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EXIT TIMES FOR SYMMETRIC STABLE PROCESSES IN $\mathbb{R}^n$

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Let X_t be a symmetric stable process of index α in $\mathbb{R}^n$ and $\tau = \inf\{t: X_t \notin D\}$ where D is a connected open region in $\mathbb{R}^n$. If $0 < p < \alpha$ two sided L^p inequalities are obtained between $\tau^{1/\alpha}$ and the maximal function $X_\tau^* = \sup_{t<\tau}|X_t|$. Analytic conditions for $\tau^{1/\alpha} \in L^p$ are given in terms of domination of $|x|^p$, $x \in D^c$ by a function $u(x)$ α-harmonic in D. Also, the boundary behavior of α-harmonic functions is studied by obtaining two-sided L^p inequalities, $0 < p < \infty$, between a random and deterministic maximal function of non-negative α-harmonic functions.

1. Introduction. Let X be a Brownian motion in $\mathbb{R}^n$, D an open connected subset of $\mathbb{R}^n$, $x \in D$ and define

$$\tau = \inf\{t > 0: X_t \notin D\}.$$

If $X_T^* = \sup_{0\le t\le T}|X_t|$ for T a stopping time, we have the following results of Burkholder (1977).

(1) Given $0 < p < \infty$, $E^x\tau^{p/2} < \infty$ if and only if there is a function u harmonic in D such that $|x|^p \le u(x)$, $x \in D$.

(2) Given a continuous function $\Phi: [0, \infty] \to [0, \infty]$ with $\Phi(0) = 0$ and

$$\Phi(2\lambda) \le a\Phi(\lambda), \quad \lambda > 0$$

and T a stopping time of X, then

$$cE^x\Phi([T + |x|^2]^{1/2}) \le E^x\Phi(X_T^*) \le CE^x\Phi([T + |x|^2]^{1/2})$$

where c and C depend only on Φ and n.

The symmetric stable processes of index α, $0 < \alpha < 2$, share scaling properties similar to Brownian motion but fail to have continuous paths. These properties, scaling and sample path continuity, were key ingredients in Burkholder's proof of 2). In the present work 1) and 2) are extended to stable processes of index α. This is done in Section 3.

In another direction, we investigate boundary behavior of α-harmonic functions on the unit ball in $\mathbb{R}^n$. The main result of Section 4 gives an L^p equivalence, $0 < p < \infty$, between the symmetric stable maximal function

$$u^*(\omega) = \sup_{t<\tau}|u(X_t(\omega))|,$$

where $\tau = \inf\{t > 0: |X_t| > 1\}$, and X is a symmetric stable process of index α, and the maximal function

$$N_\sigma u(x) = \sup_{y\in\Gamma_\sigma(x)}|u(y)|, \quad |x| > 1,$$

where $\Gamma_\sigma(x)$ is a "Stolz-like" domain in $B(0, 1) = \{y: |y| < 1\}$. When u is positive and α-harmonic in $B(0, 1)$ there is a two-sided L^p-inequality between u^* and N_σ. This is analogous to the Brownian motion case treated in Burkholder, Gundy and Silverstein (1971).

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2. Preliminaries. We introduce some notation and a few known results. Let (X_t, P^x) be a symmetric stable process of index α, $0 < \alpha < 2$, with values in $\mathbb{R}^n$. This process has

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infinitesimal generator $\Delta^{\alpha/2}$ where

$$\Delta^{\alpha/2}u(x) = A(n, \alpha)\int_{R^n} \frac{u(x+y)-u(x)}{|y|^{n+\alpha}}\,dy, \quad A(n, \alpha) = \pi^{\alpha-n/2}\Gamma((n-\alpha)/2)/\Gamma(\alpha/2). \tag{2.1}$$

A function u is called α-harmonic in an open set $D \subset R^n$ if $\Delta^{\alpha/2}u(x) = 0$ for $x \in D$. Note first that u must be defined in all of R^n and secondly u should satisfy $\int_{|y|>1} (|u(y)|/|y|^{n+\alpha})\,dy < \infty$. The justification for calling the generator $\Delta^{\alpha/2}$ is due to M. Riesz (1938).

