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ESTIMATE ON MOMENTS OF THE SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS IN THE PLANE[1]

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Let $(\Omega, \mathscr{F}, P; \mathscr{F}_{s,t})$ be a probability space with a family of sub-σ-algebras indexed by $(s, t) \in [0, \infty) \times [0, \infty)$, satisfying the usual conditions. Let $X(s, t)$ be a solution of a stochastic differential equation in the plane with respect to the Wiener-Yeh process. Under one of the usual conditions used to guarantee existence and uniqueness of a solution to the equation, it is shown that the absolute moments of $X(s, t)$ grow at most exponentially in st. The estimate is based on a version of the two parameter Ito formula and on an extension of Gronwall's inequality to functions of two variables.

1. Introduction and notation. In this paper we are concerned with the two-parameter stochastic differential equation

$$dX(s, t) = e(s, t, X)\, dB(s, t) + f(s, t, X)\, dm(s, t) \tag{1.1}$$

where $B(s, t)$, $(s, t) \in D \equiv [0, \infty) \times [0, \infty)$ is the two-parameter Wiener-Yeh process, and m is Lebesgue measure on D. Conditions on the coefficients e and f that assure existence and uniqueness of a solution to (1.1) are known and will be reviewed below. Our purpose is to derive an a priori estimate on the moments of such a solution. In fact, we show that for any $n \geq 4$ there is a constant $C > 0$ such that the nth absolute moment of $X(s, t)$ grows at most on the order of $\exp(Cst)$. The constant C depends only on n and on the parameters appearing in the conditions on e and f, i.e., C does not depend on the solution $X(s, t)$ itself.

The probabilistic apparatus is as follows. All processes have indices in D, which is given the partial ordering $(s, t) \leq (s', t')$ iff $s \leq s'$ and $t \leq t'$. The probability space $(\Omega, \mathscr{F}, P)$ is complete and $\{\mathscr{F}_{s,t} \mid (s, t) \in D\}$ is a system of sub-σ-algebras of $\mathscr{F}$ satisfying the usual axioms as introduced in Cairoli and Walsh [1]. The latter paper also contains the definitions of martingale and weak martingale that we use.

In Yeh [6], stochastic differential equations are solved in the above setting; we outline the results here. The Brownian motion $B(s, t)$ can be assumed to be adapted to $\mathscr{F}_{s,t}$ and to have the property that $B(R)$ is independent of $\mathscr{F}_{s,t}$ where R is any rectangle in D disjoint from $(0, s] \times (0, t]$. Let W be the space of continuous real-valued functions on D. Let $\mathscr{B}(W)$ be the σ-algebra on W generated by sets of the form $\{w \in W \mid w(s, t) \in E\}$, for some $(s, t) \in D$ and $E \in \mathscr{B}(R)$. Let $\mathscr{B}_{s,t}(W)$ be the σ-algebra on W generated by sets of the form $\{w \in W \mid w(u, v) \in E\}$ for some $(u, v) \leq (s, t)$ and $E \in \mathscr{B}(R)$. The coefficients e and f are assumed to satisfy

(a) e is a measurable map from $(D \times W, \mathscr{B}(D) \times \mathscr{B}(W))$ into $(R, \mathscr{B}(R))$;
(b) For every $(s, t) \in D$, $e(s, t, \cdot)$ is a measurable map from $(W, \mathscr{B}_{s,t}(W))$ into $(R, \mathscr{B}(R))$,

and similarly for f. Then by a solution to (1.1) we mean a continuous process $X(s, t)$ such that

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(a) $E[\int_{[0,T]\times[0,T]} |e(s, t, X)|^2 \, dm(s, t)] < \infty$, and

$$E\left[\int_{[0,T]\times[0,T]} |f(s, t, X)| \, dm(s, t)\right] < \infty \quad \text{for every} \quad T > 0 \tag{1.2}$$

(b) for all $(s, t) \in D$, $X(s, t) - X(s, 0) - X(0, t) + X(0, 0) = \int_{[0,s]\times[0,t]} e(u, v, X) \, dB(u, v) + \int_{[0,s]\times[0,t]} f(u, v, X) \, dm(u, v)$.

The stochastic integral in (1.2b) can be defined as a continuous, square-integrable martingale because of the integrability condition on e in (1.2a). Sufficient conditions on e and f are given for existence and uniqueness of a solution to (1.1). One of these is the following growth condition: there is a Borel measure λ on D such that for every $T > 0$ there exists a constant $L_T > 0$ such that

$$e(s, t, w)^2 + f(s, t, w)^2 \leq L_T\left(1 + w(s, t)^2 + \int_{[0,s]\times[0,t]} w(u, v)^2 \, d\lambda(u, v)\right) \tag{1.3}$$

for all $(s, t) \leq (T, T)$ and $w \in W$. The measure λ gives finite mass to finite rectangles.

After some preliminary results we prove our main estimate in Section 4: if X is a solution to (1.1) where e and f satisfy (1.3), and if X is constant on the axes, then for $n = 2$ or $n \geq 4$, and for every $T > 0$ there is a constant $C_T > 0$, depending only on n, L_T, and $K_T \equiv \lambda([0, T] \times [0, T])$ such that

$$E(|X(s, t)|^n) \leq (1 + E(|X(0, 0)|^n))\exp(C_T st) - 1 \tag{1.4}$$

for every $(s, t) \leq (T, T)$.

We use the following notation throughout. The rectangle $[0, s] \times [0, t] \subseteq D$ is denoted by $R_{s,t}$. If $f(s, t)$ is any function of two variables, and if $R = [s, s'] \times [t, t']$ is any rectangle in D, then $f(R) = f(s', t') - f(s', t) - f(s, t') + f(s, t)$.

