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BROWNIAN MOTION WITH LOWER CLASS MOVING BOUNDARIES WHICH GROW FASTER THAN $t^{1/2}$

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Upper and lower bounds are obtained for $P(|W(t)| \le f(t),\ t \le u)$ and $P(|S(n)| \le f(n),\ n \le N)$, u, N large, where $W(t)$ is a Brownian motion, $S(n)$ is a random walk with $ES(1) = 0$, $E|S(1)|^{2+2\eta} < \infty$, and $f(t)$ is a deterministic function growing faster than $t^{1/2}$ but slower than $(2t \ln\ln t)^{1/2}$.

1. Introduction. Let $W(t)$ be a standard Brownian motion, $f(t)$ a deterministic increasing function. How does $\mathscr{P}_u = P(|W(t)| \le f(t),\ t \le u)$ behave for large u? A closely related problem is the behavior of $\mathscr{P}_N = P(|S(n)| \le f(n),\ n \le N)$, where $S(n)$ is a random walk.

When $f(t) = ct^{1/2}$, this problem has been considered by Breiman (1965), Gundy and Siegmund (1967), and Brown (1969), among others. When $f(t) = o(t^{1/2})$, results have been obtained by Lai (1977), Portnoy (1978), Kesten (1978), and Novikov (1981), to mention only a few. Upper class functions f have been studied by Cuzick (1981) and Jennen and Lerche (1981).

Here we consider the remaining case, when f grows faster than $t^{1/2}$ but more slowly than $(2t \ln\ln t)^{1/2}$. In contrast to the $o(t^{1/2})$ case, where $\ln \mathscr{P}_u$ is asymptotically $-c \int_0^u f(t)^{-2}\, dt$, we get that $\ln \mathscr{P}_u$ behaves like $-\int_0^u f(t)^{-2} \exp(-cf(t)^2/2t)\, dt$ and a similar result for random walks.

More precisely, we get the following:

Suppose $f(t)$ is bounded away from 0, and for $t \ge u_1 > 0$, for some u_1, $f(t) = t^{1/2}L(t)$, where $L(t)$ is strictly positive, nondecreasing, continuous, slowly varying, $L(t) = o((\ln\ln t)^{1/2})$, and $L(+\infty) = +\infty$. Let $\alpha = 1/3000$.

THEOREM 1. *Given $\varepsilon > 0$, there exist constants c_1, c_2, and u_0, depending only on ε and f, such that if $u \ge u_0$,*

$$c_1 \exp\left(-\int_0^u f(t)^{-2} e^{-\alpha f(t)^2/2t}\, dt\right) \le \mathscr{P}_u \le c_2 \exp\left(-\int_0^u f(t)^{-2} e^{-(1+\varepsilon) f(t)^2/2t}\, dt\right).$$

THEOREM 2. *Let X_i, $i = 1, 2, \cdots$ be iid random variables, $S(n) = \sum_{n=1}^n X_i$. Suppose $EX_1 = 0$, $EX_1^2 = 1$, and $E|X_1|^{2+2\eta} < \infty$ for some $\eta > 0$. Then there exist constants c_1, c_2, and N_0 such that if $N \ge N_0$,*

$$c_1 \exp\left(-\sum_{i=1}^N f(i)^{-2} e^{-\alpha f(i)^2/2i}\right) \le \mathscr{P}_N \le c_2 \exp\left(-\sum_{i=1}^N f(i)^{-2} e^{-(1+\varepsilon) f(i)^2/2i}\right).$$

Our techniques still work for $f(t) = c(2t \ln\ln t)^{1/2}$, but the results are of no interest unless $c < \alpha$.

Theorem 1 is proved in Section 2. Theorem 2 is proved in Section 3.

2. Brownian Motion. We start by getting an estimate on $Q(t, x, k) = P^x(|W(s)| \le k,\ s \le t)$. Note that $Q(t, -x, k) = Q(t, x, k)$ by symmetry.

Let $d_\varepsilon \ge 4$ be such that $d^{-1}\exp(-d^2/2) \le d^{-2}\exp(-(1-\varepsilon)d^2/2)$ whenever $d \ge d_\varepsilon$.