The process X_t has jumps, so if $B(0, r) = \{x : |x| < r\}$ and $T = \inf\{t > 0 : X_t \notin B(0, r)\}$, then the P^x distribution of X_T has density

$$\hat{P}_r(x, y) = \Gamma\left(\frac{n}{2}\right)\pi^{-n/2-1}\sin\frac{\pi\alpha}{2}\cdot\left[\frac{r^2-|x|^2}{|y|^2-r^2}\right]^{\alpha/2}|x-y|^{-n} \tag{2.2}$$

for $|x| < r$, $|y| > r$. Similarly when $S = \inf\{t > 0 : X_t \in B(0, r)\}$ the P^x distribution of X_S has density

$$\check{P}_r(x, y) = \Gamma\left(\frac{n}{2}\right)\pi^{-n/2-1}\sin\frac{\pi\alpha}{2}\left[\frac{|x|^2-r^2}{r^2-|y|^2}\right]^{\alpha/2}|x-y|^{-n}, \tag{2.3}$$

for $|x| > r$, $|y| < r$. These, of course, are the analogues of the Poisson kernel for the interior and exterior of the sphere respectively. The justification for this may be found, for example, in N. S. Landkof (1972).

We shall need the following two consequences of (2.2) and (2.3). The first was observed by M. Riesz (1938). The proof follows exactly as in the classical case of $\alpha = 2$ and will be omitted.

LEMMA 2.1. (*Harnack's Inequality*). *If u is positive and α-harmonic in $B(0, r)$, then*

$$u(x) \le \left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2}\left|\frac{r-|x|}{r+|y|}\right|^{-n}u(y)$$

for each $x, y \in B(0, r)$.

LEMMA 2.2. *If $|x| = R \ge 2r > 0$ there exists a constant $c = c(\alpha, n) > 0$ such that*

$$P^x[|X_t| < r \text{ for some } t > 0] \le c\left(\frac{r}{R}\right)^{n-\alpha}.$$

PROOF. Using (2.3)

$$P^x(|X_t| < r \text{ for some } t > 0) = \Gamma\left(\frac{n}{2}\right)\pi^{-n/2-1}\sin\frac{\pi\alpha}{2}\int_{B(0,r)}\left[\frac{|x|^2-r^2}{r^2-|y|^2}\right]^{\alpha/2}|x-y|^{-n}\,dy.$$

Observing that

$$|x|^2 - r^2 \le R^2, \quad |y| \le r,$$

and

$$|x-y| \ge |x| - |y| \ge R - r \ge R/2,$$

we switch to polar coordinates and get

$$P^x(|X_t| < r \text{ for some } t > 0) \le c(\alpha, n)R^\alpha\left(\frac{2}{R}\right)^n r^{n-2}\int_0^r \frac{s}{[r^2-s^2]^{\alpha/2}}\,ds$$

$$\le c(\alpha, n)\left(\frac{r}{R}\right)^{n-\alpha} \text{ as desired. } \square$$

DEFINITION 2.3. A continuous nondecreasing function $\Phi: [0, \infty] \to [0, \infty]$, $\Phi(0) = 0$, is said to have moderate growth if there is a constant c such that $\Phi(2\lambda) \le c\Phi(\lambda)$ for all $\lambda > 0$. A function $\Phi: [0, \infty) \to [0, \infty)$ will be said to grow more slowly than λ^p, $p > 0$, if there are constants c and $q < p$ such that $\Phi(a\lambda) < ca^q\Phi(\lambda)$ for all $\lambda > 0$ and all a larger than some a_0.

The following lemma due to Burkholder (1973) shall be useful in what follows; we refer the reader to the above reference for the proof.

LEMMA 2.4. *Suppose that f and g are nonnegative measurable functions on a probability space and $\beta > 1$, $\delta > 0$, $\varepsilon > 0$ are real numbers such that*

$$P(g > \beta\lambda, f \le \delta\lambda) \le \varepsilon P(g > \lambda), \quad \lambda > 0. \tag{2.4}$$

Let γ and η be real numbers satisfying

$$\Phi(\beta\lambda) \le \gamma\Phi(\lambda), \Phi(\delta^{-1}\lambda) \le \eta\Phi(\lambda), \quad \lambda > 0. \tag{2.5}$$

Finally, suppose that $\gamma\varepsilon < 1$. Then

$$E\Phi(g) \le \gamma\eta(1 - \gamma\varepsilon)^{-1}E\Phi(f). \tag{2.6}$$

Condition (2.5) *is satisfied when Φ has moderate growth.*

3. Exit times of symmetric stable processes in $\mathbb{R}^n$. Throughout this section X_t is a symmetric stable process of index α, $0 < \alpha < 2$, in $\mathbb{R}^n$, $n \ge 2$. The case $n = 1$ will not be considered here. Define $X_t^* = \sup_{s \le t}|X_s|$, $X_{t-}^* = \sup_{s<t}|X_s|$, and we may similarly define X_T^* and X_{T-}^* for stopping times T.