2. Ito's formula. Let $L_\infty(\mathscr{F}_{s,t})$ denote the class of two-parameter adapted, measurable processes which are uniformly bounded. In [4] we prove the following theorem.

THEOREM 2.1. *Let $a, b \in L_\infty(\mathscr{F}_{s,t})$ and denote*

$$M(s, t) = \int_{[0,s]\times[0,t]} a(u, v) \, dB(u, v), \quad J(s, t) = \int_{[0,s]\times[0,t]} a(u, v)^2 \, dm(u, v),$$

and

$$L(s, t) = \int_{[0,s]\times[0,t]} b(u, v) \, dm(u, v),$$

for $(s, t) \in D$. Let $Y(s, t)$ be an adapted, continuous process satisfying

$$Y(s, t) - Y(0, 0) = M(s, t) + L(s, t), \tag{2.1}$$

for $(s, t) \in D$, with $Y(0, 0)$ bounded. Let $g \in C^4(\mathbb{R}^1)$ and suppose there exist constants $A, B > 0$ such that $|g^{(k)}(x)| \leq A \exp(B|x|)$ for $x \in \mathbb{R}^1$ and $k = 0, 1, 2, 3, 4$. Then

$$g(Y(s, t)) - \int_{[0,s]\times[0,t]} T_{u,v} g(Y(u, v)) \, dm(u, v) \tag{2.2}$$

is a weak $\mathscr{F}_{s,t}$-martingale, where the operators $T_{s,t}$, $(s, t) \in D$, are defined by

$$\begin{aligned} T_{s,t} g(x) = {} & L_{st}(s, t) g'(x) + (L_s L_t + \tfrac{1}{2} J_{st})(s, t) g''(x) \\ & + \tfrac{1}{2}(L_s J_t + L_t J_s)(s, t) g^{(3)}(x) \\ & + \tfrac{1}{4} J_s J_t(s, t) g^{(4)}(x). \end{aligned} \tag{2.3}$$

EXAMPLE. If $a \equiv 1$, $b \equiv 0$, and $Y(s, t) = B(s, t)$, then

$$g(B(s, t)) - \int_{[0,s]\times[0,t]} \left(\frac{1}{2} g''(B(u, v)) + \frac{uv}{4} g^{(4)}(B(u, v))\right) dm(u, v)$$

is a weak martingale for any g of exponential growth.

The fact that an expression of the form (2.2) is a weak martingale can be inferred from recent work of Guyon and Prum [2]; indeed, the weak martingale is identified as the sum of stochastic integrals of a and b. However, certain integrability conditions on $g^{(i)}(Y(s, t))$ are assumed there; these are replaced in our result by the growth condition on g. Theorem 2.1 is obtained using a "martingale approach" adapted from Stroock and Varadhan [5] which we intend to treat in another paper. Another result proved in [4] by similar techniques and which we shall need in Section 4 is the following.

THEOREM 2.2. *Let $a(s, t) \in L_\infty(\mathscr{F}_{s,t})$ and let*

$$M_{s,t} = \sup_{(u,v)\le(s,t)} \sup_{\omega\in\Omega} |a(u, v, \omega)|$$

for $(s, t) \in D$. Then

$$E\left[\exp\left(\int_{[0,s]\times[0,t]} a(u, v)\, dB(u, v)\right)\right] \le 2 \exp\left(\frac{st}{2} M_{s,t}^2\right).$$

3. An extension of Gronwall's inequality. One of the key ingredients in the derivation of the one-parameter analogue of the estimate (1.4) is Gronwall's inequality (see, for example, [3], Theorem 4.6). Our goal in this section is to obtain a two-parameter version of this inequality.

THEOREM 3.1. *Let $(S, T) \in D$ and let f and g be bounded, measurable functions on $[0, S] \times [0, T]$. Suppose there exists a constant $C > 0$ such that*

$$f(s, t) \le g(s, t) + C \int_{[0,s]\times[0,t]} f(u, v)\, dm(u, v), \tag{3.1}$$

for all $(s, t) \le (S, T)$. Then

$$f(s, t) \le g(s, t) + C \int_{[0,s]\times[0,t]} g(u, v) J(C(t-v)(s-u))\, dm(u, v), \tag{3.2}$$

for all $(s, t) \le (S, T)$, where

$$J(x) = \sum_{j=0}^{\infty} \frac{x^j}{(j!)^2}, \quad x \in R. \tag{3.3}$$

PROOF. We write $du\, dv$ for $dm(u, v)$. First, using estimate (3.1) for the integrand in (3.1) we obtain

$$\begin{aligned} f(s, t) \le{}& g(s, t) + C \int_0^s \int_0^t g(u, v)\, du\, dv \\ &+ C^2 \int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} f(u_2, v_2)\, du_2\, dv_2\, du_1\, dv_1, \end{aligned} \tag{3.4}$$

for $(s, t) \le (S, T)$. We shall show by induction that for all $n = 1, 2, 3, \cdots$ we have

$$f(s, t) \leq g(s, t) + C \int_0^s \int_0^t g(u, v) \sum_{k=0}^{n-1} \frac{(C(t-v)(s-u))^k}{(k!)^2} du\, dv$$

$$(3.5) \qquad + C^{n+1} \int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} \cdots \int_0^{u_n} \int_0^{v_n} f(u_{n+1}, v_{n+1})$$

$$\cdot\, du_{n+1}\, dv_{n+1} \cdots du_1\, dv_1, \quad \text{for} \quad (s, t) \leq (S, T).$$

