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SETS WHICH DETERMINE THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM

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Rates of convergence in the central limit theorem are frequently described in terms of the uniform metric. However, statisticians often apply the central limit theorem only at symmetric pairs of isolated points, such as the 5% points of the standard normal distribution, ±1.645. In this paper we study rates of convergence on sets of the form $\{-\theta, \theta\}$, where $\theta \geq 0$. It is shown that the rate of convergence on the 5% points is the same as the rate uniformly on the whole real line, up to terms of order $n^{-1/2}$. Curiously, the rate of convergence on the 1% points ±2.326 can be faster than the rate on the whole real line.

1. Introduction. The probability literature contains a large body of very elegant mathematical theory which describes the rate of convergence in the central limit theorem. These results often involve a uniform measure of the rate of convergence; see for example the characterisations given by Ibragimov (1966) and Heyde (1967). Statisticians are sometimes rather skeptical of such theory, pointing out that it is disjoint from the more practical problems which they encounter. Frequently they are only interested in the rate of convergence at isolated points, such as the upper and lower 5% points of the standard normal distribution, ±1.645. Some of their queries can be presented in simplified form in the following question: "If I spend all my time making normal approximations at the points ±1.645, will the rate of convergence in my case be sometimes faster than that which is described by your limit theorems for the uniform metric?"

The literature contains very little material which can be used to solve this problem. If third moments are finite and non-zero, and if the underlying distribution is non-lattice, then an asymptotic expansion can be used to answer the statistician's question in the negative. See for example Theorem 3, page 541 of Feller (1971). However, in the more interesting case where third moments are infinite, the problem seems much deeper. Our aim in the present paper is to provide a more general solution to this question. In Section 2 we introduce the notion of sets which determine the rate of convergence to order $n^{-1/2}$, or to order n^{-1}. Then we examine symmetric doublets which are candidates for such sets. One corollary of the results in Section 3 is that if the summand distribution is either lattice or satisfies Cramér's continuity condition, then the rate of convergence on the 5% points ±1.645 is the same as the rate of convergence uniformly on the whole real line, up to terms of order n^{-1}. However, there exist distributions for which the rate of convergence on the 2½% points ±1.960, or on the 1% points ±2.236, is *faster* than the rate of convergence on the whole real line.

All the proofs are placed together in Section 4.

2. Rate-of-convergence determining sets. Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables with zero mean and unit variance, and define $S_n = \sum_{j=1}^n X_j$. The uniform measure of the rate of convergence in the central limit theorem is provided by

$$\Delta_n = \sup_{-\infty<x<\infty} | P(S_n \leq n^{1/2}x) - \Phi(x) |,$$

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where Φ is the standard normal distribution function. Define

$$\delta_n = E\{X^2 I(|X| > n^{1/2})\} + n^{-1}E\{X^4 I(|X| \le n^{1/2})\} + n^{-1/2}|E\{X^3 I(|X| \le n^{1/2})\}|.$$

It follows from results of Osipov (1968), Rozovskii (1978a, 1978b) and Hall (1980) that the sequence $(\Delta_n + n^{-1/2})/(\delta_n + n^{-1/2})$ is bounded away from zero and infinity as $n \to \infty$. That is, the uniform "error", Δ_n, is of precise order δ_n, up to terms of order $n^{-1/2}$. The set $\mathscr{S}$ will be said to be (rate of) convergence determining to order $n^{-1/2}$ if, no matter what standardised distribution is chosen for X, the sequence

$$\{\sup_{x\in\mathscr{S}}|P(S_n \le n^{1/2}x) - \Phi(x)| + n^{-1/2}\}/(\delta_n + n^{-1/2}), \qquad n \ge 1,$$

is bounded away from zero and infinity as $n \to \infty$. In other words, $\mathscr{S}$ is convergence determining to order $n^{-1/2}$ if the rate of convergence on $\mathscr{S}$ is the same as the rate of convergence on the whole real line, up to terms of order $n^{-1/2}$.

