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HARMONIC FUNCTIONS AND THE DIRICHLET PROBLEM FOR REVIVAL MARKOV PROCESSES

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Harmonic functions and the Dirichlet problem on an open set G are defined for a pieced-out or revival Markov process constructed from a continuous base process. A one-to-one correspondence is obtained between the bounded harmonic functions of the revival process and those of the base process. The harmonic functions of the revival process are shown to be continuous on G under certain conditions and to coincide with the solutions of $\mathscr{U}f = 0$ on G where $\mathscr{U}$ is the characteristic operator. These results are applied to random evolution processes and branching processes.

1. Introduction. Ikeda, et al. in [7] develop a general method of "restarting" a given Markov process (here called the base process) at the end of its lifetime. The restarting is done with an "instantaneous" distribution in such a way that the new process, called a pieced-out or revival process, still has the Markov property. By specializing the base process and the instantaneous distribution, many interesting types of Markov processes can be obtained. For example, revival processes are used to construct branching Markov processes in [8] and are used to construct random evolution processes with feedback in [12]. Several other interesting applications are given in [9].

One of the main uses of these new processes has been the probabilistic analysis of certain types of equations. For example, Miyamoto [10], Heath [6], and Griego and Hersh [3], [4] use random evolutions to study certain finite systems of equations. Nagasawa [11] uses a branching process to solve a type of nonlinear Dirichlet problem.

In this paper, harmonic functions and the Dirichlet problem are studied for a general revival process with a continuous base process. A bounded, nearly Borel function f is defined to be harmonic on an open set G if it satisfies the usual mean value property on open subsets of G. But since the revival process is discontinuous, it is necessary that f be defined on the entire state space (not just on $\bar{G}$) and of course the classical theorems (see [2]) do not apply. The main result of this paper is Theorem (4.6) which gives a one-to-one correspondence between the harmonic functions of the base process and the harmonic functions of the revival process. Since the base process is continuous, this correspondence is then used to obtain analogues of the classical theorems for the revival process. In particular, it is shown that under appropriate conditions, the harmonic functions are continuous on G and coincide with the zeros of the characteristic operator on G, and the Dirichlet problem with a given "exterior function" is solved.

The results are applied to random evolution processes and branching processes. The random evolution processes considered are more general than those used by previous authors and allow feedback and an uncountable number of evolution modes. Moreover, the main theorem yields additional insights. In the random evolution setting it gives a correspondence between solutions of the coupled system of equations and solutions of the associated uncoupled system. In the branching process setting of Nagasawa [11] it gives a correspondence between solutions of the nonlinear equation and solutions of the associated linear equation.

2. Preliminaries. A locally compact Hausdorff space with a countable base will be called a semicompact. If E is a semicompact, $\mathscr{C}(E)$ will denote the topology of E, $\mathscr{B}(E)$

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$= \sigma\mathscr{C}(E)$ the σ-algebra of Borel sets, and $\mathscr{B}^*(E)$ the completion of $\mathscr{B}(E)$ with respect to the set of all probability measures on $(E, \mathscr{B}(E))$. If $x \in E$, δ_x denotes point mass at x. If $f \in \mathscr{B}^*(E)/\mathscr{B}(\mathbb{R})$ is bounded, $\| f \|$ denotes the sup norm of f.

We will use the usual notation and results for a standard Markov process $X = (\Omega, \mathscr{M}, \mathscr{M}(t), X(t), \theta(t), P^x)$ with semicompact state space $(E, \mathscr{B}(E))$, argumented by the usual dead state Δ (see [1]). In particular, ζ denotes the lifetime of X, $T(A) = \inf\{t \geq 0 : X(t) \notin A\}$ the first exit time of X from $A \subset E$, and $T^+(A) = \inf\{t > 0 : X(t) \notin A\}$ the first exit time of X from A after 0. A function $f \in \mathscr{B}^*(S)/\mathscr{B}(R)$ is automatically extended to $S \cup \{\Delta\}$ by $f(\Delta) = 0$. The semigroup of X is denoted $T(t)$, $t \geq 0$. The process X is said to be strong Feller if $T(t)$ takes bounded $\mathscr{B}(E)/\mathscr{B}(\mathbb{R})$ functions into continuous functions and X is $\hat{\mathscr{C}}(E)$ if $T(t)$ preserves the space of bounded continuous functions "vanishing at ∞" (see [2]). The characteristic operator $\mathscr{U}$ of X is defined by

$$\mathscr{U}f(x) = \lim_{V \downarrow x} \frac{E^x[f(X(T(V)))] - f(x)}{E^x[T(V)]} \tag{2.1}$$

with domain $\mathscr{D}(\mathscr{U}; x)$ at x, where the limit is taken over open sets V with compact closure shrinking to x (see [2]).

In the sequel, if a Markov process has a subscript or other distinguishing mark, objects (such as σ-algebras, characteristic operators, etc.) with the same subscript or mark will refer to this process.

Now let $\tilde{Z}$ be a fixed, standard, continuous Markov process with state space $(S, \mathscr{B}(S))$ such that $0 < \tilde{\zeta} < \infty$ a.s. $\tilde{P}^z$ for $z \in S$. $\tilde{Z}$ will be the base process. Let μ be an instantaneous distribution for $\tilde{Z}$ in the sense of [7], i.e.,

(2.2) (a) μ is a transition probability from $(\tilde{\Omega}, \tilde{\mathscr{F}})$ to $(S, \mathscr{B}(S))$.