Indeed, for $n = 1$ (3.5) is just (3.4). Inductively, suppose that (3.5) holds for n, and let I_1 denote the third term on the right side of (3.5). Using (3.1) for $f(u_{n+1}, v_{n+1})$ we get

$$I_1 \leq C^{n+1} \int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} \cdots \int_0^{u_n} \int_0^{v_n} g(u_{n+1}, v_{n+1})\, du_{n+1}\, dv_{n+1} \cdots du_1\, dv_1$$

$$(3.6) \qquad + C^{n+2} \int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} \cdots \int_0^{u_{n+1}} \int_0^{v_{n+1}} f(u_{n+2}, v_{n+2})$$

$$\cdot\, du_{n+2}\, dv_{n+2} \cdots du_1\, dv_1$$

$$\equiv I_2 + I_3, \quad \text{for} \quad (s, t) \leq (S, T).$$

Next, upon interchanging order of integration (justified by the boundedness of g), we find

$$C^{n+1} \int_0^s \int_0^t g(u_{n+1}, v_{n+1}) \left(\int_{u_{n+1}}^s \int_{v_{n+1}}^t \int_{u_n}^s \int_{v_n}^t \cdots \int_{u_2}^s \int_{v_2}^t du_1\, dv_1 \cdots du_n\, dv_n \right) du_{n+1}\, dv_{n+1}$$

$$(3.7) \qquad = C^{n+1} \int_0^s \int_0^t g(u_{n+1}, v_{n+1}) \frac{((s - u_{n+1})(t - v_{n+1}))^n}{(n!)^2} du_{n+1}\, dv_{n+1}$$

$$= C \int_0^s \int_0^t g(u, v) \frac{(C(s-u)(t-v))^n}{(n!)^2} du\, dv, \quad (s, t) \leq (S, T).$$

Using (3.6) and (3.7) in (3.5) we have the statement that (3.5) holds for $n + 1$, and this completes the induction proof.

Next we note that

$$I_3 \leq C^{n+2} \sup_{(u,v) \leq (S,T)} f(u, v) \int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} \cdots \int_0^{u_{n+1}} \int_0^{v_{n+1}} du_{n+2}\, dv_{n+2} \cdots du_1\, dv_1$$

$$= C^{n+2} \sup_{(u,v) \leq (S,T)} f(u, v) \frac{(st)^{n+2}}{((n+2)!)^2}, \quad (s, t) \leq (S, T).$$

Using this fact, and (3.3) in (3.5) (with n replaced by $n + 1$) we obtain

$$(3.8) \qquad f(s, t) \leq g(s, t) + C \int_0^s \int_0^t g(u, v) J(C(t-v)(s-u))\, du\, dv$$

$$+ \sup_{(u,v) \leq (S,T)} f(u, v) \frac{(Cst)^{n+2}}{((n+2)!)^2}, \quad (s, t) \leq (S, T)$$

for all $n = 1, 2, 3, \cdots$. But $f(u, v)$ is bounded on $[0, S] \times [0, T]$, so

$$\sup_{(u,v)\leq(S,T)} f(u, v) \frac{(Cst)^{n+2}}{((n+2)!)^2} \to 0 \quad \text{as} \quad n \to \infty$$

for all $(s, t) \leq (S, T)$. Thus (3.8) implies (3.2) and we are done. □

Corollary 3.2. *Let $h(s,t)$ be a non-negative continuous function on $R_{S,T}$, and suppose there exist $a \in R$ and $C \geq 1$ such that*

$$h(s, t) \leq a + C\left(st + \frac{s^2t^2}{4}\right) + C \int_0^s \int_0^t (1 + uv) \sup_{(u',v')\leq(u,v)} h(u', v')\, dm(u, v), \tag{3.9}$$

for all $(s, t) \leq (S, T)$. Then

$$h(s, t) \leq (1 + a)e^{2Cst} - 1, \quad (s, t) \leq (S, T). \tag{3.10}$$

Proof. Denote $dm(u, v) = du\, dv$,

$$f(s, t) = \sup_{(u,v)\leq(s,t)} h(u, v),$$

and

$$g(s, t) = a + C\left(st + \frac{s^2t^2}{4}\right)$$

so that (3.9) becomes

$$h(s, t) \leq g(s, t) + C \int_0^s \int_0^t (1 + uv) f(u, v)\, du\, dv, \quad (s, t) \leq (S, T).$$

(Note that f is continuous and non-negative since h is.) Since $g(s, t)$ and $\int_0^s \int_0^t (1 + uv) f(u, v)\, du\, dv$ are both increasing functions of (s, t) we actually have

$$f(s, t) \leq g(s, t) + C \int_0^s \int_0^t (1 + uv) f(u, v)\, du\, dv, \quad (s, t) \leq (S, T).$$

In the latter integral we shall make the change of variables $U(u, v) = (u + u^2v/2, v)$. We note that U is a one-to-one continuously differentiable transformation, with Jacobian $1 + uv$. The continuous inverse transformation is given by

$$U^{-1}(x, y) = \begin{cases} \left(\dfrac{-1 + \sqrt{1 + 2xy}}{y}, y\right), & y \neq 0 \\ (x, y), & y = 0. \end{cases} \tag{3.11}$$

Thus we have

$$f(s, t) \leq g(s, t) + C \iint_{U(R_{s,t})} f(U^{-1}(x, y))\, dx\, dy, \quad \text{for} \quad (s, t) \leq (S, T). \tag{3.12}$$

We now define, for $(x, y) \leq (S + \frac{1}{2}S^2T, T)$

$$\hat{f}(x, y) = \begin{cases} f(U^{-1}(x, y)), & (x, y) \in U(R_{S,T}) \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\hat{g}(x, y) = \begin{cases} g(U^{-1}(x, y)), & (x, y) \in U(R_{S,T}) \\ 0, & \text{otherwise}. \end{cases}$$