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PROPOSITION 2.1. *Suppose* $0 \le x \le k$. *If* $(k-x)^2/t \ge d_\varepsilon^2$ *then*

$$1-(4t(2\pi)^{-1/2}/(k-x)^2)\exp(-(1-\varepsilon)(k-x)^2/2t) \le Q(t,x,k) \le 1-(t(2\pi)^{-1/2}/k^2)\exp(-k^2/2t).$$

PROOF OF 2.1. Letting $d = k/t^{1/2}$ and using the inequality

$$\int_d^\infty e^{-y^2/2}\,dy \ge (d^{-1}-d^{-3})e^{-d^2/2} \ge \frac{1}{2}d^{-1}e^{-d^2/2}$$

for $d \ge 2$, (Feller, 1968, page 175) we have

$$Q(t,x,k) \le P^x(|W(t)| \le k) \le P^0(|W(t)| \le k) = 1-2(2\pi t)^{-1/2}\int_k^\infty e^{-y^2/2t}\,dy \le 1-(2\pi)^{-1/2}d^{-1}e^{-d^2/2}$$

if $k^2/t \ge 4$. Since $d^{-1} \ge d^{-2}$ for $d \ge 1$, this gives the right hand inequality.

Let $q(t,x,y,k)$ be the transition density for Brownian motion started at x and killed on leaving $[-k,k]$. Of course, $Q(t,x,k) = \int_{-k}^{k} q(t,x,y,k)\,dy$.

A simple change of variables applied to a formula of Feller (1971), page 341, gives

$$(2\pi t)^{1/2}q(t,x,y,k) = \sum_{j=-\infty}^{\infty}[\exp(-(y-x+4jk)^2/2t)-\exp(-(y+x+4jk+2k)^2/2t)].$$

The infinite series on the right is absolutely convergent and of alternating sign. Pairing terms, if $j \ge 1$, $\exp(-(y-x+4jk)^2/2t)-\exp(-(y+x+4jk+2k)^2/2t) \ge 0$ since $x \ge -k$. If $j \le -1$, $\exp(-(y-x+4jk)^2/2t)-\exp(-(y+x+4(j-1)k+2k)^2/2t) \ge 0$ since $x \le k$. Therefore

$$(2\pi t)^{1/2}q(t,x,y,k) \ge e^{-(y-x)^2/2t}-e^{-(y+x-2k)^2/2t}-e^{-(y+x+2k)^2/2t}.$$

Integrating y from $-k$ to k,

$$\begin{aligned} Q(t,x,y) &\ge P^x(|W(t)| \le k)-P^{2k-x}(|W(t)| \le k)-P^{-x-2k}(|W(t)| \le k) \\ &\ge 1-P^0(W(t) > k-x)-P^0(W(t) < -k-x) \\ &\quad -P^0(-3k-x \le W(t) \le -k-x)-P^0(k-x \le W(t) \le 3k-x) \\ &\ge 1-4P^0(W(t) \ge k-x). \end{aligned}$$

Let $d = (k-x)/t^{1/2}$ and use the inequality

$$\int_d^\infty e^{-y^2/2}\,dy \le d^{-1}e^{-d^2/2}$$

to get $Q(t,x,k) \ge 1-(4/(2\pi)^{1/2}d)e^{-d^2/2}$. If $d \ge d_\varepsilon$, we thus get the left hand inequality. □

PROOF OF THEOREM 1.

Upper bound. Let $0 < \varepsilon < 1/2$. Let $q > 2$ be chosen so that $(1-q^{-1})^{-1} \le 1+\varepsilon/4$. Let u_0 be chosen equal to q^I for some I and large enough so that if $s \ge u_0/q^3$, $q^2 s \ge t \ge qs$, then

(i) $[f(t)-f(s)]^2/(t-s) > d_\varepsilon$,

(ii) $f(s)/f(t) \ge (2q)^{-1}$,

(iii) $(4(2\pi)^{1/2}q^2)^{-1}\exp(-f(t)^2/2(t-s)) \ge \exp(-(1+\varepsilon/3)f(t)^2/2(t-s))$, and

(iv) $f(t)^2/2(t-s) \le (1+\varepsilon/3)f(s)^2/2s$.

This can be done since $L(t)$ is slowly varying, but increasing to infinity.

Suppose $u \geq u_0$. Let n be the largest integer such that $q^n \leq u$. Let $t_0 = 0$, $t_i = q^i$ for $1 \leq i < n$, $t_n = u$. Let

$$A_i = \{|W(t)| \leq f(t_i),\ t_{i-1} < t \leq t_i\}, \quad i = 1, \cdots, n,$$

and let

$$r_i = (t_i - t_{i-1})/((2\pi)^{1/2} f(t_i)^2)\exp(-f(t_i)^2/2(t_i - t_{i-1})).$$

Note $q^{-2} \leq t_{i-1}/t_i \leq q^{-1}$.