THEOREM 3.1. *There exist constants c_1, c_2, c_3, c_4 depending only on n, α and Φ such that if Φ is of moderate growth and T is a finite stopping time for X, then*

$$E^x\Phi((|x|^\alpha + T)^{1/\alpha}) \le c_1E^x\Phi(X_{T-}^*) \le c_2E^x\Phi(|X_{T-}|) \le c_2E^x\Phi(X_T^*) \le c_3E^x\Phi(|X_T|). \tag{3.1}$$

If, in addition, Φ grows more slowly than λ^α,

$$E^x\Phi(X_T^*) \le c_4E^x\Phi((|x|^\alpha + T)^{1/\alpha}). \tag{3.2}$$

PROOF. It will suffice to prove

$$E^x\Phi((T + |x|^\alpha)^{1/\alpha}) \le cE^x\Phi(X_{T-}^*), \tag{3.3}$$

$$E^x\Phi(X_{T-}^*) \le cE^x\Phi(|X_{T-}|), \tag{3.4}$$

$$E^x\Phi(X_T^*) \le cE^x\Phi(|X_T|), \tag{3.5}$$

and if Φ grows more slowly than λ^α

$$E^x\Phi(X_T^*) \le cE^x\Phi((T + |x|^\alpha)^{1/\alpha}), \tag{3.6}$$

where c is a positive constant not necessarily the same in each instance, but depending only upon n, α and Φ. The inequality (2.4) will be proven in each case (a variant in the case of (3.6)), and the results will then follow from Lemma 2.4.

PROOF OF (3.3). We need to establish that for $\beta > 1$, $\delta > 0$,

$$P^x[(T + |x|^\alpha)^{1/\alpha} > \beta\lambda, X_{T-}^* \le \delta\lambda] \le c(\delta, \beta)P^x[(T + |x|^\alpha)^{1/\alpha} > \lambda]$$

where $c(\beta, \delta) \to 0$ when either $\beta \to \infty$ or $\delta \to 0$. When $|x| > \delta\lambda$ the left-hand side is zero so assume $|x| \le \delta\lambda$. Setting $a = \lambda^\alpha - |x|^\alpha$,

$$b = (\beta\lambda)^\alpha - |x|^\alpha, \text{ (take } \delta < 1 \text{ so that } a > 0)$$

$$P^x[(T+|x|^\alpha)^{1/\alpha} > \beta\lambda, X^*_{T-} \le \delta\lambda] = P^x[T > b, X^*_{T-} \le \delta\lambda]$$
$$\le P^x[T > a, \sup_{a\le t<b}|X_t - X_a| \le 2\delta\lambda]$$
$$= E^x[P^{X_a}[\sup_{0\le t<(b-a)}|X_t - X_0| \le 2\delta\lambda]; T > a].$$

The stability of X implies

$$\sup_{y\in R^n} P^y[\sup_{0\le t<b-a}|X_t - X_0| \le 2\delta\lambda] = \sup_{y\in R^n} P^y\left[\sup_{0\le t<1}|X_t - X_0| \le \frac{2\delta}{(\beta^\alpha - 1)^{1/\alpha}}\right]$$
$$= P^0\left[X^*_{1-} \le \frac{2\delta}{(\beta^\alpha - 1)^{1/\alpha}}\right] = c(\beta, \delta, n).$$

Thus

$$P^x[(T+|x|^\alpha)^{1/\alpha} > \beta\lambda, X^*_{T-} \le \delta\lambda] \le c(\beta, \delta, n) P^x[(T+|x|^\alpha)^{1/\alpha} > \lambda], \quad \text{since } \beta > 1.$$

Observe that $c(\beta, \delta, n) \to 0$ as either $\beta \to \infty$ or $\delta \to 0$, so Lemma 2.4 applies and (3.3) is proven.

Proof of (3.4). In order to establish

$$P^x[X^*_{T-} > \beta\lambda, |X_{T-}| \le \delta\lambda] \le c(\beta, \delta) P^x[X^*_{T-} > \lambda],$$

begin by defining $U = \inf\{t > 0 : |X_{T\wedge t}| > \lambda\}$, and let $\beta = 2$. If $U < T$, then $X^*_{T-} > \lambda$, and on the set $\{X^*_{T-} > \beta\lambda, |X_{T-}| \le \delta\lambda\}$, $U < T < \infty$. Using the strong Markov property at U,

$$P^x[X^*_{T-} > \beta\lambda, |X_{T-}| \le \delta\lambda] = P^x[X^*_{T-} > \beta\lambda, |X_{T-}| \le \delta\lambda, U < T]$$
$$\le E^x[P^{X_U}[|X_t| \le \delta\lambda \text{ for some } t > 0]; U < T]$$
$$\le \sup_{|y|\ge\lambda} P^y[|X_t| \le \delta\lambda \text{ for some } t > 0] P^x[U < T].$$

By Lemma 2.2,

$$\sup_{|y|\ge\lambda} P^y[|X_t| \le \delta\lambda \text{ for some } t > 0] \le c(\alpha, n)\delta^{n-\alpha}.$$

Thus

$$P^x[X^*_{T-} > \beta\lambda, |X_{T-}| \le \delta\lambda] \le c(\alpha. n)\delta^{n-\alpha} P^x[X^*_{T-} > \lambda],$$

and on letting δ be sufficiently small, Lemma 2.4 applies, and (3.4) is proven.