Note that for $(s, t) \leq (S, T)$ we have

$$U(R_{s,t}) \subseteq U(R_{S,T}),$$

and

$$U(R_{s,t}) \subseteq \left[0, s + \frac{s^2 t}{2}\right] \times [0, t].$$

Thus for $(s', t') \leq (S + \tfrac{1}{2}S^2T, T)$

$$\hat{f}(s', t') \leq g(s', t') + \iint_{U(R_{s,t})} \hat{f}(x, y)\, dx\, dy \tag{3.13}$$

where $s' = s + s^2t/2$ and $t' = t$. Indeed, if $(s', t') \in U(R_{S,T})$ then (3.13) is just a restatement of (3.12) and if $(s', t') \notin U(R_{S,T})$ then $\hat{f}(s', t') = 0$. Since $\hat{f}(x, y)$ is non-negative (3.13) implies

$$\hat{f}(s', t') \leq \hat{g}(s', t') + \int_0^{s'} \int_0^{t'} \hat{f}(x, y)\, dx\, dy, \quad \text{for} \quad (s', t') \leq \left(S + \frac{S^2T}{2}, T\right).$$

But, $\hat{f}$ and $\hat{g}$ are bounded measurable functions since f and g are continuous on $[0, S] \times [0, T]$, so Theorem 3.1 yields

$$\begin{aligned} &\hat{f}(s', t') \leq \hat{g}(s', t') + C \int_0^{s'} \int_0^{t'} \hat{g}(x, y) J(C(s' - x)(t' - y))\, dx\, dy, \\ &\text{for} \quad (s', t') \leq \left(S + \frac{S^2T}{2}, T\right). \end{aligned} \tag{3.14}$$

Now observe that $\hat{g}(x, y) \leq a + Cxy$ for $(x, y) \in D$. Indeed, if $(x, y) \in U(R_{S,T})$ and $y \neq 0$, then $\hat{g}(x, y) = g(U^{-1}(x, y))$

$$\begin{aligned} &= g\left(\frac{-1 + \sqrt{1 + 2xy}}{y}, y\right), \quad \text{by (3.11)} \\ &= a + C\left((-1 + \sqrt{1 + 2xy}) + \frac{(-1 + \sqrt{1 + 2xy})^2}{4}\right) \\ &\leq a + C\left((-1 + \sqrt{1 + 2xy}) + \frac{(-1 + \sqrt{1 + 2xy})^2}{2}\right) \\ &= a + Cxy, \end{aligned}$$

and the other cases are trivial. Thus (3.14) becomes

$$\begin{aligned} &\hat{f}(s', t') \leq a + Cs't' + C \int_0^{s'} \int_0^{t'} (a + Cxy) J(C(s' - x)(t' - y))\, dx\, dy, \\ &\text{for} \quad (s', t') \leq \left(S + \frac{S^2T}{2}, T\right). \end{aligned} \tag{3.15}$$

It is elementary to compute that

$$C \int_0^{s'} \int_0^{t'} (a + Cxy) J(C(s' - x)(t' - y))\, dx\, dy = (a + 1)J(Cs't') - (a + Cs't') - 1,$$

so (3.15) becomes

$$\hat{f}(s', t') \leq (a+1)J(Cs't') - 1, \quad \text{for} \quad (s', t') \leq \left(S + \frac{S^2T}{2}, T\right).$$

Next $\hat{f}(s', t') = f(s, t) \geq h(s, t)$ for $(s, t) \leq (S, T)$ with $s' = s + s^2t/2$, $t' = t$, so the proof of (3.10) will be complete once we show that

$$J(Cs't') \leq e^{2Cst}, \quad (s, t) \leq (S, T). \tag{3.16}$$

But $J(Cs't') = J\left(C\left(st + \frac{s^2t^2}{2}\right)\right) \leq \max(J(2Cst), J(Cs^2t^2))$, since $J(x)$ is an increasing function of $x \geq 0$. Also it is clear from the definition of J that $J(2Cst) \leq e^{2Cst}$. Finally,

$$J(Cs^2t^2) = \textstyle\sum_{k=0}^{\infty} \left(\frac{(\sqrt{C}st)^k}{k!}\right)^2 \leq \left(\sum_{k=0}^{\infty} \frac{(\sqrt{C}st)^k}{k!}\right)^2 = e^{2\sqrt{C}st} \leq e^{2Cst}$$

since $C \geq 1$ (this is the only place we used $C \geq 1$). Hence (3.16) holds and the proof is complete. □

4. The main estimate. In this section we derive the estimate on the moments of a solution to (1.1) with coefficients satisfying (1.3). We shall need to assume that X is constant on the axes, i.e.,

$$X(s, 0) + X(0, t) - X(0, 0) = X(0, 0),$$

for all $(s, t) \in D$ and $\omega \in \Omega$. This is due to the fact that Ito's formula, Theorem 2.1, is valid only for processes which are constant on the axes, in the above sense.

We first prove an elementary lemma which is used repeatedly in the proof of Theorem 4.2.

