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ON THE IDENTIFIABILITY OF MULTIVARIATE LIFE DISTRIBUTION FUNCTIONS

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Let (T_1, T_2) and (L_1, L_2) be two independent bivariate random vectors with distributions F and H. Let $\tau_1 = \min(T_1, L_1)$, $\tau_2 = \min(T_2, L_2)$ and let $G_{0,0}(s, t) = P\{\tau_1 \le s, \tau_2 \le t, T_1 \le L_1, T_2 \le L_2\}$, $G_{0,1}(s, t) = P\{\tau_1 \le s, \tau_2 \le t, T_1 \le L_1, L_2 < T_2\}$, $G_{1,0}(s, t) = P\{\tau_1 \le s, \tau_2 \le t, L_1 < T_1, T_2 \le L_2\}$ and $G_{1,1}(s, t) = P\{\tau_1 \le s, \tau_2 \le t, L_1 < T_1, L_2 < T_2\}$. Under mild conditions the distributions F and H are expressed explicitly as functionals of $G_{0,0}$, $G_{0,1}$, $G_{1,0}$ and $G_{1,1}$. Necessary and sufficient conditions for the formulas to hold even when (T_1, T_2) and (L_1, L_2) are not independent are derived. Numerous applications are indicated. Extension of the results to p-dimensional distributions $(p > 2)$ is given.

1. Introduction and summary. By observing a series system of d components $(d = 2, 3, \cdots)$ we can only determine its *lifelength* and the *components that cause the system to fail*. In particular the lifelength of a series system that consists of a nonempty subset of the original d components is unobservable. We refer to the distribution function (d.f.) of the lifelength of this series system as an *unobservable d.f.* For example, some of the original d components, such as wires and switches, may be placed in the system to support the operation of the main components. The d.f. of the lifelength of the series system that consists only of the main components is of great importance to the theory of engineering reliability but is unobservable.

Langberg, Proschan, and Quinzi (LPQ) (1978), although addressing themselves to a different problem, present, in particular, a way to relate the unobservable d.f.'s to observable quantities (see Theorem 4.1). First, LPQ (1978) define $2^d - 1$ *observable subdistribution functions* (s.d.f.'s). The value of an unobservable s.d.f. at the point $s (s \ge 0)$ is the probability that the original series system fails at a time less than or equal to s as a result of the simultaneous failure of a particular nonempty subset of the d components and of no other components. Then LPQ (1978) express the various unobservable d.f.'s as functionals of the observed s.d.f.'s, whenever the lifelengths of the d components are independent and satisfy a mild regularity condition. Thus, they provide the researchers in the theory of engineering reliability with a *theoretical* tool to determine the unobservable d.f.'s. The problem of relating unobservable d.f.'s to observable s.d.f.'s has been considered by several other researchers such as Peterson (1975), Tsiatis (1975), and Miller (1977).

The theoretical relationship discussed in the previous paragraph suggests a "naive" method of estimating the unobservable d.f.'s. LPQ (1981) consider the problem of estimating the various unobservable d.f.'s based on data collected from n independent and identical series systems each consisting of d components. First, LPQ (1981) estimate the observable s.d.f.'s by their empirical counterparts. Then they replace the observable s.d.f.'s by their empirical estimators in the respective functionals. Using this method LPQ (1981) obtain the Product Limit Estimators (P.L.E.'s) for the observable d.f.'s, first introduced by Kaplan and Meier (1958).

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An important statistical problem is to test whether the life distribution of a component, F, is exponential versus various wearout classes of alternatives on the basis of a random sample from the d.f. F in the absence of the other $d-1$ nuisance components. Tests for these hypotheses have been proposed, for example, by Proschan and Pyke (1967), Barlow (1968), Bickel (1969), Bickel and Doksum (1969), Barlow and Doksum (1972), Hollander and Proschan (1972), (1975), (1979), and Koul (1977), (1978a), (1978b). There seems to be a growing interest among statisticians to test these hypotheses in a more realistic situation; when additional "nuisance" components are present. Koul and Susarla (1978) and Chen, Hollander, and Langberg (1980a), (1980b) suggest tests for some of the hypotheses described above based on the P.L.E.

In this paper we present a formula that relates unobservable multivariate d.f.'s to observable multivariate s.d.f.'s. This formula suggests further development of statistical estimation and testing procedures concerning multivariate d.f.'s in the presence of "nuisance" components. A recent study in this direction is the work of Campbell and Foldes (1981).