(b) If $\tilde{T}$ is an $\tilde{\mathscr{M}}(t)$-stopping time and $x \in S$ then $\mu(\tilde{\omega}, \cdot) = \mu(\tilde{\theta}(\tilde{T})(\tilde{\omega}), \cdot)$ a.s. $\tilde{P}^z$ on $\{\tilde{T} < \tilde{\zeta}\}$.

Let Z be the pieced-out or revival process associated with $\tilde{Z}$ and μ (see [7], [8], and [9]). Let τ_n, $n = 0, 1, \cdots$ denote the revival times of Z. Z is a standard process and satisfies

(2.3) (a) $(Z(t), t < \tau_1, P^z)$ is equivalent to $(\tilde{Z}(t), t < \tilde{\zeta}, \tilde{P}^z)$.

(b) For $\Lambda \in \tilde{\mathscr{F}}$, $C \in \mathscr{B}(S)$,

$$P^z\{\omega : \rho_1(\omega) \in \Lambda, Z(\tau_1)(\omega) \in C\} = \tilde{E}^z[\mu(\tilde{\omega}, C); \Lambda]$$

where ρ_1 is the natural projection from Ω to $\tilde{\Omega}$ (see [7]).

3. Characteristic operations and nearly Borel sets. The goal of this section is to establish some connections between the characteristic operators and nearly Borel sets for the base process $\tilde{Z}$ and those for the revival process Z. The results will be needed for the study of harmonic functions in Section 4.

Let $\mathscr{V}$ be the operator defined by

$$\mathscr{V}f(z) = \lim_{V \downarrow z} \frac{E^z[f(Z(\tau_1)); T(V) = \tau_1]}{E^z[T(V)]} \tag{3.1}$$

with domain $\mathscr{D}(\mathscr{V}; z) \subset \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ at $z \in S$.

(3.2) THEOREM. *Let $z \in S$ and $f \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$. Assume that*

(3.3) $\mu(\tilde{\omega}, V) = 0$ a.s. $\tilde{P}^z$ *for sufficiently small compact neighborhoods of z.*

If $f \in \mathscr{D}(\mathscr{V}; z)$ then $f \in \mathscr{D}(\mathscr{U}; z)$ if and only if $f \in \mathscr{D}(\tilde{\mathscr{U}}; z)$ and

$$\mathscr{U}f(z) = \tilde{\mathscr{U}}f(z) + \mathscr{V}f(z).$$

PROOF. Let V be an open neighborhood of z with compact closure then

(3.4) $E^z[f(Z(T(V)))] = E^z[f(Z(T(V))); T(V) < \tau_1] + E^z[f(Z(T(V))); T(V) \geq \tau_1]$.

By (2.3) the first term on the right of (3.4) is

$$\tilde{E}^z[f(\tilde{Z}(\tilde{T}(V))); \tilde{T}(V) < \tilde{\zeta}].$$

By (2.3) and (3.3),

$$P^z[T(V) > \tau_1] = \tilde{E}^z[\mu(\tilde{\omega}, V); \tilde{T}(V) = \tilde{\zeta}] = 0$$

for V sufficiently small. Hence the second term on the right of (3.4) is

$$E^z[f(Z(\tau_1)); T(V) = \tau_1].$$

Also, by (2.3) and (3.3),

$$E^z[T(V)] = \tilde{E}^z[\tilde{T}(V)].$$

Hence the result follows from the definition of the characteristic operator (2.1). □

(3.5) Remark. Condition (3.3) can be weakened. For the results in this paper, all that is necessary is

$$\frac{P^z[T(V) > \tau_1]}{\tilde{E}^z[\tilde{T}(V)]} \to 0$$

and

$$\frac{E^z[T(V)]}{\tilde{E}^z[T(V)]} \to 1$$

as $V \downarrow z$. However for the applications considered here (the random evolution process and the branching process) Condition (3.3) is satisfied.

It is straightforward to show that $C \subseteq S$ is nearly Borel for the base process Z if and only if C is nearly Borel for the revival process $\tilde{Z}$. In addition, we will need the following:

(3.6) Lemma. *Let $f \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ be bounded. Let G be open in S and $n \geq 1$. Define*

$$h(z) = E^z[f(Z(\tau_n)); \tau_n \leq T^+(G)].$$

Then h is nearly Borel.

Proof. Since $h \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ is bounded, it suffices to show that $\tilde{T}(t)h \to h$ as $t \downarrow 0$ by Theorem (5.13) of [2]. But

$$\begin{aligned}\tilde{T}(t)h(z) &= \tilde{E}^z[h(\tilde{Z}(t))] = E^z[h(Z(t)); t < \tau_1]\\ &= E^z[E^{Z(t)}[f(Z(\tau_n)); \tau_n \leq T^+(G)]; t < \tau_1]\\ &= E^z[f(Z(\tau_n)); \tau_n \leq t + T^+(G) \circ \theta(t), t < \tau_1]\end{aligned}$$

by the strong Markov property since $t + \tau_n \circ \theta(t) = \tau_n$ a.s. on $\{t < \tau_1\}$. But $t + T^+(G) \circ \theta(t) \downarrow T^+(G)$ by (10.3) of [1]. Hence by bounded convergence, $\tilde{T}(t)h \to h$. □

(3.7) Remark. In Lemma (3.6), note that $T^+(G) = T(G)$ a.s. P^z for $z \in G$ and hence for $z \in G$,

$$h(z) = E^z[f(Z(\tau_n)); \tau_n \leq T(G)].$$

4. Harmonic functions and the Dirichlet problem. The purpose of this section is to define and study the harmonic functions and Dirichlet problem of the revival process Z. The main result is Theorem (4.6) which relates the harmonic functions of Z and those of the base process $\tilde{Z}$. Since $\tilde{Z}$ is a continuous process, the standard results of [2] apply to $\tilde{Z}$. Theorem (4.6) is then used to obtain analogous results for Z.