Proof of (3.5). Let U be as defined in the proof of (3.4), $\beta > 1$, $0 < \delta < 1$. On the set $\{X^*_T > \beta\lambda, |X_T| \le \delta\lambda\}$, $U < T$. Hence

$$P^x[X^*_T > \beta\lambda, |X_T| \le \delta\lambda] \le E^x[P^{X_U}[|X_t| \le \delta\lambda \text{ for some } t > 0]; U < T]$$
$$\le \sup_{|y|\ge\lambda} P^y[|X_t| \le \delta\lambda \text{ for some } t > 0] P^x[U < T]$$
$$\le c(\alpha, n)\delta^{n-\alpha} P^x[U < T]$$
$$= c(\alpha, n)\delta^{n-\alpha} P^x[X^*_T > \lambda]$$

and since $c(\alpha, n)\delta^{n-\alpha} \to 0$ as $\delta \to 0$, (3.5) follows.

Proof of (3.6). Here it is assumed that Φ grows more slowly than λ^α. Take $\beta < 6$, $\delta < 1$. With U the same as before, $U < \infty$ on the set $\{X^*_T > \beta\lambda\}$, and on $\{U < T\}$, $X^*_T > \lambda$.

If $|x| > \delta\lambda$, then

$$P^x(X^*_T > \beta\lambda, (T+|x|^\alpha)^{1/\alpha} \le \delta\lambda) = 0.$$

If $|x| \le \delta\lambda$, then

$$P^x[X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} \le \delta\lambda] \le P^x\left[X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} \le \delta\lambda, |X_U| \le \frac{\beta}{2}\lambda\right]$$
$$+ P^x\left[X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} \le \delta\lambda, |X_U| > \frac{\beta}{2}\lambda\right]$$
$$= I + J.$$

Considering I first with $a = (\delta\lambda)^\alpha - |x|^\alpha$,

$$I = P^x\left[X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} \le \delta\lambda, |X_U| \le \frac{\beta}{2}\lambda, U < T\right]$$
$$\le P^x\left[\sup_{U \le t \le U+a} |X_t - X_U| > \frac{\beta}{2}\lambda, U < T\right]$$
$$(3.7) \qquad = E^x\left[P^{X_U}\left[\sup_{t \le a} |X_t - X_0| > \frac{\beta}{2}\lambda\right]; U < T\right]$$
$$\le \sup_y P^y\left[\sup_{t \le a} |X_t - X_0| > \frac{\beta}{2}\lambda\right] P^x[U < T]$$
$$\le c_1(\beta, \delta, n) P^x[X_T^* > \lambda], \; c_1(\beta, \delta, n) = P^0(X_1^* > \delta^{-1}\beta/2).$$

In considering J, note that if $n(dx)$ is the Lévy measure for X_t, then

$$n(B_R^c) = c \int_{B_R^c} |y|^{-(n+\alpha)}\, dy = c(R^{-\alpha})$$

where B_R is the ball of radius R centered at 0.

Now if $Y_t = \sum_{s \le t} 1_{(|\Delta X_s| > R)}$, then Y_t is a Poisson process, $Y_t - n(B_R^c)t$ is a martingale, and by optional sampling, if τ is a bounded stopping time, $E^x Y_\tau = E^x \sum_{s \le \tau} 1_{(|\Delta X_s| > R)} = n(B_R^c) E^x \tau$.

Let $\tau = U \wedge T \wedge a$, $(\delta\lambda)^\alpha - |x|^\alpha = a$. If $s < \tau \le U$, $|\Delta X_s| < 2\lambda$ by the definition of U. Thus, provided $R > 2\lambda$,

$$P^x[|\Delta X_\tau| > R] = E^x \sum_{s \le \tau} 1_{(|\Delta X_s| > R)} = n(B_R^c) E^x \tau.$$

