} a_n X_n$$

then the distribution measure $F_X(dx)$ of X satisfies

$$F_X(dx) = F_Y * F_Z(dx)$$

where $*$ denotes convolution. This shows that if any one of the X_n's has a distribution a.c. dx, then so does X and, more generally, to guarantee a distribution a.c. dx for X, it suffices to show that $X' = \sum a_{n_k} X_{n_k}$ has a distribution a.c. dx, where $\{a_{n_k}\}$ is some subsequence of $\{a_n\}$. E.g., if $\{X_n\}$ is the i.i.d sequence ± 1 with probability ½ then the distribution of $\sum (1/n) X_n$ is a.c. dx since $\{1/n\}$ contains the subsequency $\{1/2^n\}$.

For the remaining part of this section, we restrict our discussion to the i.i.d sequence which equals ± 1 with probability ½. In this case, it follows by a theorem of P. Lévy [2] quoted in [1, page 51] that the distribution of $\sum a_n X_n$ cannot be discrete.

We will now show that by restricting the decay of $\{a_n\}$, one can prove that the Fourier coefficients of the distribution measure $F_X(dx)$ are absolutely summable, which proves that the distribution of $X = \sum a_n X_n$ is a.c. dx. In Sections 2 and 3 we then show that similar techniques work for more general sequences $\{a_n\}$ and $\{X_n\}$.

Assume $\sum_{n=1}^{\infty} a_n \le \pi$ (without loss of generality we may assume $a_n > 0$) and that there

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exist two constants $0 < \mathbf{C} < \bar{C}$ and $\alpha > 1$ such that

$$\mathbf{C} \cdot n^{-\alpha} \leq a_n \leq \bar{C} \cdot n^{-\alpha}. \tag{1}$$

Then the jth Fourier coefficient of $F_X(dx)$ is

$$\hat{F}_X(j) = \int_\Omega e^{ijX} P(d\omega) = \prod_{n=1}^{\infty} \cos(j \cdot a_n). \tag{2}$$

The last equality follows from the fact that F_X is even and from the independence of the X_j's. Fix some $\varepsilon > 0$ (we will determine its size later) and let

$$r_j = \text{the number of } a_n\text{'s such that} \quad \varepsilon \leq j \cdot a_n \leq \pi - \varepsilon \quad \text{for} \quad j \geq 1. \tag{3}$$

From (1) it follows that a lower bound for r_j is

$$\text{the number of } n\text{'s such that} \quad \varepsilon \leq \mathbf{C} \cdot j \cdot n^{-\alpha} \quad \text{and} \quad \bar{C} \cdot j \cdot n^{-\alpha} \leq \pi - \varepsilon. \tag{4}$$

Now solve the two inequalities for n to obtain

$$\left(\frac{j\bar{C}}{\pi - \varepsilon}\right)^{1/\alpha} \leq n \leq \left(\frac{j\mathbf{C}}{\varepsilon}\right)^{1/\alpha} \tag{5}$$

and from this we conclude that

$$r_j \geq \left[j^{1/\alpha} \left(\left(\frac{\mathbf{C}}{\varepsilon}\right)^{1/\alpha} - \left(\frac{\bar{C}}{\pi - \varepsilon}\right)^{1/\alpha} \right) \right] \tag{6}$$

where [ ] denotes the integer part.

Choose ε between 0 and $\pi/2$ so that

$$\left(\frac{\mathbf{C}}{\varepsilon}\right)^{1/\alpha} - \left(\frac{\bar{C}}{\pi - \varepsilon}\right)^{1/\alpha} = 1 \tag{7}$$

and (6) becomes

$$r_j \geq [j^{1/\alpha}]. \tag{8}$$

Now apply (3) and (8) in (2) to give an upper bound for $|\hat{F}_X(j)|$ as

$$|\hat{F}_X(j)| \leq (\cos(\varepsilon))^{r_j} \leq (\cos(\varepsilon))^{[j^{1/\alpha}]}. \tag{9}$$

From (9) and the fact that $\hat{F}_X(j) = \hat{F}_X(-j)$ it follows that

$$\sum_{j=-\infty}^{\infty} |\hat{F}_X(j)| < \infty. \tag{10}$$

Therefore the restriction of the function

$$f(x) = (1/2\pi) \sum_{-\infty}^{\infty} \hat{F}_X(j) e^{-ijx}$$

to $[-\pi, \pi]$ is the density of $F_X(dx)$.