Assume that a pair of individuals, a wife and a husband for example, are under study. The observation of each of the two individuals is terminated in the event of death or in case of a withdrawal from the study. The joint lifelengths of the two individuals is of great importance to the theory of biostatistics but is unobservable.

In Section 2 we relate the unobservable joint life distribution of the two individuals to observable quantities. First we define 4 observable joint s.d.f.'s. The value of an observable joint s.d.f. at times t, $s (t, s \geq 0)$ is the probability that the two individuals are removed from the study at times less than or equal to t and s, respectively, as a result of a specific combination of deaths and withdrawals. Then we express the unobservable joint d.f. as a functional of the observable s.d.f.'s, whenever some mild conditions hold. Thus we provide the researchers in biostatistics with a *theoretical* tool to determine the unobservable d.f.

In Section 3 we consider a group of p individuals, $p = 1, 2, \cdots$, a family for example, and assume that each individual is exposed to d risks, $d = 2, 3, \cdots$. Using the results obtained in Section 2 we relate the joint d.f. of the times to deaths of the p individuals from risk $j, j = 1, \cdots, d$, to the observable s.d.f.'s. The value of an observable s.d.f. at times $t_1, \cdots, t_p (t_1, \cdots, t_p \geq 0)$ is the probability that the p individuals die at times less than or equal to $t_1, \cdots, t_p$, respectively, as a result of the occurrence of a specific combination of risks.

Finally, we note that although we have chosen to employ the languages of reliability theory, biostatistics and the theory of competing risks (series system, components, death, withdrawal, risks, etc.) the results presented here apply to any model where observations include (1) the time at which a particular event occurs and (2) the cause(s) of the occurrence of the event. This is the case, for example, in mortality studies (Hoel, 1972) and some mathematical epidemiology models (see Billard, Lacayo and Langberg, 1979; Lacayo and Langberg, 1980a, 1980b; and Langberg, 1980.)

Throughout we define a product over an empty set of indices as 1 and an integral over an empty set as zero.

2. The bivariate case. Consider a pair of individuals, a wife and a husband for example, with lifelengths T_1 and T_2. The observation of each of the two individuals is terminated in the event of death or in case of a withdrawal from the study. Let L_1 and L_2 denote the withdrawal times of the two individuals respectively. We assume that the random variables (r.v.'s) T_1, T_2, L_1 and L_2 are defined on a common probability space $(\Omega, \mathscr{B}, P)$.

In this section we present the joint d.f. of the lifelengths (T_1, T_2) as a functional of observable quantities. To be more specific we introduce two definitions and some notation.

DEFINITION 2.1. We say that individual i is *removed* from the study due to *death* (*withdrawal*) and write $\xi_i = 1 (\xi_i = 2)$ if $T_i \leq L_i (L_i < T_i)$, $i = 1, 2$.

DEFINITION 2.2. Let K be a function defined on $(-\infty, \infty)$. We say that K is a *subdistribution function* (s.d.f.) if K is nondecreasing, right-continuous, and assumes values in $[0, 1]$.

Let $\tau_i = \min\{T_i, L_i\}$ be the removal time of individual i, and let G_i be the d.f. of τ_i, $i = 1, 2$. Further, let $F(t, s, I_1, I_2) = P\{\tau_1 \le t, \tau_2 \le s, \xi_1 = I_1, \xi_2 = I_2\}$, $I_1, I_2 \in \{1, 2\}$ be the observable joint s.d.f.'s and let $M(t, s) = P\{T_1 \le t, T_2 \le s\}$ be the unobservable joint d.f. of the lifelengths of the two individuals under consideration. For a s.d.f. K, let $\bar{K}(t) = \lim_{s\to\infty}K(s) - K(t)$ be the *subsurvivalfunction* corresponding to K, and let $\alpha(K) = \sup\{t: \bar{K}(t) > 0\}$. In this section we present the d.f. $M(\cdot, \cdot)$ as a functional of the s.d.f.'s: $F(\cdot, \cdot, I_1, I_2)$, $I_1, I_2 \in \{1, 2\}$. Formally, G_1 and G_2 will also appear in the expression for M; notice however that G_1 and G_2 are determined by $F(\cdot, \cdot, I_1, I_2)$, $I_1, I_2 \in \{1, 2\}$.