(4.1) LEMMA. *Let G be open is S. Suppose that*

$$\inf_{z\in G}\tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}] > 0. \tag{4.2}$$

Then

(a) $\sum_{n=0}^{\infty} P^z[\tau_n \le T(G)]$ *converges uniformly and is bounded in S.*
(b) $P^z[T(G) < \zeta] = 1$ *for* $z \in S$.

PROOF. By assumption, there exists $\varepsilon > 0$ such that $\tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}] > \varepsilon$ for $z \in G$. If $z \notin G$, $\tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}] = 1$. Let $\delta = 1 - \varepsilon$. Then $0 < \delta < 1$ and $\tilde{P}^z[\tilde{T}(G) = \tilde{\zeta}] \le \delta$ for $z \in S$. For $n \ge 1$,

$$\begin{aligned}P^z[\tau_n \le T(G)] &= P^z[\tau_{n-1} + \tau \circ \theta(\tau_{n-1}) \le \tau_{n-1} + T(G) \circ \theta(\tau_{n-1}); \tau_{n-1} \le T(G)]\\ &= E^z[P^{Z(\tau_{n-1})}[\tau_1 \le T(G)]; \tau_{n-1} \le T(G)]\\ &= E^z[\tilde{P}^{Z(\tau_{n-1})}[\tilde{T}(G) = \tilde{\zeta}]; \tau_{n-1} \le T(G)]\\ &\le \delta P^z[\tau_{n-1} \le T(G)].\end{aligned}$$

By induction, then $P^z[\tau_n \le T(G)] \le \delta^n$. This proves (a). Part (b) follows immediately from (a) since $\tau_n \uparrow \zeta$. □

(4.3) DEFINITION. Let G be open in S. A bounded function $f:S \to \mathbb{R}$ is harmonic for Z on G if f is nearly Borel and if for every open V with compact closure contained in G and $z \in G$,

$$E^z[f(Z(T(V)))] = f(z). \tag{4.4}$$

(4.5) REMARKS. (a) A similar definition applies for the base process $\tilde{Z}$. Since $\tilde{Z}$ is continuous, the definition reduces to the standard one (Section 12.18 of [2]).

(b) Since Z is not continuous, it is necessary in general for f to be defined on all of S, not just on G (see however Sections 5 and 6).

(c) If f is harmonic for Z on G then it follows immediately that $\mathscr{U}f = 0$ on G.

(4.6) THEOREM. *Let G be open in S. Suppose that* (4.2) *is satisfied. The following formulas establish a one-to-one correspondence between the bounded functions g that are harmonic for* $\tilde{Z}$ *on G and the bounded functions f that are harmonic for Z on G*:

$$f(z) = \sum_{n=0}^{\infty} E^z[g(Z(\tau_n)); \tau_n \le T(G)] \tag{4.7}$$

$$g(z) = f(z) - E^z[f(Z(\tau_1)); \tau_1 \le T(G)]. \tag{4.8}$$

PROOF. Let $g:S \to \mathbb{R}$ be bounded and be harmonic for $\tilde{Z}$ on G. Define $f:S \to \mathbb{R}$ by (4.7). Note that $f = g$ on $S - G$ so f is bounded and nearly Borel on $S - G$. Also, f is bounded and nearly Borel on G by Lemmas (3.6) and (4.1). Thus we need to prove the mean value property (4.4) and formula (4.8). Let $z \in G$ and let V be open with compact closure contained in G. Then

$$E^z[f(Z(T(V)))] = \sum_{n=0}^{\infty} E^z[E^{Z(T(V))}[g(Z(\tau_n)) : \tau_n \le T(G)]]. \tag{4.9}$$

The first ($n = 0$) term of the sum in (4.9) is

$$\begin{aligned}E^z[g(Z(T(V)))] &= \sum_{k=0}^{\infty} E^z[g(Z(T(V))); \tau_k \le T(V) < \tau_{k+1}]\\ &= \sum_{k=0}^{\infty} E^z[E^{Z(\tau_k)}[g(Z(T(V))); T(V) < \tau_1]; \tau_k \le T(V)]\\ &= \sum_{k=0}^{\infty} E^z[\tilde{E}^{Z(\tau_k)}[g(\tilde{Z}(\tilde{T}(V)))]; \tau_k \le T(V)]\\ &= \sum_{k=0}^{\infty} E^z[g(Z(\tau_k)); \tau_k \le T(V)]\end{aligned}$$

where we have used the strong Markov property, the assumption that g is harmonic for $\tilde{Z}$ on G and the following identities: $\tau_k + \tau \circ \theta(\tau_k) = \tau_{k+1}$, $\tau_k + T(V) \circ \theta(\tau_k) = T(V)$ a.s. on $\{\tau_k \leq T(V)\}$. For the remaining terms of the sum in (4.9) we have

$$\begin{aligned}
&\textstyle\sum_{n=1}^{\infty} E^z[E^{Z(T(V))}[g(Z(\tau_n)); \tau_n \leq T(G)]] \\
&\quad = \textstyle\sum_{n=1}^{\infty} E^z[g(Z(\tau_n)) \circ \theta(T(V)); \tau_n \circ \theta(T(V)) \leq T(G) \circ \theta(T(V))] \\
&\quad = \textstyle\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} E^z[g(Z(T(V) + \tau_n \circ \theta(T(V)))); T(V) + \tau_n \circ \theta(T(V)) \\
&\qquad\qquad \leq T(V) + T(G) \circ \theta(T(V)), \tau_m \leq T(V) < \tau_{m+1}] \\
&\quad = \textstyle\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} E^z[g(Z(\tau_{m+n})); \tau_{m+n} \leq T(G), \tau_m \leq T(V) < \tau_{m+1}] \\
&\quad = \textstyle\sum_{k=1}^{\infty} \sum_{n=1}^{k} E^z[g(Z(\tau_k)); \tau_k \leq T(G), \tau_{k-n} \leq T(V) < \tau_{k-n+1}] \\
&\quad = \textstyle\sum_{k=1}^{\infty} E^z[g(Z(\tau_k)); \tau_k \leq T(G), T(V) < \tau_k].
\end{aligned}$$