2. The absolutely continuous case for coin tossing.

THEOREM 1. *Let $\{X_n\}$ be a sequence of* i.i.d.'s ± 1 *with probability* ½. *Let $\{a_n\}$ be a sequence such that $\sum_{n=1}^{\infty} a_n^2 < \infty$ and assume that for some $\alpha > ½$ and $0 \leq \alpha - \beta < ½$*

$$n^{-\alpha} \leq |a_n| \leq n^{-\beta}.$$

Then the distribution of $X = \sum_{n=1}^{\infty} a_n X_n$ is absolutely continuous dx.

PROOF. First assume $\sum |a_n| < \infty$. Without loss of generality we may assume $a_n > 0$ and $\sum a_n \leq \pi$.

For $j > 0$ and $\varepsilon_j > 0$ let

(1) $$r_j = \text{the number of } n\text{'s such that} \quad \varepsilon_j \leq ja_n \leq \pi - \varepsilon_j.$$

To get a lower bound for r_j note that if

(2) $$\varepsilon_j \leq jn^{-\alpha} \quad \text{and} \quad jn^{-\beta} \leq \pi - \varepsilon_j,$$

then a_n satisfies the condition in (1); therefore, solving (2) for n we get

(3) $$r_j \geq \left[j^{1/\beta}\left(\frac{j^{1/\alpha - 1/\beta}}{\varepsilon_j^{1/\alpha}} - \frac{1}{(\pi - \varepsilon_j)^{1/\beta}} \right) \right].$$

Let $\varepsilon_j = \dfrac{C}{j^{\alpha/\beta - 1}}$, where C is a positive constant chosen such that

$$\frac{j^{1/\alpha - 1/\beta}}{\varepsilon_j^{1/\alpha}} - \frac{1}{(\pi - \varepsilon_j)^{1/\beta}} \geq 1$$

for all j sufficiently large. Observe that C can be chosen independent of j.

Now (3) becomes

(4) $$r_j \geq [j^{1/\beta}] \geq (½) j^{1/\beta}$$

for all j sufficiently large, from which the jth Fourier coefficient of F_X can be estimated as

(5) $$|\hat{F}_X(j)| \leq (\cos(\varepsilon_j))^{r_j} \leq (\cos(\varepsilon_j))^{(1/2)j^{1/\beta}}.$$

Since $\lim_{x\to 0} \dfrac{1 - \cos(x)}{x^2} = ½$ it follows that for j sufficiently large

(6) $$\cos(\varepsilon_j) \leq \left(1 - \frac{\varepsilon_j^2}{4}\right).$$

Using the fact that

$$e^x \geq 1 + x \quad \text{for} \quad -1 < x < 0,$$

we obtain from (5) and (6)

(7) $$|\hat{F}_X(j)| \leq \exp(-⅛\varepsilon_j^2 j^{1/\beta}).$$

The exponent without the sign is

(8) $$\frac{1}{8}\varepsilon_j^2 j^{1/\beta} = \frac{C^2}{8} j^{1/\beta - 2\alpha/\beta + 2} = k \cdot j^\gamma$$

where $k > 0$ and $\gamma = 1/\beta - 2\alpha/\beta + 2$. $\gamma > 0$ by the assumption $\alpha - \beta < 1/2$.

Using (8) and (7) we conclude

(9) $$|\hat{F}_X(j)| \leq \exp(-k \cdot j^\gamma)$$

and therefore

(10) $$\textstyle\sum_{j=-\infty}^{\infty} |\hat{F}_X(j)| < \infty.$$

Now let

(11) $$f(x) = \begin{cases} (½\pi) \sum_{j=-\infty}^{\infty} \hat{F}(j) e^{-ijx} & \text{for} \quad -\pi \leq x \leq \pi \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f(x)$ is the density of F_X.