Since $\tau \le T$, $X_T^* \ge |X_\tau|$, and consequently,

$$J = P^x\left[X_T^* > \beta\lambda, T \le a, |X_U| > \frac{\beta}{2}\lambda, U \le T\right] \le P^x\left[|\Delta X_\tau| > \left(\frac{\beta}{2} - 1\right)\lambda\right]$$
$$= n(B^c_{(\beta/2-1)\lambda}) E^x \tau = 3^\alpha\left(\frac{\beta}{2} - 1\right)^{-\alpha} n(B^c_{3\lambda}) E^x \tau = 3^\alpha\left(\frac{\beta}{2} - 1\right)^{-\alpha} P^x[|\Delta X_\tau| > 3\lambda]$$
$$\le 3^\alpha\left(\frac{\beta}{2} - 1\right)^{-\alpha} P^x[|X_\tau| > \lambda] \le 3^\alpha\left(\frac{\beta}{2} - 1\right)^{-\alpha} P^x[X_T^* > \lambda].$$

The estimates for I and J yield,

$$P^x(X_T^* > \beta\lambda) = P^x(X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} \le \delta\lambda) + P^x(X_T^* > \beta\lambda, (T + |x|^\alpha)^{1/\alpha} > \delta\lambda)$$
$$\le (c_1(\beta, \delta, n) + c_2(\beta, \alpha)) P^x(X_T^* > \lambda) + P^x((T + |x|^\alpha)^{1/\alpha} > \delta\lambda)$$

where $c_1(\beta, \delta, n)$ is given by (3.7) and $c_2(\beta, \alpha) = 3^\alpha(\beta/2 - 1)^{-\alpha}$. Recall that Φ grows more slowly than λ^α, so there are constants c and $p < \alpha$ such that $\Phi(a\lambda) \le ca^p\Phi(\lambda)$ for all $\lambda > 0$, and all a larger than some a_0. Arguing in a manner similar to Burkholder (1973),

$$E^x\Phi(\beta^{-1}X_T^*) = \int_0^\infty P^x(X_T^* > \beta\lambda)\, d\Phi(\lambda) \le (c_1(\beta, \delta, n) + c_2(\beta, \alpha)) \int_0^\infty P^x(X_T^* > \lambda)\, d\Phi(\lambda)$$
$$+ \int_0^\infty P^x((T + |x|^\alpha)^{1/\alpha} > \delta\lambda)\, d\Phi(\lambda)$$

$$= (c_1(\beta, \delta, n) + c_2(\beta, \alpha))E^x\Phi(X_T^*) + E^x\Phi(\delta^{-1}(T + |x|^\alpha)^{1/\alpha})$$

$$\leq (c_1(\beta, \delta, n) + c_2(\beta, \alpha))E^x\Phi(X_T^*) + c\delta^{-p}E^x\Phi((T + |x|^\alpha)^{1/\alpha}).$$

On the other hand,

$$E^x\Phi(X_T^*) = E^x\Phi(\beta\beta^{-1}X_T^*) \leq c\beta^p E^x\Phi(\beta^{-1}X_T^*)$$

and consequently

$$(c^{-1}\beta^{-p} - (c_1(\beta, \delta, n) + c_2(\beta, \alpha)))E^x\Phi(X_T^*) \leq c\delta^{-p}E\Phi((T + |x|^\alpha)^{1/\alpha}).$$

The theorem will be proved provided the constant on the left can be made positive by appropriate choices of β and δ. Since $c_2(\beta, \alpha) = 3^\alpha(\beta/2 - 1)^{-\alpha}$ and $\alpha > p$, we can achieve this by first taking β large, then δ very small. □

EXAMPLES. 1. The condition that Φ grow more slowly than λ^α is necessary in (3.6) as the following example shows. If not,

$$E^x\Phi(X_T) \leq cE^x\Phi(X_{T-})$$

for all Φ of moderate growth. Considering the case $D = \{z : |z| < 1\}$ and $T = \inf\{t > 0 : X_t \notin D\}$, $|X_{T-}| < 1$, and with $\Phi(\lambda) = \lambda^p$, $E^x|X_{T-}|^p < \infty, p > 0, x \in \mathbb{R}^n$. On the other hand, using the Poisson kernel (2.2) $E^x|X_T|^p = \infty$ for $p \geq \alpha$, $x \in D$.