Combining we have

$$\begin{aligned}
E^z[f(Z(T(V)))] &= \textstyle\sum_{k=0}^{\infty} E^z[g(Z(\tau_k)); \tau_k \leq T(V)] + \sum_{k=1}^{\infty} E^z[g(Z(\tau_k)); T(V) < \tau_k \leq T(G)] \\
&= \textstyle\sum_{k=0}^{\infty} E^z[g(Z(\tau_k)); \tau_k \leq T(G)] = f(z).
\end{aligned}$$

Hence f is harmonic for Z on G. To prove (4.8), it suffices to show that

$$E^z[f(Z(\tau_1)); \tau_1 \leq T(G)] = \textstyle\sum_{n=1}^{\infty} E^z[g(Z(\tau_n)); \tau_n \leq T(G)].$$

But

$$\begin{aligned}
E^z[f(Z(\tau_1)); \tau_1 \leq T(G)] &= E^z[\textstyle\sum_{n=0}^{\infty} E^{Z(\tau_1)}[g(Z(\tau_n)); \tau_n \leq T(G)]; \tau_1 \leq T(G)] \\
&= \textstyle\sum_{n=0}^{\infty} E^z[g(Z(\tau_{n+1})); \tau_{n+1} \leq T(G), \tau_1 \leq T(G)] \\
&= \textstyle\sum_{n=1}^{\infty} E^z[g(Z(\tau_n)); \tau_n \leq T(G)].
\end{aligned}$$

Now suppose that $f: S \to \mathbb{R}$ is bounded and harmonic for Z on G. Define $g: S \to \mathbb{R}$ by (4.8). Then g is bounded since f is. Also, $g = f$ on $S - G$ so g is nearly Borel since f is and by Lemma (3.6). The mean value property can be verified by arguments similar to the previous ones. Thus we will prove (4.7). For $n \geq 1$,

$$\begin{aligned}
&E^z[g(Z(\tau_n)); \tau_n \leq T(G)] \\
&\quad = E^z[f(Z(\tau_n)); \tau_n \leq T(G)] - E^z[E^{Z(\tau_n)}[f(Z(\tau_1)); \tau_1 \leq T(G)]; \tau_n \leq T(G)] \\
&\quad = E^z[f(Z(\tau_n)); \tau_n \leq T(G)] - E^z[f(Z(\tau_{n+1})); \tau_{n+1} \leq T(G)].
\end{aligned}$$

Therefore

$$\textstyle\sum_{n=1}^{N} E^z[g(Z(\tau_n)); \tau_n \leq T(G)] = E^z[f(Z(\tau_1)); \tau_1 \leq T(G)] - E^z[f(Z(\tau_{N+1})); \tau_{N+1} \leq T(G)].$$

But the last term on the right $\to 0$ as $N \to \infty$ since f is bounded and by Lemma (4.1). This proves (4.7). □

(4.10) EXAMPLE. Let G be open in S and suppose that (4.2) is satisfied. Let $\phi: S \to \mathbb{R}$ be bounded and nearly Borel. Define $f(z) = E^z[\phi(Z(T(G)))]$ and $g(z) = \tilde{E}^z[\phi(\tilde{Z}(\tilde{T}(G)))]$. Then f and g are related as in (4.7), (4.8). Also, g is harmonic for $\tilde{Z}$ on G by Theorem 12.12 of [2] so f is harmonic for Z on G.

(4.11) THEOREM. *Let G be open in S and suppose that* (4.2) *is satisfied. Let f, g be bounded functions in $\mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ related as in* (4.7), (4.8). *Let $z \in G$ and suppose that* (3.3) *is satisfied. Then $f \in \mathscr{D}(\mathscr{U}; z)$ if and only if $g \in \mathscr{D}(\tilde{\mathscr{U}}; z)$ and*

$$\mathscr{U}f(z) = \tilde{\mathscr{U}}g(z).$$

PROOF. Let V be an open neighborhood of z with compact closure contained in G. Then using the strong Markov property,

$$\tilde{E}^z[g(\tilde{Z}(\tilde{T}(V)))] = E^z[f(Z(T(V)))] - E^z[f(Z(T(V))); \tau_1 \leq T(V)] - E^z[f(Z(\tau_1)); T(V) < \tau_1 \leq T(G)].$$