Using the Markov property,

$$\begin{aligned}\mathscr{P}_u \leq P^0(A_1A_2 \cdots A_n) &= E^0(P^{W(t_{n-1})}(|W(t)| \leq f(t_n),\ 0 < t \leq t_n - t_{n-1}); A_1 \cdots A_{n-1}) \\ &= E^0(Q(t_n - t_{n-1},\ W(t_{n-1}),\ f(t_n)); A_1 \cdots A_{n-1}) \\ &\leq (1 - r_n)P^0(A_1 \cdots A_{n-1}).\end{aligned}$$

To get the last inequality we used (i) and (2.1). Repeating

$$\mathscr{P}_u \leq \textstyle\prod_{i=I+1}^{n} (1 - r_i)P^0(A_1 \cdots A_I),$$

or

$$\ln \mathscr{P}_u \leq c + \textstyle\sum_{i=I+1}^{n} \ln(1 - r_i) \leq c - \sum_{i=I+1}^{n} r_i.$$

Using (ii), (iii), and (iv), if $t_{i-1} < t \leq t_i$

$$\begin{aligned}-r_i &\leq -(4q^2(t_i - t_{i-1})/f(t_i)^2)\exp(-(1 + \varepsilon/3)f(t_i)^2/2(t_i - t_{i-1})) \\ &\leq -((t_i - t_{i-1})/f(t_{i-1})^2)\exp(-(1 + \varepsilon/3)^2 f(t_{i-1})/2(t_{i-1})).\end{aligned}$$

Thus,

$$\ln \mathscr{P}_u \leq c - \int_{t_I}^{u} f(t)^{-2}\exp(-(1 + \varepsilon)f(t)^2/2t)\, dt = c' - \int_{0}^{u} f(t)^{-2}\exp(-(1 + \varepsilon)f(t)^2/2t)\, dt.$$

Lower bound. Let $q = 9$, $\alpha = 1/3000$. Let u_0 be chosen equal to q^I for some I and large enough so that if $s \geq u_0/q^3$, $q^2 s \geq t \geq qs$, then

(v) $[f(s) - f(s/q)]^2/(t - s) \geq 2\alpha f(t)^2/t$,

(vi) $[f(s) - f(s/q)]^2/(t - s) \geq d_{1/4}^2 \qquad (> 10)$,

(vii) $(16q^2)^{-1}\exp(-\tfrac{1}{2}[f(s) - f(s/q)]^2/2(t - s))$

$$\geq (8/(2\pi)^{1/2})\exp(-\tfrac{3}{4}[f(s) - f(s/q)]^2/2(t - s)),$$

and

(viii) $(16q^2)^{-1}[f(s) - f(s/q)]^{-2} \leq f(t)^{-2}$.

To show that u_0 can be chosen so that (v) holds, observe that the right hand side of (v) is $2\alpha L(t)^2$. But since $f(s/q)/f(s) = q^{-1/2}L(s/q)/L(s) \leq 2q^{-1/2}$ for s sufficiently large and $t/s - 1 \leq q^2$, the left side of (v) is

$$L(s)^2[1 - f(s/q)/f(s)]^2/(t/s - 1) \geq L(s)^2[1 - 2q^{-1/2}]^2/q^2 \geq 4\alpha L(s)^2 > 2\alpha L(t)^2$$

for s sufficiently large.

Since $L(t)$ increases to ∞, it follows easily from (v) that u_0 can be chosen so that (vi) and (vii) hold. The inequality (viii) is argued in a manner similar to (v).

Observe that $-2x \leq \ln(1 - x)$ if $0 \leq x \leq \tfrac{1}{2}$ and that $xe^{-x/2} \leq \tfrac{1}{4}$ if $x \geq 7$.

Suppose $u \geq u_0$. Let n be the largest integer such that $q_n \leq u$, let $t_0 = 0$, $t_i = q^i$ for $1 \leq i < n$, $t_n = u$,

$$B_i = \{|W(t)| \leq f(t_{i-1}),\ t_{i-1} < t \leq t_i\}, \quad i = 1, \cdots, n,$$

and let

$$v_i = (4(2\pi)^{-1/2}(t_i - t_{i-1})/[f(t_{i-1}) - f(t_{i-2})]^2) \times \exp(-\tfrac{3}{4}[f(t_{i-1}) - f(t_{i-2})]^2/2(t_i - t_{i-1})).$$

Note $t_{i-2} = t_{i-1}/q$.