Now suppose the distribution of X satisfies Cramér's continuity condition,

$$\text{(C)} \qquad \limsup_{t\to\infty}|E(e^{itX})| < 1.$$

Then the ratio $(\Delta_n + n^{-1})/(\delta_n + n^{-1})$ is bounded away from zero and infinity as $n \to \infty$. (See Theorems 2.3, 2.4 and 4.6 of Hall 1982.) This result is false in the case of a lattice distribution, since it does not allow for rounding errors which occur in the approximation of a discrete distribution by the continuous normal distribution. The so-called "Yates' continuity correction" (see Yates, 1934, and Pearson, 1947, page 147) is designed to account for such errors. If X is lattice with maximal span b, and if the real number a satisfies $P(X \in a + b\mathbb{Z}) = 1$, then we should replace Δ_n by

$$\Delta_n' = \sup_{-\infty<x<\infty}|P(S_n \le n^{1/2}x) - \Phi(x) - n^{-1/2}bA_n(x)\phi(x)|,$$

where $A_n(x) = [(n^{1/2}x - na)/b] - (n^{1/2}x - na)/b + \tfrac{1}{2}$, $\phi = \Phi'$ and $[z]$ denotes the integer part of z. In this case, the ratio $(\Delta_n' + n^{-1})/(\delta_n + n^{-1})$ is bounded away from zero and infinity as $n \to \infty$. (See Theorems 2.3, 2.4 and 4.6 of Hall, 1982.) These properties suggest the following definition. The set $\mathscr{S}$ is (rate of) convergence determining to order n^{-1} if, whenever the distribution of X satisfies Cramér's condition (C), the sequence

$$\{\sup_{x\in\mathscr{S}}|P(S_n \le n^{1/2}x) - \Phi(x)| + n^{-1}\}/(\delta_n + n^{-1})$$

is bounded away from zero and infinity as $n \to \infty$, and whenever the distribution of X is lattice with maximal span b and satisfies $P(X \in a + b\mathbb{Z}) = 1$, the sequence

$$\{\sup_{x\in\mathscr{S}}|P(S_n \le n^{1/2}x) - \Phi(x) - n^{-1/2}bA_n(x)\phi(x)| + n^{-1}\}/(\delta_n + n^{-1})$$

is bounded away from zero and infinity.

The notion of a convergence determining set to order n^{-1} may be slightly generalised, by incorporating the (modified) first term of a Chebyshev-Edgeworth-Cramér expansion into the error estimate. That is, the function $\Phi(x)$ should be replaced by

$$\Phi_n(x; \tau) \equiv \Phi(x) + (\tau/6n^{1/2})(1 - x^2)\phi(x),$$

where τ is an arbitrary real number. Provided Φ is changed to Φ_n and the term $n^{-1/2}\,|E\{X^3 I(|X| \le n^{1/2})\}|$ appearing in δ_n is replaced by $n^{-1/2}|E\{X^3 I(|X| \le n^{1/2})\} - \tau|$, the argument in the previous paragraphs may be conducted exactly as before. The Theorem in Section 3 continues to hold for this generalised definition of "convergence determining to order n^{-1}".

Some preliminary results concerning convergence determining sets were obtained in Hall (1982). For example, it was shown there that any set containing an infinite nonzero sequence converging to zero, is convergence determining to orders $n^{-1/2}$ and n^{-1}. In the present paper we shall examine convergence determining sets consisting of symmetric doublets. This enables us to answer the question posed in Section 1.

3. Results. Consider the equation

$$ {}_1F_1[-\tfrac{1}{2};\tfrac{3}{2};\theta_0^2/2] = 0, \tag{3.1}$$

where ${}_1F_1[a; b; \cdot]$ denotes Kummer's series for the confluent hypergeometric function; see Slater (1960). It is shown in Section 4 that equation (3.1) has a unique positive solution given by $\theta_0 = 2.1241\ldots.$

The following theorem describes the convergence determining properties of symmetric doublets.