Hence for V sufficiently small,

$$\begin{aligned}\tilde{E}^z[g(\tilde{Z}(\tilde{T}(V)))] &= E^z[f(Z(T(V)))] - E^z[f(Z(\tau_1)); \tau_1 = T(V)] - E^z[f(Z(\tau_1)); T(V) < \tau_1 \leq T(G)] \\ &= E^z[f(Z(T(V)))] + g(z) - f(z)\end{aligned}$$

from (4.8). Hence

$$\tilde{E}^z[g(\tilde{Z}(\tilde{T}(V)))] - g(z) = E^z[f(Z(T(V)))] - f(z).$$

Since $E^z[T(V)] = \tilde{E}^z[\tilde{T}(V)]$, the result follows. □

(4.12) THEOREM. *Suppose that $\tilde{Z}$ is strong Feller and $\hat{\mathscr{C}}(S)$. Let G be open in S and suppose that* (4.2) *is satisfied. If f is bounded and is harmonic for Z on G then f is continuous on G.*

PROOF. Define g by (4.8). Then g is harmonic for $\tilde{Z}$ on G and hence g is continuous on G by Theorem 13.2 of [2]. Hence it suffices to show that

$$z \to E^z[f(Z(\tau_1)); \tau_1 \leq T(G)]$$

is continuous on G. Let $t > 0$. Then

$$\begin{aligned}E^z[f(Z(\tau_1)); \tau_1 \leq T(G)] &= \tilde{E}^z\left[\int_S \mu(\tilde{\omega}, du)f(u); \tilde{T}(G) = \tilde{\zeta}\right] \\ &= \tilde{E}^z\left[\int_S \mu(\tilde{\omega}, du)f(u); \tilde{T}(G) = \tilde{\zeta}, t < \tilde{T}(G)\right] \\ &\quad + \tilde{E}^z\left[\int_S \mu(\tilde{\omega}, du)f(u); \tilde{T}(G) = \tilde{\zeta}, t \geq \tilde{T}(G)\right].\end{aligned} \tag{4.13}$$

The first term on the right of (4.13) can be written

$$\tilde{E}^z\left[\tilde{E}^{\tilde{Z}(t)}\left[\int_S \mu(\tilde{\omega}, du)f(u); \tilde{T}(G) = \tilde{\zeta}\right]; t < \tilde{T}(G)\right]$$

where we have used the fact that $\mu(\tilde{\theta}(t)(\tilde{\omega}), \cdot) = \mu(\tilde{\omega}, \cdot)$ a.s. on $\{t < \tilde{\zeta}\}$. But this last expression is continuous as a function of z on G by Theorem 13.1 of [2]. The second term on the right in (4.13) is bounded in absolute value by

$$\|f\| \tilde{P}^z[\tilde{T}(G) \leq t]$$

which $\to 0$ as $t \downarrow 0$ uniformly on open sets with compact closure contained in G (see the proof of Theorem 13.1 of [2]). □

(4.14) THEOREM. *Suppose that $\tilde{Z}$ is strong Feller and $\hat{\mathscr{C}}(S)$. Suppose that G is open in S and that* (4.2) *is satisfied. Suppose also that* (3.3) *holds on G and that for each open U with compact closure contained in G, there exists an open V with compact closure contained in G such that V has regular boundary for $\tilde{Z}$ and contains U. If $f: S \to \mathbb{R}$ is bounded and nearly Borel, then f is harmonic for Z on G if and only if f is continuous on G and $\mathscr{U}f = 0$ on G.*

PROOF. Suppose that $f: S \to \mathbb{R}$ is bounded and nearly Borel. If f is harmonic for Z on G then $\mathcal{U}f = 0$ on G as noted in Remark (4.5) and f is continuous on G by Theorem (4.12). Conversely, suppose that f is continuous on G and that $\mathcal{U}f = 0$ on G. Define g by (4.8). Then $\tilde{\mathcal{U}}g = 0$ on G by Theorem (4.11). Moreover, g is continuous on G (see the proof of (4.12)). Hence g is harmonic for $\tilde{Z}$ on G by the corollary to Theorem 13.5 of [2]. Hence f is harmonic for Z on G by Theorem (4.6). □

(4.15) DEFINITION. Let G be open in S and let $\phi: S - G \to \mathbb{R}$. We say that f is a solution of the Dirichlet problem for Z on G with exterior function ϕ if

(a) f is harmonic for Z on G
(b) $f = \phi$ on $S - G$.
(c) $\lim_{z \to a, z \in G} f(z) = \phi(a)$ *for* $a \in \partial G$.

(4.16) THEOREM. *Suppose that $\tilde{Z}$ is strong Feller and $\hat{\mathscr{C}}(S)$. Suppose that G is open in S with compact closure and that Condition* (4.2) *is satisfied. Suppose also that G has regular boundary for $\tilde{Z}$. Suppose that $\phi: S - G \to \mathbb{R}$ is bounded, nearly Borel, and continuous on ∂G. Then there exists a unique, bounded solution of the Dirichlet problem for Z on G with exterior function ϕ, given by*

$$f(z) = E^z[\phi(Z(T(G)))]. \tag{4.17}$$

PROOF. First recall that $z \to \tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}]$ is harmonic for $\tilde{Z}$ on G (see Section 12.18 of [2]). Since G has regular boundary for $\tilde{Z}$,

$$\lim_{z \to a, z \in G} \tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}] = 1 \tag{4.18}$$

for $a \in \partial G$ (see Theorem 13.1 of [2]). Let f be given by (4.17). Define g by (4.8). By example (4.9), f is harmonic for Z on G and of course $f = \phi$ on $S - G$. Moreover,

$$g(z) = \tilde{E}^z[\phi(\tilde{Z}(\tilde{T}(G)))].$$

So by Theorem 13.4 of [2],

$$\lim_{z \to a, z \in G} g(z) = \phi(a)$$

for $a \in \partial G$. On the other hand,

$$|E^z[f(Z(\tau_1)); \tau_1 \le T(G)]| \le \|f\| P^z[\tau_1 \le T(G)] = \|f\| \tilde{P}^z[\tilde{T}(G) = \tilde{\zeta}]$$

and the last term $\to 0$ as $z \to a \in \partial G (z \in G)$ by (4.18). Hence

$$\lim_{z \to a, z \in G} f(z) = \phi(a)$$

for $a \in \partial G$ so f satisfies the Dirichlet problem for Z on G with exterior function ϕ.