LEMMA 4.1. *Let $Y(s, t)$ be a non-negative measurable process, $(s, t) \in D$, and $\mu_1, \cdots, \mu_m$, $m \geq 1$ be Borel measures on D which are finite on compact subsets of D. Let n, a_i, and b_i, $1 \leq i \leq m$, be non-negative numbers with the property that*

$$\begin{cases} a_i \leq n, \ 1 \leq i \leq m, \quad \text{and} \\ \sum_{i=1}^m a_i b_i \leq n \end{cases}$$

Then for every $(s, t) \in D$ we have

$$E\left(\prod_{i=1}^m \left(\int_{R_{s,t}} Y^{a_i}\, d\mu_i\right)^{b_i}\right) \leq \prod_{i=1}^m \mu_i(R_{s,t})^{b_i} \cdot (1 + \sup_{(u,v) \leq (s,t)} E(Y(u, v)^n)). \tag{4.1}$$

(*Note*: *we interpret $Y^0 \equiv 1$*).

PROOF. Consider first the case $m = 1$. Assume that $a_1 > 0$, $b_1 > 0$; otherwise (4.1) is trivial. If $b_1 \geq 1$ then

$$E\left(\left(\int_{R_{s,t}} Y^{a_1}\, d\mu_1\right)^{b_1}\right) \leq E\left(\mu_1(R_{s,t})^{b_1 - 1} \int_{R_{s,t}} Y^{a_1 b_1}\, d\mu_1\right)$$

(by Jensen's inequality)

$$\begin{aligned} &= \mu_1(R_{s,t})^{b_1-1} \int_{R_{s,t}} EY^{a_1 b_1}\, d\mu_1 \\ &\leq \mu_1(R_{s,t})^{b_1-1} \int_{R_{s,t}} (1 + EY^n)\, d\mu_1, \end{aligned}$$

since $a_1 b_1 \le n$,

$$\le \mu_1(R_{s,t})^{b_1}(1 + \sup_{(u,v)\le(s,t)} EY(u, v)^n),$$

and if $b_1 < 1$ then

$$E\left(\left(\int_{R_{s,t}} Y^{a_1}\, d\mu_1\right)^{b_1}\right) \le \left(E\left(\int_{R_{s,t}} Y^{a_1}\, d\mu_1\right)\right)^{b_1}$$

(by Holder's inequality)

$$= \left(\int_{R_{s,t}} EY^{a_1}\, d\mu_1\right)^{b_1}$$

$$\le \left(\int_{R_{s,t}} (1 + EY^n)\, d\mu_1\right)^{b_1},$$

since $a_1 \le n$,

$$\le \mu_1(R_{s,t})^{b_1}(1 + \sup_{(u,v)\le(s,t)} EY(u, v)^n)^{b_1}$$

$$\le \mu_1(R_{s,t})^{b_1}(1 + \sup_{(u,v)\le(s,t)} EY(u, v)^n),$$

for $(s, t) \in D$. Thus (4.1) holds for $m = 1$.

For the case of $m \ge 2$, we again see that without loss of generality we may assume $a_i > 0$, $b_i > 0$, $1 \le i \le m$, since if $a_i = 0$ for some i, then a factor of $\mu_i(R_{s,t})^{b_i}$ appears on each side of (4.1), and if $b_i = 0$ for some i, then a factor of 1 appears on each side of (4.1). Now define

$$p_k = \frac{\sum_{i=1}^m a_i b_i}{a_k b_k} \ge 1, \quad 1 \le k \le m.$$

By Holder's inequality (generalized to m exponents) we have for $(s, t) \in D$

$$E\left(\prod_{i=1}^m \left(\iint_{R_{s,t}} Y^{a_i}\, d\mu_i\right)^{b_i}\right) \le \prod_{i=1}^m \left(E\left(\left(\iint_{R_{s,t}} Y^{a_i}\, d\mu_i\right)^{b_i p_i}\right)\right)^{1/p_i}. \tag{4.2}$$

Now for fixed i, $1 \le i \le m$, we have $a_i \le n$ and $a_i \cdot (b_i p_i) = \sum_{k=1}^m a_k b_k \le n$, so by the first part of the proof

$$E\left(\left(\iint_{R_{s,t}} Y^{a_i}\, d\mu_i\right)^{b_i p_i}\right) \le \mu_i(R_{s,t})^{b_i p_i}(1 + \sup_{(u,v)\le(s,t)} E(Y(u, v)^n)). \tag{4.3}$$

Combining (4.2) and (4.3), and using the fact that $\sum_{i=1}^m \frac{1}{p_i} = 1$, we obtain

$$E\left(\prod_{i=1}^m\left(\iint_{R_{s,t}} Y^{a_i}\, d\mu_i\right)^{b_i}\right)$$

$$\le \prod_{i=1}^m (\mu_i(R_{s,t})^{b_i} \cdot (1 + \sup_{(u,v)\le(s,t)} E(Y(u, v)^n))^{1/p_i})$$

$$= \prod_{i=1}^m \mu_i(R_{s,t})^{b_i} \cdot (1 + \sup_{(u,v)\le(s,t)} E(Y(u, v)^n)),$$

for $(s, t) \in D$, which is the desired result. □

THEOREM 4.2. *Let X be a solution to* (1.1) *with coefficients satisfying* (1.3) *and assume that $\partial X \equiv X(0, 0)$. Let $n = 2$ or $n \ge 4$. Then for every $T > 0$ there is a constant C_T*

> 0, *depending only on* n, K_T *and* L_T *such that*

$$E\,|X(s,t)|^n \le (1 + E(|X(0,0)|^n))e^{C_T st} - 1, \tag{4.4}$$

for $(s, t) \le (T, T)$.