2. The inequality $E^x\Phi(X_{T-}) \leq cE^x\Phi((T + |x|^\alpha)^{1/\alpha})$ also requires that Φ grow more slowly than λ^α as the following example shows. Take

$$D = \{z : |z_1 - m| < \varepsilon \text{ for some integer } m\},$$

where $z = (z_1, \cdots, z_n)$ and $\frac{1}{2} > \varepsilon > 0$ is fixed. With

$$T = \inf\{t : X_t \notin D\}$$

we show $E^xX_{T-}^{*p} = \infty$ yet $E^x(T + |x|^\alpha)^{p/\alpha} < \infty$ for $p \geq \alpha$. Since $X_t = (X_t^1, \cdots, X_t^n)$ exits D when $X_t^1 = Y_t$ exits D_1,

$$D_1 = \{\omega \in \mathbb{R} : |\omega - m| < \varepsilon \text{ for some integer } m\},$$

and $X_{T-}^* \geq Y_{T-}^*$, it suffices to consider the one dimensional symmetric stable process of index α, Y_t. By a result of Watanabe (1962) if $S = \inf\{t > 0 : Y_t \notin (-\varepsilon, \varepsilon)\}$ then

$$P^x[Y_s \in dy] = \pi^{-1}\sin\frac{\alpha\pi}{2}\left(\frac{\varepsilon^2 - x^2}{y^2 - \varepsilon^2}\right)^{\alpha/2} |y - x|^{-1}\, dy$$

for $|x| < \varepsilon$, $|y| \geq \varepsilon$. Letting $x = 0$

$$E^0Y_{T-}^{*p} \geq \pi^{-1}\sin\frac{\alpha\pi}{2}\int_{D_1\cap(-\varepsilon,\varepsilon)^c} |y|^{p-1}(y^2 - \varepsilon^2)^{-\alpha/2}\, dy$$

and the right-hand side is infinite for $p \geq \alpha$.

Define $Z(y) = k$ if $k - \frac{1}{2} < y \leq k + \frac{1}{2}$ for integer k. Let $U_0 = 0$, $U_1 = \inf\{t > 0 : |Y_t - Z(Y_0)| > \varepsilon\}$ and $U_{i+1} = U_i + U_1\circ\theta_{U_i}$. For any real y, let $I_y = (Z(y) - \varepsilon, Z(y) + \varepsilon)$. Then $\gamma = \sup_{x,y}P^x(Y_1 \in I_y) < 1$ and by the Markov property,

$$P^y(Y_1 \in I_y, \cdots, Y_n \in I_y) = E^y(P^{Y_{n-1}}(Y_1 \in I_y); Y_1 \in I_y, \cdots, Y_{n-1} \in I_y)$$

$$\leq \gamma P^y(Y_1 \in I_y, \cdots, Y_{n-1} \in I_y) \leq \gamma^n.$$

Consequently,

$$P^y(U_1 > n) \leq P^y(Y_1 \in I_y, \cdots, Y_n \in I_y) \leq \gamma^n$$

and

$$\zeta_r = \sup_y E^y U_1^r \le \sum_n n^r \gamma^{n-1} < \infty, \quad E^y(U_1 \circ \theta_{U_i})^r = E^y(E^{Y_{U_i}}(U_1^r)) \le \zeta_r.$$

If $r \ge 1$, by Minkowski's inequality,

$$E^y(U_{i+1}^r)^{1/r} \le (E^y U_i^r)^{1/r} + (E^y(U_1 \circ \theta_{U_i})^r)^{1/r}.$$

By induction, $E^y U_i^r \le i^r \zeta_r$.

Using Watanabe's result

$$\lambda = \sup_y P^y(Y_{U_1} \in D_1) < 1,$$

and by an argument similar to the above

$$P^x(T > U_i) \le P^x(Y_{U_1} \in D_1, \cdots, Y_{U_i} \in D_1) \le \lambda^i.$$

Finally, suppose $p \ge \alpha$ and let $r = 2p/\alpha$. Cauchy-Schwartz implies

$$\begin{aligned} E^x T^{p/\alpha} &= \textstyle\sum_{i=1}^\infty E^x(U_i^{p/\alpha};\, T = U_i) \\ &\le \textstyle\sum_{i=1}^\infty (E^x(U_i^r))^{1/2}(P^x(T = U_i))^{1/2} \le \sum_{i=1}^\infty i^{r/2}\zeta_r^{1/2}(\lambda^{1/2})^{i-1} < \infty. \end{aligned}$$

As pointed out by the referee, $T \equiv 1$ provides another example of a stopping time (but not an exit time) where Φ growing slower than λ^α is required.

The next theorem is the analogue for symmetric stable processes of Theorem 3.1 in Burkholder (1977) for Brownian motion. In the Brownian motion case $|x|^p$ is subharmonic, so the majorization of $|x|^p$ by a harmonic function $u(x)$ on D is achieved if $u(x) \ge |x|^p$ for $x \in \partial D$. For symmetric stable processes D^c acts as the boundary, since the process has jumps, and the majorization condition must change accordingly. The proof is so similar to its Brownian analogue, Theorem 3.1, Burkholder (1977), that it is omitted.