We will assume that:

(A.1) The lifelengths (T_1, T_2) and the withdrawal times (L_1, L_2) are independent random vectors, and that

(A.2) For $i = 1, 2$, the functions $P\{\tau_i \le t, \xi_i = 1\}$ and $P\{\tau_i \le t, \xi_i = 2\}$ are continuous in $t \in (-\infty, \infty)$.

REMARK. Assumption (A.2) can be weakened by assuming only that, for $i = 1, 2$, the two sets of discontinuity points of $P\{\tau_i \le \cdot, \xi_i = 1\}$ and of $P\{\tau_i \le \cdot, \xi_i = 2\}$ are disjoint. Generalizations of the results below are then possible and are available in Langberg and Shaked (1980).

First, we reduce the problem of obtaining the desired functional that relates the observable joint s.d.f.'s to the unobservable joint d.f. concerning two individuals to a similar, but simpler problem, concerning only one individual. Note that by Assumption (A.1)

$$\begin{aligned} P\{T_1 > t, T_2 > s\} &= P\{T_1 > t \mid T_2 > s\}P\{T_2 > s\} \\ &= P\{T_1 > t \mid \tau_2 > s\}P\{T_2 > s\} \quad \text{for} \quad t \in (-\infty, \infty), \quad s \in (-\infty, \alpha(G_2)), \end{aligned} \tag{2.1}$$

$$\text{The r.v.'s } T_2 \text{ and } L_2 \text{ are independent,} \tag{2.2}$$

and that

$$\text{For } s \in (-\infty, \alpha(G_2)) \text{the conditional r.v.'s } \{T_1 \mid \tau_2 > s\} \quad \text{and} \quad \{L_1 \mid \tau_2 > s\} \quad \text{are independent.} \tag{2.3}$$

Further, by (A.2):

$$\text{For } s \in (-\infty, \alpha(G_2)) \text{ functions } P\{\tau_1 \le \tau, \xi_1 = 1 \mid \tau_2 > s\} \quad \text{and} \quad P\{\tau_1 \le \tau, \xi_1 = 2 \mid \tau_2 > s\} \quad \text{are continuous in } t \in (-\infty, \infty). \tag{2.4}$$

Observe that to express $M(\cdot, \cdot)$ as a functional of the s.d.f.'s: $F(\cdot, \cdot, I_1, I_2)$, $I_1, I_2 = 1, 2$, it suffices (a) to express $P\{T_2 > \cdot\}$ as a functional of the s.d.f.'s: $P\{\tau_2 \le \cdot, T_2 \le L_2\}$, $P\{\tau_2 \le \cdot, L_2 < T_2\}$, (b) to express $P\{T_1 > \cdot \mid \tau_2 > s\}$ as a functional of the s.d.f.'s: $P\{\tau_1 \le \cdot, T_1 \le L_1 \mid \tau_2 > s\}$, $P\{\tau_1 \le \cdot, L_1 < T_1 \mid \tau_2 > s\}$ for every $s \in (-\infty, \alpha(G_2))$ and (c) to insert these functionals in Equation (2.1). This is done in Theorem 2.4, but first we need some preliminaries.

Let T and L be two independent r.v.'s representing, respectively, the lifelength and withdrawal time of a single individual. Further, let $\tau = \min\{T, L\}$ be the removal time of the individual from the study and let G denote the d.f. of τ. We express now the unobservable survival function: $P\{T > \cdot\}$ as a functional of the observed s.d.f.'s: $P\{\tau \le \cdot, T \le L\}$, $P\{\tau \le \cdot, L < T\}$. The result is a restatement of Equation (3.5) of LPQ (1978).

LEMMA 2.3. *Assume that*

(A.3) *The* r.v.'s *T and L are independent and that*
(A.4) *The functions $P\{\tau \le t, T \le L\}$ and $P\{\tau \le t, L \le T\}$ are continuous. Then, for $t \in (-\infty, \alpha(G))$,*

$$P\{T > t\} = \exp\left\{-\int_{-\infty}^{t} [\bar{G}(a)]^{-1}\, dP\{\tau \le a, T \le L\}\right\}. \tag{2.5}$$

We return now to our original problem, and present the functional that relates the joint d.f. of the lifelengths of the two individuals considered to their observable s.d.f.'s.