THEOREM. *If $0 < \theta < \sqrt{3} \simeq 1.7321$, and if $\theta \neq 1$, then the pair $\{-\theta, \theta\}$ is convergence determining to orders $n^{-1/2}$ and n^{-1}. If $\theta = 0, 1$ or $\sqrt{3}$, or if $\theta \geq \theta_0$, then the pair $\{-\theta, \theta\}$ is not convergence determining to order $n^{-1/2}$. If $\theta = 0$ or 1, or if $\theta \geq \sqrt{3}$, then the pair $\{-\theta, \theta\}$ is not convergence determining to order n^{-1}.*

The 5% points ± 1.645 are convergence determining to order $n^{-1/2}$, but the 1% points ± 2.326 are not. The convergence determining nature to order $n^{-1/2}$ of the 2½% points ± 1.960, remains undetermined. However, the 2½% points are not convergence determining to order n^{-1}.

4. Proofs. Define the function

$$L_n(x) = nE\{\Phi(x - X/n^{1/2}) - \Phi(x)\} - \tfrac{1}{2}\phi'(x), \qquad -\infty < x < \infty.$$

The following lemma may be deduced from Theorems 2.2 and 4.6 of Hall (1982).

LEMMA. *In the notation introduced in Section 2, we have*

$$\sup_{-\infty<x<\infty} | P(S_n \leq n^{1/2}x) - \Phi(x) - L_n(x) | = O(\delta_n^2 + n^{-1/2})$$

as $n \to \infty$. If the underlying distribution satisfies Cramér's condition (C) *then the term $n^{-1/2}$ on the right hand side may be replaced by n^{-1}, and if the distribution is lattice with maximal span b and satisfies $P(X \in a + b\mathbb{Z}) = 1$,* then

$$\sup_{-\infty<x<\infty} | P(S_n \leq n^{1/2}x) - \Phi(x) - L_n(x) - n^{-1/2}bA_n(x)\phi(x) | = O(\delta_n^2 + n^{-1})$$

as $n \to \infty$.

It follows from the lemma that the convergence determining properties of a set $\mathcal{S}$ depend on the behaviour of $L_n(x)$ on $\mathcal{S}$. It is easily proved (see Theorem 2.3 of Hall, 1982) that

$$\sup_{-\infty<x<\infty} | L_n(x) | = O(\delta_n),$$

and so the set $\mathcal{S}$ is convergence determining to orders $n^{-1/2}$ and n^{-1} if

$$\liminf_{n\to\infty} \{\sup_{x\in\mathcal{S}} | L_n(x) |\}/\delta_n > 0,$$

for all standardised choices of the distribution of X. Conversely, if there exists an absolutely continuous distribution such that

$$\{\sup_{x\in\mathcal{S}} | L_n(x) | + n^{-j/2}\}/(\delta_n + n^{-j/2}) \to 0$$

as $n \to \infty$, then the set $\mathcal{S}$ is not convergence determing to order $n^{-j/2}$ ($j = 1, 2$).

The remainder of our proof is divided into three parts. First we prove that any set of the form $\{-\theta, \theta\}$, where $0 < \theta < 1$ or $1 < \theta < \sqrt{3}$, is convergence determining to orders $n^{-1/2}$ and n^{-1}. Then we show that if $\theta > \theta_0$, the set $\{-\theta, \theta\}$ is not convergence determining to order $n^{-1/2}$, and if $\theta > \sqrt{3}$, the set $\{-\theta, \theta\}$ is not convergence determining to order n^{-1}. Finally, we prove that the sets $\{0\}$, $\{-1, 1\}$ and $\{-\sqrt{3}, \sqrt{3}\}$ are not convergence determining. The symbol C denotes a generic positive constant.

Part (*i*). Define the function $f(x) = f(x;\, \theta) \equiv \Phi(\theta - x) + \Phi(\theta + x) - 2\Phi(\theta) - x^2\phi'(\theta)$, where $\theta > 0$ is fixed and $-\infty < x < \infty$. We shall prove that if $0 < \theta \le \sqrt{3}$ then $f(x) > 0$ for all $x \neq 0$. Since f is even, and $f(0) = 0$, it will suffice to show that $f'(x) > 0$ for $x > 0$. It is easily checked that $m!6^m \le (2m+1)!$ for $0 \le m \le 4$, and strict inequality obviously holds for $m \ge 5$. Consequently

$$e^{u^2/6} - (e^u - e^{-u})/2u = \textstyle\sum_{m=0}^{\infty} u^{2m}\{1/m!6^m - 1/(2m+1)!\} > 0$$