THEOREM 3.2. *Let D be a region in $\mathbb{R}^n$ and $T = \inf\{t : X_t \notin D\}$. Then for $0 < p < \alpha$, $T^{1/\alpha} \in L^p$ if and only if there is a function u defined on $\mathbb{R}^n$ which is α-harmonic on D and $u(x) \ge |x|^p$ for all x.*

REMARK. If $T^{1/\alpha} \in L^p$ for some $p < \alpha$, then by Theorem 3.1, $E^x X_T^{*p} < \infty$ and $u(x) = E^x |X_T|^p$ defines a function which is α-harmonic on D and majorizes $|x|^p$. As the following example may show, finding such a u for a given domain D is not easy.

EXAMPLE (An open problem). Here we set $D = \{(r, \theta) : |\theta| < \theta_0\} \subset \mathbb{R}^2$. The problem is to find the critical value p' for which $T^{1/\alpha} \in L^p$ for $p < p'$ when θ_0 and α are given. To find suitable u, we need only consider $u(x)$ of the form

$$u(x) = |x|^p g(\theta) \tag{3.8}$$

for we know $u(x) = E^x |X_T|^p$ dominates $|x|^p$ and is α-harmonic in D when $T^{1/\alpha} \in L^p$; by the stability property of X_t such a u may be written $u(x) = |x|^p g(\theta)$. The condition that u be α-harmonic on D is exactly

$$\Delta^{\alpha/2} u(x) = c \int_{\mathbb{R}^2} \frac{u(x+y) - w(y)}{|y|^{2+\alpha}}\, dy = 0 \text{ for } x \in D$$

which after a change of variable and substitution of (3.8) becomes

$$\int_{-\pi}^{\pi} (g(\gamma + \theta) - g(\theta)) K_{p-\alpha}(\gamma)\, d\gamma + A_{p,\alpha} g(\theta) = 0, \quad |\theta| < \theta_0. \tag{3.9}$$

That $u(x) \ge |x|^p$ yields

$$g(\theta) \ge 1. \tag{3.10}$$

The quantities $K_{p,\alpha}(\gamma)$, $A_{p,\alpha}$ in (3.9) are given by

$$K_{p,\alpha}(\gamma) = \int_0^\infty \frac{\rho^{p+1}}{[\rho^2 - 2\rho\cos\gamma + 1]^{1+\alpha/2}}\,d\rho$$

$$A_{p,\alpha} = \int_{-\pi}^{\pi}\int_0^\infty \frac{\rho(\rho^p - 1)}{[\rho^2 - 2\rho\cos\gamma + 1]^{1+\alpha/2}}\,d\rho\,d\gamma.$$

The kernel $K_{p,\alpha}(\gamma)$ is the Mellin transform of $(\rho^2 - 2\rho\cos\gamma + 1)^{-(1+\alpha/2)}$; by Oberhettinger (1974),

$$K_{p,\alpha}(\gamma) = 2^{1+\alpha/2}\Gamma\left(\frac{3+\alpha}{2}\right)(\sin\gamma)^{-(1+\alpha/2)}B(p+2, \alpha - p)\cdot P^{-1+\alpha/2}_{p+(1-\alpha/2)}(\cos\gamma)$$

which is valid for $p < \alpha$.

Here $B(x, y)$ is the Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

and $P^\mu_\nu(z)$ is the Legendre function

$$P^\mu_\nu(z) = [\Gamma(1-\mu)]^{-1}\left(\frac{z+1}{z-1}\right)^{(1/2)\mu} {}_2F_1\left(-r, \nu+1; 1-\mu; \frac{1}{2} - \frac{1}{2}z\right),$$

${}_2F_1(a, b; c; z)$ is Gauss' hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}, \quad |z| < 1.$$

The problem (3.9)–(3.10) may be viewed as an eigenvalue problem for the integral operator determined by the first term in (3.9). This is analogous to the case $\alpha = 2$ treated by Burkholder (1977) where separation of variables as in (3.8) leads to the problem

$$g''(\theta) + p^2 g(\theta) = 0, \quad |\theta| < \theta_0 \tag{3.11}$$

$$g(\theta_0) = g(-\theta_0) = 1. \tag{3.12}$$

The authors do not know how to solve the problem (3.9)–(3.10). It would be of interest to solve this when $\alpha = 1$. T. McConnell has pointed out that the following Dirichlet problem for Δ in $\mathbb{R}^3$ is equivalent to solving (3.9)–(3.10) for $\alpha = 1$ and $D = \{(r, \theta) : |\theta| < \theta_0\}$. Consider the problem