If $\sum_{n=1}^{\infty} a_n = \infty$, consider the measures

(12) $$F_L(dx) = P(\textstyle\sum_{n=1}^{\infty} a_n X_n \bmod 2^L(2\pi) \in dx)$$

on $[-2^L\pi, 2^L\pi]$ for $L = 2, 3, \cdots$.

Since the jth Fourier coefficient of $F_L(dx)$ is

$$\hat{F}_L(j) = \prod_{n=1}^{\infty} \cos\left(\frac{ja_n}{2^L}\right),$$

it is clear that we can use the previous method to show that $F_L(dx)$ has a density

$$f_L(x) = \begin{cases} (1/2\pi) \sum_{j=-\infty}^{\infty} \hat{F}_L(j)e^{-ijx} & \text{on } [-2^L\pi, 2^L\pi] \\ 0 & \text{elsewhere.} \end{cases}$$

Now, for any L, $L + 1$ and ω such that

$$\sum_{n=1}^{\infty} a_n X_n \bmod 2^{L+1}(2\pi) \in [-2^L\pi, 2^L\pi]$$

it follows, since 2^{L+1} is divisible by 2^L, that

$$\sum_{n=1}^{\infty} a_n X_n \bmod 2^{L+1}(2\pi) = \sum_{n=1}^{\infty} a_n X_n \bmod 2^L(2\pi)$$

and therefore for any measurable $B \subset [-2^L\pi, 2^L\pi]$

$$P(\sum_{n=1}^{\infty} a_n X_n \bmod 2^L(2\pi) \in B) \geq P(\sum_{n=1}^{\infty} a_n X_n \bmod 2^{L+1}(2\pi) \in B)$$

from which we conclude that

$$f_L(x) \geq f_{L+1}(x) \quad \text{for a.e.} \quad x \in [-2^L\pi, 2^L\pi]. \tag{13}$$

Let

$$f(x) = \lim_{L\to\infty} f_L(x). \tag{14}$$

From

$$P(|\sum_{n=1}^{\infty} a_n X_n| > 2^L(2\pi)) \leq \frac{\sum_{n=1}^{\infty} a_n^2}{(2^{L+1}\pi)^2}$$

we see that

$$\lim_{L\to\infty} F_L(dx) = F_X(dx)$$

which together with (14) implies that $f(x)$ is the density of F_X. □

3. Some generalizations of the absolutely continuous case.

THEOREM 2. *Let $\{X_n\}$ be a sequence of independent random variables (not necessarily distributed on a countable set) such that*

(A) $\sup_n E|X_n| < \infty$

or

(A′) $\sup_n E(X_n^2) < \infty$ *and* $E(X_n) = 0$,

(B) *there are numbers* $0 < \lambda_1 < \lambda_2$ *such that*

$$\liminf_n P(|X_n| \leq \lambda_1) > 0 \quad \textit{and}$$
$$\liminf_n P(|X_n| \geq \lambda_2) > 0,$$

(C) $\sum_{n=1}^{\infty} |a_n|^{\ell} < \infty$, $\ell = 1, 2$ *for* (A), (A') *respectively, and*

(D) $n^{-\alpha} \leq |a_n| \leq n^{-\beta}$ *where* $0 \leq \alpha - \beta < 1/2$ *and* $\alpha > 1$, $1/2$ *for* (A), (A') *respectively.*

Then the distribution of $X = \sum a_n X_n$ is absolutely continuous dx.

PROOF. We will assume $a_n > 0$ and $X_n \geq 0$. The other cases are dealt with similarly. As before

$$\hat{F}_X(j) = \prod_{n=1}^{\infty} E(e^{ija_nX_n}).$$

Let
$$A_n = \{\omega | X_n \leq \lambda_1\}, \qquad B_n = \{\omega | \lambda_2 \leq X_n \leq M\}$$

where M is chosen so large that

$$\liminf_n P(B_n) > 0 \tag{1}$$

and let $C_n = \Omega - (A_n \cup B_n)$. Then

$$|E(e^{ija_nX_n})| \le P(C_n) + \left| \int_{A_n} e^{ija_nX_n} P(d\omega) + \int_{B_n} e^{ija_nX_n} P(d\omega) \right|. \tag{2}$$