Conversely, suppose that f is a bounded solution of the Dirichlet problem for Z on G with exterior function ϕ. Define g by (4.8). Then g is bounded and is harmonic for $\tilde{Z}$ on G by Theorem (4.6). Moreover,

$$\lim_{z \to a, z \in G} E^z[f(Z(\tau_1)); \tau_1 \le T(G)] = 0 .$$

for $a \in \partial G$ as above so it follows that

$$\lim_{z \to a, z \in G} g(z) = \phi(a)$$

for $a \in \partial G$. Hence g solves the Dirichlet problem for $\tilde{Z}$ on G with boundary function ϕ by Theorem 13.4 of [2]. Hence

$$g(z) = \tilde{E}^z[\phi(\tilde{Z}(\tilde{T}(G)))].$$

By Example (4.9), f is given by (4.17). □

5. The random evolution process. Let E, F be semicompacts and let $S = E \times F$.

If $C \subset S$ and $y \in F$, C_y denotes the cross section $\{x \in E:(x, y) \in C\}$. Similarly, if f is a function on S and $y \in F$, f_y denotes the function on E defined by $f_y(x) = f(x, y)$. With the product topology $\mathscr{C}(S)$, S is a semicompact. We will also need the topology $\mathscr{C}'(S) = \{G \subset S: G_y \in \mathscr{C}(E) \text{ for } y \in F\}$. If F is countable, then F is automatically given the discrete topology so that $\mathscr{C}(S) = \mathscr{C}'(S)$. In general, however, $\mathscr{C}(S) \subset \mathscr{C}'(S)$.

For $y \in F$, let X_y be a standard, conservative, continuous process with state space $(E, \mathscr{B}(E))$. We assume that for $C \in \mathscr{B}(S)$,

$$(5.1) \qquad (x, y) \to P_y^x[X_y(t) \in C_y] \in \mathscr{B}(S)/\mathscr{B}[0, 1].$$

Let $q \in \mathscr{B}(S)/\mathscr{B}(0, \infty)$ be bounded and such that q_y is continuous on E for each $y \in F$. Let M_y be the multiplicative functional of X_y given by

$$M_y(t) = \exp\left(-\int_0^t q_y(X_y(s))\, ds\right).$$

Let $\tilde{X}_y$ be the killed process associated with X_y and M_y (see Chapter III of [1]). Let $\tilde{Z} = (\tilde{X}, \tilde{Y})$ be the composite process on $(S, \mathscr{B}(S))$ associated with $\{\tilde{X}_y: y \in F\}$ i.e., $(\tilde{Z}(t), t < \tilde{\zeta}, \tilde{P}^{(x,y)})$ is equivalent to $((\tilde{X}_y, y), t < \tilde{\zeta}_y, \tilde{P}_y^x)$ for $(x, y) \in S$. $\tilde{Z}$ will be the base process.

Let Q be a probability kernel on $(S, \mathscr{B}(S))$ such that for $(x, y) \in S$,

$$(5.2) \qquad Q((x, y), E \times \{y\}) = 0.$$

We will also denote by Q the corresponding operator

$$Qf(z) = \int_S Q(z, du) f(u)$$

defined on a domain of functions in $\mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$. Define an instantaneous distribution μ for $\tilde{Z}$ by

$$(5.3) \qquad \mu(\tilde{\omega}, C) = Q(\tilde{Z}(\tilde{\zeta}-)(\tilde{\omega}), C).$$

The revival process $Z = (X, Y)$ associated with $\tilde{Z}$ and μ is called the random evolution process (REP) associated with the basic data $\{X_y: y \in F\}$, q, and Q (see [14] for details). Condition (5.1) is necessary for the existence of $\tilde{Z}$ and Z. Note that (5.1) is automatic if F is countable.

Z has the following intuitive description: X switches at random among the processes X_y, $y \in F$ and $Y = y$ whenever X is evolving according to X_y. The function q is a rate function for the random times between jumps. The kernel Q is the probability law for the change in evolution rule and the change in evolution state at a jump time. Feedback is incorporated by allowing q and Q to depend on the evolution state x. If Q satisfies

$$(5.4) \qquad Q((x, y), \{x\} \times F) = 1$$

for $(x, y) \in S$ then X is continuous and we will say that the evolution is continuous (but of course Z is still discontinuous because of jumps in Y).

In this setting, it is natural to consider the characteristic operator of $\tilde{Z}$ and Z in the $\mathscr{C}'(S)$ topology. That is, in definitions (2.1), (3.1) with $z = (x, y)$, V is restricted to be of the form $V = U \times \{y\}$ where U is an open neighborhood of x in E with compact closure (for $\tilde{Z}$, the characteristic operators in the $\mathscr{C}(S)$ and $\mathscr{C}'(S)$ topologies are the same since the second component of $\tilde{Z}$ is constant). Note then that condition (3.3) is true at each $z \in S$ by (5.2).