for all $u > 0$. Therefore $e^{u^2/2\theta^2} - (e^u - e^{-u})/2u > 0$ for all $u > 0$ and all $0 < \theta \le \sqrt{3}$. Setting $u = x\theta$ we see that $2x\theta e^{x^2/2} + e^{-x\theta} - e^{x\theta} > 0$ for $x > 0$. But $f'(x) = \phi(\theta)e^{-x^2/2}(2x\theta e^{x^2/2} + e^{-x\theta} - e^{x\theta})$, and so $f'(x) > 0$, as had to be proved.

Observe that as $x \to 0$,

$$f(x) = \tfrac{1}{12}\, x^4\phi^{(3)}(\theta) + \tfrac{1}{360}\, x^6\phi^{(5)}(\theta) + o(x^6), \tag{4.1}$$

while as $x \to \infty$, $f(x) \sim x^2\theta\phi(\theta)$. Therefore if $0 < \theta < \sqrt{3}$, there exist positive constants $C_1(\theta)$ and $C_2(\theta)$ such that

$$C_1x^2\min(1, x^2) \le f(x) \le C_2x^2\min(1, x^2)$$

for all x. Consequently

$$\begin{aligned} L_n(\theta) - L_n(-\theta) &= nE\{\Phi(\theta - X/n^{1/2}) + \Phi(\theta + X/n^{1/2}) - 2\Phi(\theta) - (X/n^{1/2})^2\phi'(\theta)\} \\ &\ge C_1[E\{X^2I(|X| > n^{1/2})\} + n^{-1}E\{X^4I(|X| \le n^{1/2})\}]. \end{aligned} \tag{4.2}$$

Furthermore,

$$L_n(\theta) + L_n(-\theta) = nE\{\Phi(\theta - X/n^{1/2}) - \Phi(\theta + X/n^{1/2}) + 2(X/n^{1/2})\phi(\theta)\},$$

and the function $g(x) \equiv \Phi(\theta - x) - \Phi(\theta + x) + 2x\phi(\theta)$ satisfies $g(x) = -\tfrac{1}{3}\, x^3\phi''(\theta) + O(x^5)$ as $x \to 0$. Consequently

$$|g(x) + \tfrac{1}{3}x^3\phi''(\theta)| \le C_3x^4$$

whenever $0 < x \le 1$, and obviously $|g(x)| \le C_4x^2$ for $x > 1$. Therefore

$$\begin{aligned} |\tfrac{1}{3}n^{-1/2}E\{X^3I(|X| \le n^{1/2})\}\phi''(\theta)| &\le C_3n^{-1}E\{X^4I(|X| \le n^{1/2})\} + C_4E\{X^2I(|X| > n^{1/2})\} \\ &+ |L_n(\theta) + L_n(-\theta)| \le C_1^{-1}\max(C_3, C_4)|L_n(\theta) - L_n(-\theta)| + |L_n(-\theta) + L_n(\theta)|. \end{aligned}$$

From this result and (4.2) it follows that if $0 < \theta < \sqrt{3}$ and $\theta \neq 1$ then $\sup_{x\in\{-\theta,\theta\}}|L_n(x)| > C_5\delta_n$, and so the set $\{-\theta, \theta\}$ is convergence determining.

Part (*ii*). We shall first prove that for each $\theta > \theta_0$, there exists a distribution $X = X(\theta)$ such that $n^{1/2}\Delta_n \to \infty$ as $n \to \infty$ but

$$\sup_{x\in\{-\theta,\theta\}} |P(S_n \le n^{1/2}x) - \Phi(x)|/\Delta_n \to 0. \tag{4.3}$$