THEOREM 2.4. *Assume (T_1, T_2, L_1, L_2) satisfies Conditions* (A.1) *and* (A.2) *and let $\alpha_{1,s} = \sup\{t: P(\tau_1 > t \mid \tau_2 > s) > 0\}$, $s \in (-\infty, \alpha(G_2))$. Then for $(t, s) \in \{(u, v): v \in (-\infty, \alpha(G_2)), u \in (\infty, \alpha_{1,v})\}$:*

$$\begin{aligned} P\{T_1 > t, T_2 > s\} &= \exp\left\{-\int_{-\infty}^{s} [\bar{G}_2(u)]^{-1}\, dP\{\tau_2 \le u, T_2 \le L_2\}\right\} \\ &\quad \cdot \exp\left\{-\int_{-\infty}^{t} [P\{\tau_1 > v \mid \tau_2 > s\}]^{-1}\, dP\{\tau_1 \le v, T_1 \le L_1 \mid \tau_2 > s\}\right\}. \end{aligned} \tag{2.6}$$

PROOF. First note that, by Assumptions (A.1) and (A.2), Conditions (A.3) and (A.4) hold for $(T, L) = (T_2, L_2)$. Thus, by Lemma 2.3 for $s \in (-\infty, \alpha(G_2))$:

$$P\{T_2 > s\} = \exp\left\{-\int_{-\infty}^{s} [\bar{G}_2(u)]^{-1}\, dP\{\tau_2 \le u, T_2 \le L_2\}\right\}. \tag{2.7}$$

Let $s \in (-\infty, \alpha(G_2))$. Note that by Assumptions (A.1) and (A.2), Conditions (A.3) and (A.4) hold for the random pair (T, L) that is stochastically equal to the conditional random pair $[(T_1, L_1) \mid \tau_2 > s]$. Thus, by Lemma 2.3 for $t \in (-\infty, \alpha_{1,s})$:

$$P\{T_1 > t \mid \tau_2 > s\} = \exp\left\{-\int_{-\infty}^{t} [P\{\tau_1 > v \mid \tau_2 > s\}]^{-1}\, dP\{\tau_1 \le v, T_1 \le L_1 \mid \tau_2 > s\}\right\}. \tag{2.8}$$

Consequently the desired result follows by Equations (2.1), (2.7) and (2.8) by insertion. □

Next we note that the relationship between the unobservable d.f. and the observable s.d.f.'s given by Equation (2.6) holds for some cases where the independence asssmption, given by (A.1), does not hold. First, we present a necessary and sufficient condition for the relationship given by (2.6) to hold. Then we give an example for which Assumption (A.1) does not hold, however the desired relationship, given by Equation (2.6), holds. We need the following lemma.

LEMMA 2.5. (LPQ (1981), Theorem 4.4). *Let (L, T) be a pair of* r.v.'s *defined on a common probability space as in Lemma* 2.3, *but now do not assume Condition* (A.3) *or* (A.4). *If $G(\cdot)$ is continuous on $(-\infty, \infty)$ then Equation* (2.5) *holds if and only if the following condition is satisfied*:

(A.5) *For $t \in (-\infty, \alpha(G))$, $P\{L \ge t \mid T = t\}$*
$= P\{L > t \mid T > t\}$ a.s. *with respect to $P\{T \le \cdot\}$.*

We are ready to present the necessary and sufficient conditions.

THEOREM 2.6. *Let* T_1, L_1, T_2, *and* L_2 *be* r.v.'s *defined on a common probability space. Assume that Condition* (2.1) *holds, that* $P\{T_2 \le \cdot\}$ *is continuous on* $(-\infty, \infty)$ *and that* $P\{T_1 \le \cdot, \tau_2 > s\}$ *is continuous on* $(-\infty, \infty)$ *for every* $s \in (-\infty, \alpha(G_2))$. *Then Equation* (2.6) *holds if and only if the following conditions hold*:

(A.6) *For* $s \in (-\infty, \alpha(G_2))$, $P\{L_2 \ge s \mid T_2 = s\}$

$= P\{L_2 > s \mid T_2 > s\}$ a.s. *with respect to* $P\{T_2 \le \cdot\}$,

(A.7) *For* $t \in (-\infty, \alpha_{1,s})$, $s \in (-\infty, \alpha(G_2))$,

$$P\{L_1 \ge t \mid \tau_2 > s, T_1 = t\} = P\{L_1 > t \mid \tau_2 > s, T_1 > t\}$$

a.s. *with respect to* $P\{T_1 \le \cdot \mid \tau_2 > s\}$.