PROOF. We may assume $E\,|X(0,0)|^n < \infty$; otherwise (4.4) is trivial (and uninteresting). Let $N > 0$ be given. For $(s, t) \in D$, let

$$I_N(s,t,\omega) = \begin{cases} 1, & \text{if} \quad \sup_{(u,v)\le(s,t)} |X(u,v,\omega)| \le N, \\ 0, & \text{otherwise,} \end{cases} \tag{4.5}$$

so that $|I_N X| \le N$ for all (s, t), ω. Note that

$$I_N(s,t) = I_N(u,v) I_N(s,t), \quad \text{for} \quad (u,v) \le (s,t), \tag{4.6}$$

since if $I_N(s,t) = 1$ then $I_N(u,v) = 1$. Also, note that $I_N \in L_\infty(\mathscr{T}_{s,t})$ since X is an adapted continuous process.

We introduce the coefficients

$$\begin{cases} a_N(s,t,\omega) = I_N(s,t,\omega) e(s,t,X(\cdot,\cdot,\omega)) \\ b_N(s,t,\omega) = I_N(s,t,\omega) f(s,t,X(\cdot,\cdot,\omega)), \end{cases} \tag{4.7}$$

and the process

$$\begin{aligned} Y_N(s,t) &= I_N(0,0)X(0,0) + \int_0^s \int_0^t a_N(u,v)\, dB(u,v) \\ &\quad + \int_0^s \int_0^t b_N(u,v)\, dm(u,v), \end{aligned} \tag{4.8}$$

for $(s, t) \in D$. Note first that $Y_N(0,0) = I_N(0,0)X(0,0)$ is bounded by N. Next, we claim that $a_N, b_N \in L_\infty(\mathscr{T}_{s,t})$. Indeed, the required measurability conditions follows from those of I_N, e, and f, and the boundedness follows from

$$\begin{aligned} a_N^2(s,t) + b_N^2(s,t) &= I_N(s,t)(e(s,t,X)^2 + f(s,t,X)^2) \\ &\le I_N(s,t) L_T\left(1 + X(s,t)^2 + \iint_{R_{s,t}} X(u,v)^2\, d\lambda(u,v)\right) \\ &\le L_T(1 + N^2 + N^2 K_T) < \infty, \end{aligned}$$

for $(s, t) \le (T, T)$, by (4.7), (1.3), and (4.6). Next, we claim that

$$I_N(s,t) Y_N(s,t) = I_N(s,t) X(s,t), \quad (s,t) \in D. \tag{4.9}$$

In view of (4.6), (4.7), (4.8), and (1.2b) and the fact that $\partial X \equiv X(0,0)$, (4.9) will follow from

$$I_N(s,t) \int_0^s \int_0^t e(u,v,X)\, dB(u,v) = I_N(s,t) \int_0^s \int_0^t I_N(u,v) e(u,v,X)\, dB(u,v),$$

and

$$I_N(s,t) \int_0^s \int_0^t f(u,v,X)\, dm(u,v) = I_N(s,t) \int_0^s \int_0^t I_N(u,v) f(u,v,X)\, dm(u,v),$$

for $(s, t) \in D$. But these are easy: on $\{\omega \mid I_N(s,t,\omega) = 0\}$ the equalities are trivial, and on $\{\omega \mid I_N(s,t,\omega) = 1\}$ we have $I_N(u,v,\omega) = 1$ for $(u,v) \le (s,t)$ and again the equalities hold.

Now let $n = 2$ or $n \geq 4$ be fixed. Define

$$\begin{cases} g(x) = x^n \\ h(x) = |x|n, \ x \in R. \end{cases} \tag{4.10}$$

Then $h \in C^4(R)$, in fact for $n = 2$, $h \equiv g$, and for $n \geq 4$,

$$h^{(k)}(x) = \begin{cases} g^{(k)}(|x|), & k = 2, 4 \\ g^{(k)}(|x|)\mathrm{Sign}(x), & k = 1, 3. \end{cases} \tag{4.11}$$

Also, it is clear that h and its derivatives are bounded above by an exponential function. Thus we may apply Theorem 2.1 to the process Y_N and the function h, obtaining that

$$h(Y_N(s, t)) - \int_0^s \int_0^t (T_{u,v}h)(Y_N(u, v))\, dm(u, v)$$

is a weak martingale (with the notation of Theorem 2.1). In particular, noting that $h(Y_N(0, 0)) \leq h(X(0, 0))$, we have

$$\begin{aligned} Eh(Y_N(s, t)) &= Eh(Y_N(0, 0)) + E \int_0^s \int_0^t (T_{u,v}h)(Y_N(u, v))\, dm(u, v) \\ &= Eh(Y_N(0, 0)) + \int_0^s \int_0^t E(T_{u,v}h)(Y_N(u, v))\, dm(u, v) \\ &\leq Eh(X(0, 0)) + \int_0^s \int_0^t E|(T_{u,v}h)(Y_N(u, v))|\, dm(u, v) < \infty, \quad \text{for} \quad (s, t) \in D, \end{aligned} \tag{4.12}$$

where the finiteness is by Theorem 2.2. Now from (2.3) with a and b replaced by a_N and b_N, respectively,

$$\begin{cases} E|(T_{s,t}h)(Y_N(s, t))| \leq E|L_{st}(s, t)h'(Y_N(s, t))| \\ \qquad + E|L_sL_t(s, t)h''(Y_N(s, t))| \\ \qquad + E|\tfrac{1}{2} J_{st}(s, t)h''(Y_N(s, t))| \\ \qquad + E|\tfrac{1}{2} L_s J_t(s, t)h^{(3)}(Y_N(s, t))| \\ \qquad + E|\tfrac{1}{2} L_t J_s(s, t)h^{(3)}(Y_N(s, t))| \\ \qquad + E|\tfrac{1}{4} J_s J_t(s, t)h^{(4)}(Y_N(s, t))| \\ \qquad \equiv c_1(s, t) + c_2(s, t) + \cdots + c_6(s, t), \end{cases} \tag{4.13}$$