Let X have unit variance and be absolutely continuous and symmetric with regularly varying tails of order α, $2 < \alpha < 3$. Thus,

$$P(|X| > x) \sim x^{-\alpha}L(x), \tag{4.4}$$

where L is slowly varying at infinity. It may be deduced from results of Höglund (1970), or from Theorem 4.10 of Hall (1982), that in this special case,

$$L_n(x) = nP(|X| > n^{1/2})\xi(x) + r_n(x) \tag{4.5}$$

as $n \to \infty$, where

$$\xi(x) = \{2(\alpha - 1)(\alpha - 2)\}^{-1}\int_0^{\infty} u^{2-\alpha}\{\phi''(x + u) - \phi''(x - u)\}\, du \tag{4.6}$$

and $\sup_{-\infty<x<\infty}|r_n(x)| = o\{nP(|X| > n^{1/2})\}$ as $n \to \infty$. Standard techniques in the theory of regular variation (see Theorem 1, page 281 of Feller, 1971, and also Seneta, 1976) enable us to prove that $\delta_n \sim CnP(|X| > n^{1/2})$ for a positive constant C. Therefore if $x(>0)$ is a solution of the equation

$$\int_0^\infty u^{2-\alpha}\{\phi''(x+u) - \phi''(x-u)\}\, du = 0, \tag{4.7}$$

then the set $\{-x, x\}$ will not be convergence determining.

Equation (4.7) can be written as

$$(d/dx)^2[e^{-x^2/4}\{D_{\alpha-3}(x) - D_{\alpha-3}(-x)\}] = 0,$$

where $D_{\alpha-3}$ is Whittaker's parabolic cylinder function. Using the differentiation formula on page 327 of Magnus, Oberhettinger and Soni (1966), we see that (4.7) is equivalent to

$$D_{\alpha-1}(x) - D_{\alpha-1}(-x) = 0. \tag{4.8}$$

In view of the relationship between the confluent hypergeometric function and parabolic cylinder functions (see pages 686 and 687 of Abramowitz and Stegun, 1965), equation (4.8) is equivalent to

$${}_1F_1[1 - \alpha/2; 3/2; x^2/2] = 0. \tag{4.9}$$

For $2 < \alpha < 4$, this equation has a unique positive solution; see page 103 of Slater (1960). The solution diverges to $+\infty$ as $\alpha \downarrow 2$, since the function on the left in (4.9) converges to the constant 1. In the special cases $\alpha = 3$ and $\alpha = 4$, the solutions are $(2 \times 2.2559296)^{1/2} = 2.12411$ and $\sqrt{3}$, respectively (see Appendix I of Slater, 1960). Since the hypergeometric function is continuous in its parameters and argument, the following proposition holds: for each $\theta \ge \theta_0$ there exists $\alpha = \alpha(\theta) \in (2, 3]$ such that (4.9) holds with $x = \theta$, and for each $\theta > \sqrt{3}$ there exists $\alpha \in (2, 4)$ such that (4.9) holds with $x = \theta$. From this follows (4.3). Result (4.5) easily extends to the case $2 < \alpha < 4$, and so (4.3) also holds for $\theta > \sqrt{3}$.

Part (iii). The singleton $\{0\}$ is obviously not convergence determining. (Consider the case where X has a smooth, symmetric distribution.) We shall prove next that the set $\{-1, 1\}$ is not convergence determining.

Let Y be a random variable with $P(Y \ge 0) = 1$, $E(Y) = \mu$, $\mathrm{Var}(Y) = 1$ and $P(Y > x) = x^{-3}$ for large x. Set $X = Y - \mu$. It can be shown that in this case,

$$\delta_n = n^{-1/2}|E\{Y^3 I(Y \le n^{1/2})\}| + O(n^{-1/2}) \sim (3/2)n^{-1/2}\log n \tag{4.10}$$

as $n \to \infty$. A short Taylor expansion shows that

$$L_n(1) = nE[\{-1/6(X/n^{1/2})^3\phi''(1) + 1/24(X/n^{1/2})^4\phi'''(Y_n)\}I(|X| \le n^{1/2})]$$
$$+ nE[\{\Phi(1 - X/n^{1/2}) - \Phi(1) + (X/n^{1/2})\phi(1) - 1/2(X/n^{1/2})^2\phi'(1)\}I(|X| > n^{1/2})],$$

where the random variable Y_n takes only values between 0 and 2. Consequently

$$|L_n(1)| \le C[n^{-1}E\{X^4 I(|X| \le n^{1/2})\} + E\{X^2 I(|X| > n^{1/2})\}] = O(n^{-1/2})$$

as $n \to \infty$, and a similar inequality holds for $|L_n(-1)|$. When these bounds are compared with (4.10) we see that the rate of convergence on $\{-1, 1\}$ is faster than that on the whole real line, and so $\{-1, 1\}$ is not convergence determining.