PROOF. Note that

$$P\{T_2 \in B, T_2 \le L_2\} = \int_B P\{L_2 \ge u \mid T_2 = u\}\, dP\{T_2 \le u\} \tag{2.9}$$

for all Borel sets in $(-\infty, \alpha(G_2))$ and that

$$P\{T_1 \in B, T_1 \le L_1 \mid \tau_2 > s\} = \int_B P\{L_1 \ge u \mid \tau_2 > s, T_1 = u\}\, dP\{T_1 \le u \mid \tau_2 > s\} \tag{2.10}$$

for all Borel sets in $(-\infty, \alpha(G_1))$.

First, assume that Conditions (A.6) and (A.7) hold. Then Equation (2.6) follows by substitution and by Equations (2.5), (2.9) and (2.10).

Now, assume that Equation (2.6) holds. Then, in particular, the equation holds for $t = 0$, $s \in (-\infty, \alpha(G_2))$. Condition (A.6) follows now by Lemma 2.5. Since from (A.6) the pair (T_2, L_2) satisfies Equation (2.5), it follows from Equations (2.1) and (2.6) that the conditional pair $(\{T_1 \mid \tau_2 > s\}, \{L_1 \mid \tau_2 > s\})$, $s \in (-\infty, \alpha(G_2))$ satisfies Equation (2.5). Thus, Condition (A.7) follows by Lemma 2.5. □

Finally we present an example where the independence assumption, given by (A.1), does not hold, but the desired relationship given by Equation (2.6) is satisfied. The following definition is needed.

DEFINITION 2.7. (Marshall and Olkin, 1967). Let λ_I, $\phi \ne I \subset \{1, \cdots, n\}$ be nonnegative real numbers, $\sum_{\phi \ne I \subset \{1,\cdots,n,\}} \lambda_I > 0$, and let V_I, $\phi \ne I \subset \{1, \cdots, n\}$ be independent exponential r.v.'s with rates λ_I, $\phi \ne I \subset \{1, \cdots, n\}$, respectively. The random vector $\langle U_1, \cdots, U_n \rangle$ has a *Marshall-Olkin Multivariate Exponential Distribution* (M.O.M.E.D.) with parameters λ_I, $\phi \ne I \subset \{1, \cdots, n\}$ if the two random vectors $\langle U_1, \cdots, U_n \rangle$ and $\langle \min\{V_I{:}I, 1 \in I\}, \cdots, \min\{V_I{:}I, n \in I\} \rangle$ are stochastically equal.

EXAMPLE 2.8. Let $\langle U_1, \cdots, U_4 \rangle$ have a M.O.M.E.D. with parameters λ_I, $\phi \ne I \subset \{1, \cdots, 4\}$, and let us assume that $\langle T_1, T_2, L_1, L_2 \rangle$ and $\langle U_1, \cdots, U_4 \rangle$ are stochastically equal. Let $\alpha_{1,s}$, $s \in (-\infty, \alpha(G_2))$ be as in Theorem 2.4. Note that $\alpha(G_2) = \alpha_{1,s} = \infty$, and that the two random pairs (T_1, T_2) and (L_1, L_2) are not necessarily independent. We want to show that Formula (2.6) holds. To obtain the desired result it suffices, by Theorem 2.6, to verify Equation (2.1) and Conditions (A.6) and (A.7). These two conditions follow by some simple calculation with independent exponential r.v.'s. Equation (2.1) does not hold in general, however if $\lambda_{\{1,3,4\}} = \lambda_{\{1,4\}} = 0$, then Equation (2.1) holds.

Note that by Assumption (A.1)

$$P\{T_1 > t, T_2 > s\} = P\{T_2 > s \mid \tau_1 > t\} P\{T_1 > t\} \tag{2.11}$$

Thus, Theorems 2.4 and 2.6 hold when we exchange T_1 and T_2, L_1 and L_2 and (2.1) and (2.11).

Finally, note that if we exchange T_1 and L_1 and T_2 and L_2 then Theorems 2.4 and 2.6 hold for the random pair (L_1, L_2).

3. The multivariate case. Consider a group of p individuals, a family for example, $p = 1, 2, \cdots$, and assume that each individual is exposed to d risks, $d = 2, 3, \cdots$. Let us denote the time to death of individual i from risk j by a nonnegative r.v. $T_{i,j}$, $i = 1, \cdots, p$, and let $T_{0,j} = \infty$, $j = 1, \cdots, d$. Assume that all these r.v.'s are defined on a common probability space $(\Omega, \mathscr{B}, P)$.