(5.5) THEOREM. *Let $f \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ and $z = (x, y) \in S$. If $(Qf)_y$ is defined in a neighborhood of x and is continuous at x, then $f \in \mathscr{D}(\mathscr{U}; (x, y))$ if and only if $f_y \in \mathscr{D}(\tilde{\mathscr{U}}_y; x)$ and*

$$\mathscr{U}f(x, y) = \tilde{\mathscr{U}}_y f_y(x) + q(x, y) Qf(x, y).$$

Moreover, if $f_y \in \mathscr{D}(\mathscr{U}_y; x)$ *then* $f_y \in \mathscr{D}(\tilde{\mathscr{U}}_y; x)$ *and*

$$\tilde{\mathscr{U}}_y f_y(x) = \mathscr{U}_y f_y(x) - q(x, y) f(x, y).$$

PROOF. From Theorem (3.2), to prove the first statement it suffices to show that $f \in \mathscr{D}(\mathscr{V}; (x, y))$ and

$$\mathscr{V} f(x, y) = q(x, y) Q f(x, y).$$

Let $V = U \times \{y\}$ with U an open neighborhood of x with compact closure. Then from the construction of Z,

$$E^z[f(Z(\tau_1)); T(V) = \tau_1] = E_y^x\left[\int_0^{T_y(U)} q_y(X_y(t))(Qf)_y(X_y(t)) M_y(t)\, dt\right]$$

and

$$E^z[T(V)] = \tilde{E}_y^x \tilde{T}(U) = E_y^x\left[\int_0^{T_y(U)} M_y(t)\, dt\right].$$

Here the desired result follows from the definition of $\mathscr{V}$ and the continuity assumptions on q_y and $(Qf)_y$. The second statement in Theorem (5.5) follows from a result in [5]. □

With the $\mathscr{C}'(S)$ topology, Definition (4.3) is modified by taking G and V nearly Borel and $\mathscr{C}'(S)$ open (as before, the distinction is unimportant for $\tilde{Z}$).

(5.6) LEMMA. *Let* $G \in \mathscr{C}'(S)$ *be nearly Borel. If there exists* $t > 0$ *and* $\varepsilon > 0$ *such that*

$$P_y^x[T_y(G_y) < t] > \varepsilon \tag{5.7}$$

for $(x, y) \in G$, *then* (4.2) *holds.*

PROOF. Let $z = (x, y) \in G$. Then

$$\begin{aligned}\tilde{P}^z[\tilde{T}(G) < \tilde{\zeta}] &= \tilde{P}_y^x[\tilde{T}_y(G_y) < \tilde{\zeta}_y] = E_y^x[M_y(T_y(G_y))] \\ &\geq E_y^x[\exp(-\|q\| T_y(G_y)); T_y(G_y) < t] \\ &\geq \varepsilon\, e^{-\|q\| t}. \;\square\end{aligned}$$

Let $G \in \mathscr{C}'(S)$ be nearly Borel and suppose G_y has compact closure for each y. Suppose that (5.7) is satisfied and that G_y has regular boundary for X_y (equivalently $\tilde{X}_y$) for each y. Suppose that X_y (hence $\tilde{X}_y$) is strong Feller and $\mathscr{C}(E)$ for each y. Then the conclusions of Theorems (4.6), (4.11), (4.12), (4.14), and (4.16) hold except that limit and continuity statements must be interpreted relative to the $\mathscr{C}'(S)$-topology. In particular, if Q preserves the space of bounded, nearly Borel, $\mathscr{C}'(S)$ continuous functions, then Theorem (4.6) gives a one-to-one correspondence between the bounded solutions of the "coupled" equation

$$\tilde{\mathscr{U}}_y f_y(x) + q(x, y) Q f(x, y) = 0, \quad (x, y) \in \dot{G}$$

and the "uncoupled" equation

$$\tilde{\mathscr{U}}_y g_y(x) = 0, \quad (x, y) \in G.$$

An interesting special case occurs if the evolution is continuous ((5.4)) and $G = U \times F$ with U open in E. In this case, a harmonic function need only be defined on $\bar{G} = \bar{U} \times F$ and the theory in Section 4 is more analogues to the classical theory.

6. The branching process. Let D be a compact Hausdorff space with a countable base and let $S = \bigcup_{n=0}^{\infty} D^n$ where $D^0 = \{\partial\}$ (∂ a new point) and where D^n is the n-fold symmetric product of D for $n \geq 1$. For $f \in \mathscr{B}^*(D)/\mathscr{B}(\mathbb{R})$ define $\hat{f} \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ by $\hat{f}(\partial) = 1$ and $\hat{f}[x_1, x_2, \cdots, x_n] = f(x_1)f(x_2)\cdots f(x_n)$ for $[x_1, x_2, \cdots, x_n] \in D^n$.

Let X_1 be a standard, continuous, conservative process on $(D, \mathscr{B}(D))$. Let $q \in \mathscr{B}(D)/\mathscr{B}(0, \infty)$ be bounded and continuous and let M_1 be the multiplicative functional of X_1 given by

$$M_1(t) = \exp\left(-\int_0^t q(X_1(s))\, ds\right).$$

Let $\tilde{X}_1$ be the killed process associated with X_1 and M_1. Let $\tilde{Z}$ be the direct sum over all n of the symmetric n-fold products of $\tilde{X}_1$ (see [10]). The process $\tilde{Z}$ with state space $(S, \mathscr{B}(S))$ will be the base process.

Let Q be a transition probability from $(D, \mathscr{B}(D))$ to $(S, \mathscr{B}(S))$ such that for $x \in D$,

$$Q(x, D) = 0. \tag{6.1}$$

We will also denote by Q the transformation from $\mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ functions to $\mathscr{B}^*(D)/\mathscr{B}(\mathbb{R})$ functions:

$$Qf(x) = \int_S Q(x, dz)f(z).$$

Let μ be the instantaneous distribution for $\tilde{Z}$ associated with Q as in [10].