for $(s, t) \in D$. We shall show that for $1 \leq k \leq 6$ there is a constant $C_{k,T}$ depending only on L_T, K_T and n such that

$$c_k(s, t) \leq C_{k,T}(1 + st)(1 + \sup_{(u,v)\leq(s,t)} Eh(Y_N(u, v))), \tag{4.14}$$

for $(s, t) \leq (T, T)$. Before verifying (4.14), we can finish the proof. Let

$$C_T = (6 \max_{1\leq k\leq 6} C_{k,T}) \vee 1.$$

Combining (4.12), (4.13), (4.14) we get

$$\begin{aligned} Eh(Y_N(s, t)) \leq Eh(X(0, 0)) + C_T\left(st + \frac{s^2t^2}{4}\right) \\ + C_T \int_0^s \int_0^t (1 + uv)\sup_{(u',v')\leq(u,v)} Eh(Y_N(u', v'))\, dm(u, v), \end{aligned}$$

for $(s, t) \le (T, T)$. We see from (4.12) that $Eh(Y_N(s, t))$ is a continuous function of (s, t). Thus Corollary 3.2 applies, yielding

$$Eh(Y_N(s, t)) \le (1 + Eh(X(0, 0)))e^{2C_T st} - 1, \tag{4.15}$$

for $(s, t) \le (T, T)$. Now by (4.5) and the continuity of X we have that $I_N(s, t) \uparrow 1$ as $N \to \infty$, so by (4.9) $Y_N(s, t) \to X(s, t)$ as $N \to \infty$, for $(s, t) \in D$. Thus by (4.10), Fatou's Lemma, and (4.15)

$$\begin{aligned} E|X(s, t)|^n &= E(\lim_{N\to\infty}|Y_N(s, t)|^n) \le \liminf_{N\to\infty} Eh(Y_N(s, t)) \\ &\le (1 + E|X(0, 0)|^n)e^{2C_T st} - 1, \end{aligned}$$

for $(s, t) \le (T, T)$. Thus we have obtained (4.4) with C_T replacing $2C_T$.

The proof of Theorem 4.2 is complete except for the derivation of (4.14). Let us introduce the following notation

$$\alpha(s, t) = \iint_{R_{s,t}} Y_N(u, v)^2 \, d\lambda(u, v)$$

$$\sigma_1(s, t) = \int_0^s |Y_N(u, t)| \, du, \quad \sigma_2(s, t) = \int_0^s Y_N(u, t)^2 \, du$$

$$\tau_1(s, t) = \int_0^t |Y_N(s, v)| \, dv, \quad \tau_2(s, t) = \int_0^t Y_N(s, v)^2 \, dv$$

$$q_k = |Y_N(s, t)|^{n-k}, \quad 0 \le k \le 4,$$

and

$$Q = 1 + \sup_{(u,v)\le(s,t)} E|Y_N(u, v)|^n.$$

Since N and T are fixed in this part of the proof we denote Y_N, a_N, b_N, I_N, K_T, and L_T by Y, a, b, I, K, and L, respectively. Also, all functions on D shall be evaluated at (s, t) unless otherwise indicated.

To begin the derivation of (4.14) first note that from (4.10) and (4.11) we have

$$\begin{cases} g(x) &= x^n \\ |h^{(k)}(x)| &= g^{(k)}(|x|), \quad k = 1, 2, 3, 4, \\ \text{and} \quad h(x) &= g(|x|) = |x|^n. \end{cases} \tag{4.16}$$

Next, fix $(s, t) \le (T, T)$. Then

$$\begin{cases} |b| = I|f(s, t, X)| \quad \text{by (4.7)} \\ \le IL^{1/2}\left(1 + X^2 + \iint_{R_{s,t}} X(u, v)^2 \, d\lambda(u, v)\right)^{1/2}, \quad \text{by (1.3)} \\ \le IL^{1/2}\left(1 + |X| + \left(\iint_{R_{s,t}} X(u, v)^2 \, d\lambda(u, v)\right)^{1/2}\right) \\ \le IL^{1/2}\left(1 + |Y| + \left(\iint_{R_{s,t}} Y(u, v)^2 \, d\lambda(u, v)\right)^{1/2}\right) \\ \le L^{1/2}(1 + |Y| + \alpha^{1/2}). \end{cases} \tag{4.17}$$

Similarly,

$$a^2 \leq L(1 + Y^2 + \alpha). \tag{4.18}$$

We now derive (4.14) for $1 \leq k \leq 6$. The proofs for each case are similar but we write them all out for the sake of completeness.

(i) Estimate for $c_1 = E\,|L_{st}h'(Y)|$. First,

$$|L_{st}h'(Y)| = |b|\;|h'(Y)| \leq L^{1/2}(1 + |Y| + \alpha^{1/2})g'(|Y|),$$
$$= nL^{1/2}(q_1 + q_0 + q_1\alpha^{1/2}), \quad \text{by (4.16), (4.17).}$$