Next we examine the set $\{-\sqrt{3}, \sqrt{3}\}$. If we let $x = \sqrt{3}$ in (4.1), and then conduct the argument of Part (i) as before, we may deduce that for positive constants C_1 and C_2 we have for any symmetric X,

$$\begin{aligned} &C_1[E\{X^2 I(|X| > n^{1/2})\} + n^{-2}E\{X^6 I(|X| \le n^{1/2})\}] \\ &\quad \le \sup_{x \in \{-\sqrt{3}, \sqrt{3}\}}|L_n(x)| \le C_2[E\{X^2 I(|X| > n^{1/2})\} + n^{-2}E\{X^6 I(|X| \le n^{1/2})\}]. \end{aligned} \tag{4.11}$$

We shall construct an example in which, along a subsequence diverging to infinity, the terms within square brackets in (4.11) are of order $n^{-1/2}$ but $\delta_n \sim C(n^{-1}\log n)^{1/2}$ for a positive constant C. From this it follows that the set $\{-\sqrt{3}, \sqrt{3}\}$ is not convergence determining.

Set $a_n = 2^{(n+1)^2}$ and $p_n = 2^{(2n+1)^2}$, and let Y be an absolutely continuous, nonnegative random variable with density

$$f(y) = \begin{cases} p_n y^{-5} & \text{for } a_{2n-1} < y \le a_{2n} \\ 0 & \text{for } a_{2n} < y \le a_{2n+1}, \end{cases}$$

for large n. Observe that

$$\int_{a_{2n-1}}^{a_{2n+1}} y^{2i} f(y)\,dy \sim r_i(n) \equiv \begin{cases} 2^{-4n^2+4n} & \text{if } i = 1 \\ (8\log 2)n2^{4n^2+4n} & \text{if } i = 2 \\ 2^{2(2n+1)^2+4n^2+4n} & \text{if } i = 3. \end{cases}$$

Consequently $\int_{a_{2n}}^{\infty} y^2 f(y)\,dy \sim r_1(n+1)$, and

$$\int_0^{a_{2n}} y^{2i} f(y)\,dy \sim \begin{cases} r_2(n) & \text{if } i = 2 \\ r_3(n) & \text{if } i = 3. \end{cases}$$

From these results we may deduce that if $x(n)$ is a sequence of constants satisfying $x = x(n) > a_{2n}$ and $x \sim a_{2n}$ as $n \to \infty$, then

$$E\{Y^2 I(Y > x)\} \sim 2^{-4(n^2+n)}, \qquad x^{-2}E\{Y^4 I(Y \le x)\} \tag{4.12}$$
$$\sim (2\log 2)n2^{-4(n^2+n)} \quad \text{and} \quad x^{-4}E\{Y^6 I(Y \le x)\} \sim 2^{-4(n^2+n)-2}.$$

Set $c^2 = E(Y^2)$, and let X be a symmetric random variable such that $|X|$ has the distribution of Y/c. Let $n = n(m)$ be a sequence of positive integers diverging to infinity and such that $cn^{1/2}(m) > a_{2m}$ and $cn^{1/2}(m) \sim a_{2m}$ as $m \to \infty$. Then along this subsequence,

$$\delta_{n(m)} = c^{-2}E\{Y^2 I(Y > cn^{1/2})\} + c^{-2}(c^2 n)^{-1}E\{Y^4 I(Y \le cn^{1/2})\}$$
$$\sim c^{-2}(2\log 2)m2^{-4(m^2+m)} \sim 2^{1/2}c^{-3}(\log 2)(n^{-1}\log n)^{1/2},$$

using (4.12). However, (4.12) implies that

$$E\{X^2 I(|X| > n^{1/2})\} + n^{-2}E\{X^6 I(|X| \le n^{1/2})\} = O(n^{-1/2})$$

as $n \to \infty$. This produces the counter-example described in the previous paragraph.

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