The revival process Z associated with $\tilde{Z}$ and μ is called the branching process (BP) associated with X_1, q, and Q (see [10] for details). Z has the following intuitive descripton: Starting at $x \in D$, a particle moves according to X_1 for a random time τ_1 with rate function q. The particle then disappears or splits into a number of new particles with new positions according to the probability law Q.

Suppose Q satisfies

$$Q(x, \cdot) = \textstyle\sum_{n=0}^{\infty} p_n(x)\delta_{[x,x,\cdots,x]}(\cdot) \tag{6.2}$$

where $p_n \in \mathscr{B}(D)/\mathscr{B}[0, 1]$, and $\sum_{n=0}^{\infty} p_n(x) = 1$, $p_1(x) = 0$ for all x, and where $[x, x, \cdots, x]$ is the element of D^n all of whose coordinates are x. Then we say that the particles branch continuously.

Note that Condition (3.3) holds at each $x \in D \subset S$ by (6.1). The proof of the following theorem is similar to the proof of (5.5).

(6.3) THEOREM. *Let $f \in \mathscr{B}^*(S)/\mathscr{B}(\mathbb{R})$ and let $x \in D$. If Qf is defined in a neighborhood of x and continuous at x, then $f \in \mathscr{D}(\mathscr{U}; x)$ if and only if $f|_D \in \mathscr{D}(\tilde{\mathscr{U}}_1; x)$ and*

$$\mathscr{U}f(x) = \tilde{\mathscr{U}}_1 f(x) + q(x)Qf(x).$$

Moreover if $f \in \mathscr{D}(\mathscr{U}_1; x)$ then $f \in \mathscr{D}(\tilde{\mathscr{U}}_1; x)$ and

$$\tilde{\mathscr{U}}_1 f(x) = \mathscr{U}_1 f(x) - q(x)f(x).$$

(6.4) REMARK. In (6.3), if Q has the form given in (6.2), and $f = \hat{g}$ for $g \in \mathscr{B}^*(D)/\mathscr{B}(\mathbb{R})$ then we have

$$\mathscr{U}\hat{g}(x) = \tilde{\mathscr{U}}_1 g(x) + q(x)\textstyle\sum_{n=0}^{\infty} p_n(x)g^n(x).$$

The proof of the following is identical to the proof of (5.6).

(6.5) LEMMA. *Let G be open in D (and hence S). If there exists $t > 0$ and $\varepsilon > 0$ such that*

$$P_1^x[T_1(G) < t] > \varepsilon \tag{6.6}$$

for $x \in G$. Then (4.2) holds.

Now let G be open in D. Suppose that (6.6) holds and that G has regular boundary for X_1. Suppose that X_1 is strong Feller (and hence also $\hat{\mathscr{C}}(D)$). Note that $(\tilde{Z}, \tilde{P}^x)$ is equivalent to $(\tilde{X}_1, \tilde{P}_1^x)$ for $x \in D$. The conclusions of Theorems (4.6), (4.11), (4.12), (4.14) and (4.16) hold.

An interesting special case occurs if the particles branch continuously ((6.2)). In this case, a function harmonic on $G \subset D$ need only be defined on $\cup_{n=0}^{\infty} (\bar{G})^n$.

Nagasawa in [11] considers the following setting: Let X_0 be a standard, continuous, conservative, strong Feller process on a semicompact $(E, \mathscr{B}(E))$. Let G be open in E with compact closure. Suppose (6.6) holds for X_0 on G and that G has regular boundary for X_0. Let $D = \bar{G}$ and let X_1 be the process on D obtained by stopping X_0 on ∂G. Let Q satisfy (6.2) (continuous branching) and suppose further that p_n is continuous for each n. Now let $\tilde{X}_1, \tilde{Z}, Z$ be as in this section relative to this X_1. Let ϕ be a continuous function on ∂G with $\|\phi\| \leq 1$. Then $\hat{\phi}$ is defined on $\cup_{n=0}^{\infty} (\partial G)^n$ and $\|\hat{\phi}\| \leq 1$. Nagasawa studies the function

$$f(z) = E^z[\hat{\phi}(V)] \tag{6.7}$$

where V is the random vector of positions of all particles when they hit (and stop at) the boundary of G. Using a semigroup representation of f rather than (6.7), he shows that f solves the nonlinear boundary value problem

$$\mathscr{U}_0 f(x) - q(x)f(x) + q(x) \textstyle\sum_{n=0}^{\infty} p_n(x) f^n(x) = 0, \quad x \in G \tag{6.8}$$

$$\lim_{x \to a, x \in G} f(x) = \phi(a), \quad a \in \partial G.$$

Let g be defined from f by (4.8). It is not hard to show that for $x \in G$,

$$g(x) = \tilde{E}_1^x[\hat{\phi}(\tilde{X}_1(\tilde{T}_1(G)))].$$

Thus, f is harmonic for Z on G in the sense of Definition (4.3). Moreover, f is continuous on G and (6.8) is just $\mathscr{U} f(x) = 0$, $x \in G$. Also, g satisfies the linear boundary value problem

$$\mathscr{U}_0 g(x) - q(x)g(x) = 0, \quad x \in G$$

$$\lim_{x \to a, x \in G} g(x) = \phi(a), \quad a \in \partial G.$$

It is important to realize, however, that f is strongly influenced by the values of g on $S - G$. On $S - G$, there does not seem to be a simple way to express g in terms of the base process $\tilde{Z}$.

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