$$\begin{aligned}\mathcal{P}_u &\geq P^0(B_1 B_2 \cdots B_n)\\ &= E^0(P^{W(t_{n-1})}(|W(t)| \leq f(t_{n-1}), 0 < t \leq t_n - t_{n-1}); B_1 \cdots B_{n-1})\\ &= E^0(Q(t_n - t_{n-1}, W(t_{n-1}), f(t_{n-1})); B_1 \cdots B_{n-1})\\ &\geq (1 - v_n)P^0(B_1 \cdots B_{n-1})\end{aligned}$$

by (2.1), (vi), the fact that e^{-x}/x is decreasing, and the fact that on B_{n-1}, $|W(t_{n-1})| \leq f(t_{n-2})$. Repeating, $\mathcal{P}_u \geq \prod_{i=I+1}^{n} (1 - v_i)P^0(B_1 \cdots B_I)$, or

$$\ln \mathcal{P}_u \geq c + \sum_{i=I+1}^{n} \ln(1 - v_i) \geq c - 2\sum_{i=I+1}^{n} v_i,$$

using (vi) again.

Using (v), (vii), and (viii), if $t_{i-1} < t \leq t_i$,

$$\begin{aligned}-2v_i &\geq -((16q^2)^{-1}(t_i - t_{i-1})/[f(t_{i-1}) - f(t_{i-2})]^2)\\ &\qquad \times \exp(-\tfrac{1}{2}[f(t_{i-1}) - f(t_{i-2})]^2/2(t_i - t_{i-1}))\\ &\geq -((t_i - t_{i-1})/f(t_i)^2)\exp(-\alpha f(t_i)^2/2t_i),\end{aligned}$$

or

$$\ln \mathcal{P}_u \geq c - \int_{t_I}^{u} f(t)^{-2}\exp(-\alpha f(t)^2/2t)dt = c' - \int_0^u f(t)^{-2}\exp(-\alpha f(t)^2/2t)\, dt. \qquad \square$$

COMMENT. The value of α comes from (v). Replacing the 2's in the derivation of (v) by $1 + \delta$, δ small, and varying q allow one to improve the value of α. But the best value of α one could possibly hope for from our method would be

$$\begin{aligned}&\sup_q \lim \sup_{t=qs, s\to\infty}[1 - f(s/q)/f(s)]^2/(t/s - 1)\\ &\qquad \leq \sup_q (1 - q^{-1/2})^2/(q - 1) = \tfrac{1}{2}(5\sqrt{5} - 11) \approx .09,\end{aligned}$$

which is still substantially less than 1. The difficulty comes from the fact that in the proof of the lower bound, $|W(t_{n-1})|$ is close to $f(t_{n-2})$ with nonnegligible probability.

3. Random walk. Let X_i, $i = 1, 2, \cdots$ be iid, mean 0, variance 1 random variables with $E|X_1|^{2+2\eta} < \infty$ for some $\eta > 0$. Let $S(n) = \sum_{i=1}^{n} X_i$. Define $Q(N, x, k) = P^x(|S(n)| \leq k, n \leq N)$.

PROPOSITION 3.1. *Let $\frac{1}{2} > \varepsilon > 0$, $k \geq |x| \geq 0$. There exist constants M and N_0 such that if $N \geq N_0$, $(k - |x|)/N^{1/2} \geq M$, and $k \leq (2N \ln \ln N)^{1/2}$, then*

$$\begin{aligned}&1 - (N/(k - |x|)^2)\exp(-(1 - \varepsilon)(k - |x|)^2/2N)\\ &\qquad \leq Q(N, x, k) \leq 1 - (N/k^2)\exp(-(1 + \varepsilon)k^2/2N).\end{aligned}$$

PROOF. Suppose $x \geq 0$. The other case is similar.

Use Skorokhod imbedding to find a Brownian motion $W(t)$ and stopping times $U_1, \cdots, U_N$ such that $U_1, U_2 - U_1, \cdots, U_N - U_{N-1}$ are independent, and $(S(1), \cdots, S(N))$ is equal in law to $(W(U_1), \cdots, W(U_N))$, and $EU_1^{1+\eta} < \infty$ (see Skorokhod, 1965, for example).

If we let $U_0 = 0$, $Y_i = U_i - U_{i-1} - 1$, $i = 1, \cdots, N$, the Y_i's are iid random variables, $EY_1 = EU_1 - 1 = EW(U_1)^2 - 1 = EX_1^2 - 1 = 0$, and $EY_1^{1+\eta} < \infty$.