In this section we present the joint d.f. of the times to death of the p individuals from risk j: $P\{T_{1,j} \le \cdot, \cdots, T_{d,j} \le \cdot\}$ as functional of observable quantities. To be more specific we introduce some notation. Throughout we fix the risk considered and denote it by j. Let $L_{ij} = \min\{T_{iq}, q = 1, \cdots, d, q \ne j\}$, let $\tau_i = \min\{L_{i,j}, T_{i,j}\}$ be the lifelength of individual i and let G_i be the d.f. of τ_i, $i = 1, \cdots, p$. Further, let $\xi_{i,j} = 1$ on the set $\{T_{i,j} \le L_{i,j}\}$, and $= 2$ on the set $\{L_{i,j} < T_{i,j}\}$, $i = 1, \cdots, p$, let $M_j(t_1, \cdots, t_p) = P\{T_{i,j} \le t_i, i = 1, \cdots, p\}$ be the probability that the p individuals die at times less than or equal to $t_1, \cdots, t_p$, respectively, from risk j in the *absence* of all the other risks, and let $F_j(t_1, \cdots, t_p, I_1, \cdots, I_p) = P\{\tau_i \le t_i, \xi_{i,j} = I_i, i = 1, \cdots, p\}$, $I_i = 1, 2$, $i = 1, \cdots, p$, be the observed s.d.f.'s: the probability that the p individuals die at times less than or equal to $t_1, \cdots, t_p$, respectively, as a result of a specific combination of risks. In this section we present each of the unobservable d.f.'s $M_j(\cdot, \cdots, \cdot)$ as a functional of the observed s.d.f.'s: $F_j(\cdot, \cdots, \cdot, I_1, \cdots, I_p)$, provided that:

(B.1) The random vectors $\langle T_{1,q}, \cdots T_{p,q}\rangle$, $q = 1, \cdots, d$ are independent, and that
(B.2) For $r = 1, \cdots, p$ the functions

$P\{\tau_r \le \cdot, \xi_{r,q} = 1\}$ and $P\{\tau_r \le \cdot, \xi_{r,q} = 2\}$ are continuous on $(-\infty, \infty)$.

THEOREM 3.1. *Let*

$$\alpha_{1,t_0} \equiv \sup\{t: P\{\tau_1 > t\} > 0\}, \qquad \alpha_{r,t_0,\cdots,t_{r-1}} \equiv \sup\{t: P\{\tau_r > t \mid \tau_1 > t_1, \cdots, \tau_{r-1} > t_{r-1}\}\},$$

$$t_1 < \alpha_{1,t_0}, \qquad t_2 < \alpha_{2,t_0,t_1}, \cdots, t_r < \alpha_{r,t_0,\cdots t_{r-1}}, \qquad r = 2, \cdots, p, \; t_0 = -\infty.$$

Assume Conditions (B.1) *and* (B.2) *hold. Then for* $(t_0, \cdots, t_p) \in \{(u_0, \cdots, u_p): u_0 = -\infty, u_r < \alpha_{r,u_0,\cdots,u_{r-1}}, r = 1, \cdots, p\}$,

$$\begin{aligned} &P\{T_{i,j} > t_1, \cdots, T_{p,j} > t_p\} \\ &= \exp\Big\{-\textstyle\sum_{r=1}^{p} \displaystyle\int_{-\infty}^{t_p} [P\{\tau_r > a, \tau_i > t_i, i = 0, \cdots, r-1\}]^{-1} \\ &\qquad \cdot dP\{\tau_r \le a, \xi_{r,j} = 1, \tau_i > t_i, i = 0, \cdots, r-1\}\Big\}. \end{aligned} \tag{3.1}$$

PROOF. First, note that by Condition (B.1)

$$\begin{aligned} P\{T_{i,j} > t_1, \cdots, T_{p,j} > t_p\} &= \textstyle\prod_{r=1}^{p} P\{T_{r,j} > t_r \mid T_{i,j} > t_i, i = 0, \cdots, r-1\} \\ &= \textstyle\prod_{r=1}^{p} P\{T_{r,j} > t_r \mid \tau_i > t_i, i = 0, \cdots, r-1\}. \end{aligned}$$

Consequently the desired result follows now by Condition (B.2) and Theorem 2.4 applied to the conditional pairs $[\{T_{r,j}, L_{r,j}\} \mid \tau_i > t_i, i = 0, \cdots, r-1]$ $r = 1, \cdots, p$. □

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