Hence

$$c_1 \leq nL^{1/2}(Eq_1 + Eq_0 + Eq_1\alpha^{1/2}). \tag{4.19}$$

But $Eq_1 \leq Q$ by Lemma 4.1 with $m = 1$, $\mu_1 = \delta_{s,t}$, $a_1 = n - 1$, and $b_1 = 1$. Similarly, $Eq_0 \leq Q$. (Of course we don't need a lemma for these trivial inequalities, but Lemma 4.1 is definitely needed for later estimates; it also covers the trivial cases.) By Lemma 4.1 we also have $Eq_1\alpha^{1/2} \leq \lambda(R_{s,t})^{1/2}Q$ (take $m = 2$, $\mu_1 = \delta_{s,t}$, $\mu_2 = \lambda$, $a_1 = n - 1$, $b_1 = 1$, $a_2 = 2$ and $b_2 = \frac{1}{2}$). Using these estimates in (4.19) we obtain $c_1 \leq nL^{1/2}(2 + K^{1/2})Q$. Thus (4.14) holds for $k = 1$ with

$$C_{1,T} = nL^{1/2}(2 + K^{1/2}).$$

(ii) Estimate for $c_2 = E|L_sL_th''(Y)|$. Using (4.16) and (4.17) we obtain

$$|L_sL_th''(Y)| = \left|\int_0^t b(s,v)\,dv\right| \cdot \left|\int_0^s b(u,t)\,du\right| \cdot \left|h''(Y)\right|$$
$$\leq n(n-1)L\left(\int_0^t [1 + |Y(s,v)| + \alpha(s,v)^{1/2}]\,dv\right)$$
$$\cdot \left(\int_0^s [1 + |Y(u,t)| + \alpha(u,t)^{1/2}]\,du\right) \cdot q_2$$
$$\leq n(n-1)L(t + \tau_1 + t\alpha^{1/2})(s + \sigma_1 + s\alpha^{1/2})\,q_2$$

since $\alpha(s,t)$ is an increasing function of (s,t). Thus multiplying out the factors in the last expression and taking expectation we find

$$c_2 \leq n(n-1)L(stEq_2 + sE\tau_1q_2 + stEq_2\alpha^{1/2} + tE\sigma_1q_2 + E\sigma_1\tau_1q_2$$
$$+ tE\sigma_1q_2\alpha^{1/2} + stEq_2\alpha^{1/2} + sE\tau_1\alpha^{1/2}q_2 + stEq_2\alpha).$$

Now by Lemma 4.1 we find

$$Eq_2 \leq Q, \quad E\tau_1q_2 \leq tQ, \; Eq_2\alpha^{1/2} \leq \lambda(R_{s,t})^{1/2}Q, \quad E\sigma_1q_2 \leq sQ,$$
$$E\sigma_1\tau_1q_2 \leq stQ, \quad E\sigma_1q_2\alpha^{1/2} \leq s\lambda(R_{s,t})^{1/2}Q, \; E\tau_1q_2\alpha^{1/2} \leq t\lambda(R_{s,t})^{1/2}Q,$$

and

$$E\alpha q_2 \leq \lambda(R_{s,t})Q.$$

Thus $c_2 \leq n(n-1)Lst(4 + 4K^{1/2} + K)Q$, since $\lambda(R_{s,t}) \leq K$. So, (4.14) holds for $k = 2$ with

$$C_{2,T} = n(n-1)L(4 + 4K^{1/2} + K).$$

(iii) Estimate for $c_3 = E\,|\frac{1}{2}J_{st}h''(Y)|$. Using (4.16) and (4.18),

$$|J_{st}h''(Y)| = a^2g''(|Y|) \leq Ln(n-1)(1 + Y^2 + \alpha)q_2.$$

Thus $c_3 \leq \frac{1}{2}\,n(n-1)L(Eq_2 + Eq_0 + E\alpha q_2) \leq \frac{1}{2}n(n-1)L(Q + Q + KQ)$, again by Lemma 4.1. Thus (4.14) holds for $k = 3$ with

$$C_{3,T} = \tfrac{1}{2}n(n-1)(2+K).$$

(iv) Estimate for $c_4 = E\,|\,\tfrac{1}{2}L_sJ_th^{(3)}(Y)\,|$. By the same technique as in (i) – (iii) we arrive at

$$c_4 \le \tfrac{1}{2}\, n(n-1)(n-2)L^{3/2}[stEq_3 + sEq_3\tau_1 + stEq_3\alpha^{1/2} + tEq_3\sigma_2 + Eq_3\sigma_2\tau_1 + tEq_3\sigma_2\alpha^{1/2} + stEq_3\alpha + sEq_3\tau_2\alpha + stEq_3\alpha^{3/2}]$$

$$\le \tfrac{1}{2}\, n(n-1)(n-2)L^{3/2}st[1+1+K^{1/2}+1+1+K^{1/2}+K+K+K^{3/2}]Q.$$

Thus (4.14) holds for $k = 4$ with

$$C_{4,T} = \tfrac{1}{2}n(n-1)(n-2)L^{3/2}(4 + 2K^{1/2} + 2K + K^{3/2}).$$

(v) Estimate for $c_5 = E\,|\,\tfrac{1}{2}L_tJ_sh^{(3)}(Y)\,|$. Since (v) is the same as (iv) with s and t interchanged, we obtain (4.14) for $k = 5$ with

$$C_{5,T} = C_{4,T}.$$

Finally,

(vi) Estimate for $c_6 = E\,|\,\tfrac{1}{4}J_sJ_th^{(4)}(Y)\,|$.

$$c_6 \le \tfrac{1}{4}\, n(n-1)(n-2)(n-3)L^2[stEq_4 + sEq_4\tau_2 + 2\, st\, Eq_4\alpha + tEq_4\sigma_2 + Eq_4\sigma_2\tau_2 + tEq_4\sigma_2\alpha + sEq_4\tau_2\alpha + stq_4\alpha^2]$$

$$\le \tfrac{1}{4}\,(n)(n-1)(n-2)(n-3)L^2st[1+1+2K+1+1+K+K+K^2]Q.$$

Thus (4.14) holds for $k = 6$ with

$$C_{6,T} = \tfrac{1}{4}\, n(n-1)(n-2)(n-3)L^2(4 + 4K + K^2). \qquad \square$$

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