Then by Petrov (1975), page 283, for each δ, $P(|\sum_{i=1}^{N} Y_i|/N > \delta) = o(N^{-\eta})$. So for N sufficiently large, $P^0(|U_N - N| > \delta N) \le N^{-\eta}$.

$$
\begin{aligned}
P^x(|S(n)| \le k, n \le N) &= P^x(|W(U_n)| \le k, n \le N) \le P^x(|W(U_N)| \le k) \\
&\le P^x(|W(U_N)| \le k, |U_N - N| \\
&\quad \le \delta N, \sup_{|t-N|\le\delta N} |W(t) - W(N)| < \varepsilon k/3) \\
&\quad + P^X(\sup_{|t-N|\le\delta N} |W(t) - W(N)| \\
&\quad \ge \varepsilon k/3) + P^x(|U_N - N| > \delta N) \\
&\le P^x(|W(N)| < (1+\varepsilon/3)k) + 2P^0(\sup_{t\le\delta N} |W(t)| > \varepsilon k/3) \\
&\quad + P^0(|U_N - N| > \delta N).
\end{aligned}
$$

As in the proof of (2.1), the first term on the right is

$$
\begin{aligned}
&\le 1 - (N/((2\pi)^{1/2}(1+\varepsilon/3)^2 k^2))\exp(-(1+\varepsilon/3)^2 k^2/2N) \\
&\le 1 - (3N/k^2)\exp(-(1+\varepsilon)k^2/2N),
\end{aligned}
$$

if k^2/N is large enough. The second term is

$$
\begin{aligned}
2(1 - P^0(|W(s)| \le \varepsilon k/3, s \le \delta N)) &\le (72\, N/((2\pi)^{1/2}\varepsilon^2 k^2)))\exp(-(1-\varepsilon/3)\varepsilon^2 k^2/(18\delta N)) \\
&\le (N/k^2)\exp(-(1+\varepsilon)k^2/N)
\end{aligned}
$$

if we take $\delta \le \varepsilon^2/72$. Finally, since $k \le (2N \ln\ln N)^{1/2}$, $(N/k^2)\exp(-(1+\varepsilon)k^2/2N) \ge \exp(-2k^2/2N) \ge \exp(-2\ln\ln N) = (\ln N)^{-2}$, which is much larger than $N^{-\eta}$ if N and K^2/N are sufficiently large. So for N, k chosen appropriately, the third term will also be $\le (N/k^2)\exp(-(1+\varepsilon)k^2/N)$.

Summing, $P^x(|S(n)| \le k, n \le N) \le 1 - (N/k^2)\exp(-(1+\varepsilon)k^2/N)$. To get the other inequality

$$
\begin{aligned}
P^x(|W(t)| \le k, t \le (1+\varepsilon/3)N) &\le P^x(|W(t)| \le k, t \le U_N) + P^x(U_N > (1+\varepsilon/3)N) \\
&\le P^x(|W(U_n)| \le k, n \le N) + P^0(|U_N - N| > \varepsilon N/3),
\end{aligned}
$$

or

$$
\begin{aligned}
P^x(|S(n)| \le k, n \le N) &= P^x(|W(U_n)| \le k, n \le N) \\
&\ge P^x(|W(t)| \le k, t \le (1+\varepsilon/3)N) - P^0(|U_N - N| > \varepsilon N/3).
\end{aligned}
$$

The first term on the right, by (2.1), is

$$
\begin{aligned}
&\ge 1 - (4(1+\varepsilon/3)N/((2\pi)^{1/2}(k-x)^2))\exp(-(1+\varepsilon/3)(k-x)^2/2(1+\varepsilon/3)N) \\
&\ge 1 - (N/2(k-x)^2)\exp(-(1-\varepsilon)(k-x)^2/2N)
\end{aligned}
$$

if $(k-x)^2/N$ is large, while the second term is $\le N^{-\eta}$ for N large, which in turn is $\le (N/2(k-x)^2)\exp(-(1-\varepsilon)(k-x)^2/2N)$ by the upper bound on k, as above.

Hence for N large, $(k-x)/N^{1/2}$ large,

$$
Q(N, x, k) \ge 1 - (N/(k-x)^2)\exp(-(1-\varepsilon)(k-x)^2/2N).
$$

Proof of Theorem 2. Let $q = 9$ in the lower bound case, q a large integer in the upper bound case. Using (3.1) in place of (2.1), the proof is virtually identical to the proof of Theorem 1. □

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