Assume $j > 0$ and

$$ja_n \cdot M \le \frac{\pi}{2}, \tag{3}$$

then

$$0 \le \arg \int_{A_n} e^{ija_nX_n} P(d\omega) \le \lambda_1 \cdot a_n \cdot j \tag{4}$$

and

$$\lambda_2 a_n j \le \arg \int_{B_n} e^{ija_nX_n} P(d\omega) \le M \cdot a_n \cdot j;$$

also

$$\left| \int_{A_n} e^{ija_nX_n} P(d\omega) \right| \le P(A_n) \tag{5}$$

and

$$\left| \int_{B_n} e^{ija_nX_n} P(d\omega) \right| \le P(B_n).$$

Now it follows from (3), (4) and (5) that to obtain the greatest length we should make the argument of the first integral in (4) as large as possible, and the argument of the second integral as small as possible, i.e.,

$$\left| \int_{A_n} e^{ija_nX_n} P(d\omega) + \int_{B_n} e^{ija_nX_n} P(d\omega) \right| \le |P(A_n)e^{ija_n\lambda_1} + P(B_n)e^{ija_n\lambda_2}| \tag{6}$$

which combined with (2) gives

$$|E(e^{ija_nX_n})| \le P(C_n) + |P(A_n) + P(B_n)e^{i \cdot ja_n(\lambda_2 - \lambda_1)}|. \tag{7}$$

Now for the three positive numbers p, q, r with $p + q + r = 1$, we have

$$\lim_{x\to 0} \frac{1 - (p + |q + re^{ix}|)}{x^2} = \lim_{x\to 0} \frac{(1-p) - \sqrt{(1-p)^2 - 2qr(1-\cos(x))}}{x^2} = \frac{q \cdot r}{2(1-p)}. \tag{8}$$

Since p, q, r stand for the probability of C_n, A_n, B_n respectively and since we have lower and upper bounds for these quantities, we conclude (as before) that

$$|E(e^{ija_nX_n})| \le \begin{cases} 1 - S(ja_n(\lambda_2 - \lambda_1))^2 & \text{for } (ja_n(\lambda_2 - \lambda_1)) \in [-\delta, \delta] \\ 1 - S\delta^2 & \text{for } \delta \le (ja_n(\lambda_2 - \lambda_1)) \le \dfrac{\pi(\lambda_2 - \lambda_1)}{2M} \end{cases} \tag{9}$$

where $S, \delta > 0$ are independent of j and n and the $\pi(\lambda_2 - \lambda_1)/2M$ bound from (3). Now we conclude the proof as in Theorem 1 by estimating the size of $|\hat{F}_X(j)|$ from the number of

n's which satisfy

$$\varepsilon_j \leq ja_n(\lambda_2 - \lambda_1) \leq \frac{\pi(\lambda_2 - \lambda_1)}{2M}. \qquad \square$$

REMARK. Note in particular that if $\{X_n\}$ is a sequence of i.i.d's which takes more than one value with positive probability, then $\{X_n\}$ satisfies hypotheses A and B of Theorem 2, assuming EX_1 is finite.

4. Some results on singularity. The results in this section are based on the following simple observations: suppose $\{X_n\}$ is a sequence of i.i.d's which take two values and the weights $a_n = 1/3^n$ for $n = 1, 2 \ldots$. We want to show the distribution of $X = \sum_{n=1}^{\infty} X_n/3^n$ is singular dx.

For $N \geq 1$ let $\mathscr{A}_N = \left\{x \mid x = \sum_{n=1}^{N} \frac{X_n}{3^n}\right\}$. It is clear that cardinality $(\mathscr{A}_N) \leq 2^N$. Around each x in $\mathscr{A}_N$ construct an interval I_x of size

$$2\cdot\max|X_1| \cdot \sum_{N+1}^{\infty} \frac{1}{3^n} = \frac{\max|X_1|}{3^N} \text{ with center } x.$$

Now let

$$O_N = \cup_{x\in\mathscr{A}_N} I_x$$

then

(A)
$$|O_N| \leq 2^N \frac{\max|X_1|}{3^N}$$

and

(B)
$$P(X \in O_N) = 1$$

(where $|\ \ |$ denotes Lebesgue measure). Therefore

$$|\cap_{N=1}^{\infty} O_N| = 0$$

and

$$P(X \in \cap_{N=1}^{\infty} O_N) = 1.$$

Similarly one can prove the following.

THEOREM 3. *Let $\{X_n\}$ be a uniformly bounded sequence of independent random variables, where each X_n takes at most two values. If*

$$\liminf 2^N \sum_{N+1}^{\infty} |a_n| = 0$$

then the distribution of $X = \sum a_n X_n$ is singular dx.

In the case of mean zero we have the following.

THEOREM 4. *If in addition to the hypotheses of Theorem 3, the random variables satisfy $E(X_n) = 0$ for all n and if*

$$\liminf 2^N \sqrt{\textstyle\sum_{N+1}^{\infty} a_n^2} = 0$$

then the distribution of $X = \sum a_n X_n$ is singular dx.

PROOF. Without loss of generality we may assume $|X_n| \leq 1$. Let $\{N_k\}$ be a sequence of integers such that

(1)
$$2^{N_k}\sqrt{\textstyle\sum_{n=N_k+1}^{\infty} a_n^2} \leq \frac{1}{k^3} \quad \text{for} \quad k = 1, 2, \cdots$$

and let

$$\mathscr{A}_k = \{x \mid x = \textstyle\sum_{n=1}^{N_k} a_n X_n\}.$$

Clearly

$$\text{cardinality of } \mathscr{A}_k \leq 2^{N_k}.$$

Around each distinct point x in $\mathscr{A}_k$ construct an interval I_x of length $2k\sqrt{\sum_{N_k+1}^{\infty} a_n^2}$ with center x.

Then letting

$$O_k = \cup_{x \in \mathscr{A}_k} I_x, \tag{2}$$

we see from (1) and the size of I_x that

$$|O_k| \leq \frac{2}{k^2}. \tag{3}$$

Now

$$\begin{aligned} P(X \in O_k) &= \textstyle\sum_{x \in \mathscr{A}_k} P((X \in O_k) \cap (\sum_{n=1}^{N_k} a_n X_n = x)) \\ &\geq \textstyle\sum_{x \in \mathscr{A}_k} P((X \in I_x) \cap (\sum_{n=1}^{N_k} a_n X_n = x)) \\ &= \textstyle\sum_{x \in \mathscr{A}_k} P((\sum_{N_k+1}^{\infty} a_n X_n \in I_x - x) \cap (\sum_{n=1}^{N_k} a_n X_n = x)) \\ &= P(|\textstyle\sum_{N_k+1}^{\infty} a_n X_n| \leq k\sqrt{\sum_{N_k+1}^{\infty} a_n^2}). \end{aligned} \tag{4}$$

By Chebychev's inequality

$$P(|\textstyle\sum_{N_k+1}^{\infty} | a_n X_n| > k\sqrt{\sum_{N_k+1}^{\infty} a_n^2}) \leq \dfrac{1}{k^2}$$

which together with (4) combines to

$$P(X \in O_k) \geq 1 - \frac{1}{k^2}. \tag{5}$$

Now from (3) and (5)

$$|\cup_{k=M}^{\infty} O_k| = O\left(\frac{1}{M}\right)$$

and

$$P(X \in \cup_{k=M}^{\infty} O_k) = 1$$

from which we conclude

$$P(X \in \cap_{M=1}^{\infty} (\cup_{k=M}^{\infty} O_k)) = 1$$

and

$$|\cap_{M=1}^{\infty} (\cup_{k=M}^{\infty} O_k)| = 0. \qquad \square$$

5. Some open questions.

1. Can we find conditions involving the distributions of the X_n's which would tell us when a sequence $\{a_n\}$ is "admissible" for $\sum a_n X_n$ to have a distribution a.c. dx? Our results do not seem to give any insights to this problem.

2. Fix a sequence $\{a_n\}$ and let $\mathscr{F}$ be the class of distribution measures generated by $\Sigma\, a_n X_n$ where $\{X_n\}$ satisfies the conditions of Theorem 2. Can we use functional theoretic methods to show that $\mathscr{F}$ is contained in L^1? This could prove Theorem 2 for much more general $\{a_n\}$'s. Note that it is not hard to show that $\mathscr{F}$ is closed under convolution.

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