$$\Delta u = 0 \text{ in } \mathbb{R}^3 \backslash F \tag{3.13}$$

$$u(x) = |x|^p, x \in F = \{(r, \theta, z) : \pi \geq |\theta| > \theta_0, z = 0\} \tag{3.14}$$

where $x = (r, \theta, z)$ is given in cylindrical coordinates. The solution has the stochastic representation

$$u(x) = E^x|B_T|^p$$

where B_t is a Brownian motion in $\mathbb{R}^3$ and $T = \inf\{t > 0 : B_t \in F\}$. The Brownian motion will hit F only if $B_t^3 = 0$; so it suffices to consider B_t only when $B_t^3 = 0$ or to consider $X_t = (B^1_{\gamma_t}, B^2_{\gamma_t})$ where γ_t is the inverse local time of B_t^3 at 0. Since γ_t has index ½ and is independent of (B_t^1, B_t^2), X_t is the symmetric Cauchy process, $\alpha = 1$, on $\mathbb{R}^2$. Problem (3.13)–(3.14) may be then replaced by (3.9)–(3.10). Thus the original problem in $\mathbb{R}^3$ is now a one dimensional problem. Whether this is a reduction is not clear.

4. An L^p equivalence between maximal functions. In this section we establish an L^p equivalence between a deterministic and a probabilistic maximal function for positive α-harmonic functions on the unit ball on $\mathbb{R}^n (n \geq 2)$.

Let X_t be symmetric stable, $T = \inf\{t > 0 : |X_t| > 1\}$ and define the probabilistic maximal function by

$$u^*(\omega) = \sup_{t<T} |u(X_t(\omega))|.$$

The deterministic maximal function requires a slightly different Stolz domain than in the classical case $\alpha = 2$. If we begin with $z \in \mathbb{R}^n$, $2 > |z| > 1$, define $w(z)$ to be the point on the ray through the origin and z such that $1 - |w(z)| = |z| - 1$. Now fix σ, $\frac{1}{2} > \sigma > 0$ and let

$$\Gamma_\sigma(z) = \text{the interior of the convex hull of } B(0, \sigma) \cup \{w(z)\}.$$

If $2 - \sigma > |z| > 1$, $\Gamma_\sigma(z)$ looks like the ordinary Stolz domain but does not touch $\{x : |x| = 1\}$. When $|z| \geq 2 - \sigma$, $\Gamma_\sigma(z) = B(0, \sigma)$. We set

$$N_\sigma u(z) = \sup_{x \in \Gamma_\sigma(z)} |u(x)|.$$

Finally, defining the measure

$$m(dy) = \hat{P}_1(0, y)\, dy,$$

we have the following theorem:

THEOREM 4.1. *There exist positive constants $c_{p,\alpha}$, $C_{p,\alpha}$, $0 < p < \infty$, such that if u is nonnegative and α-harmonic in $B(0, 1)$ then*

$$c_{p,\alpha} \int_{\mathbb{R}^n} (N_\sigma u(z))^p m(dz) \leq E^0(u^*)_p \leq C_{p,\alpha} \int_{\mathbb{R}^n} (N_\sigma u(z))^p m(dz).$$

PROOF. We require two lemmas. Expressions of the form $c(\sigma)$, $c(\sigma, \alpha, n)$ will denote constants depending only on the indicated parameters, not necessarily the same from use to use.

LEMMA 4.2. *There exists a constant $c_1 = c_1(\sigma, \alpha, n) > 0$ such that if $\lambda > 0$, $E = \{y \in B(0, 1)^c : N_\sigma u(y) > \lambda\}$, $G = \cup_{z \in E^c} \Gamma_\sigma(z)$, then*

$$\inf_{y \in G^c \cap B(0,1)} P^y[X_T \in E] \geq c_1.$$

PROOF OF LEMMA 4.2. When $E = \phi$, $G^c \cap B(0, 1) = \phi$ and there is nothing to prove. If $E \neq \phi$, take $y \in G^c \cap B(0, 1)$ and observe that $|y| \geq \sigma$. Let $\rho = 1 - |y|$ and consider the point w on the ray through the origin and y such that $|w| = 1 + \rho$. Let C_y be the reflection of $B(0, \sigma)$ through y onto $\partial B(0, 1)$, i.e.,

$$C_y = \{z \in \partial B(0, 1) : \text{the line through } y \text{ and } z \text{ intersects } B(0, \sigma)\}.$$

This gives a set whose diameter exceeds $c(\sigma)\rho$, where $c(\sigma)$ depends on σ alone. The convex hull of $C_y \cup \{w\} \subset E$; otherwise the contradiction $y \in G$ occurs. Let

$$F = \{z \in \mathbb{R}^n : 1 \leq |z| \leq 1 + \rho/3,\ (1 + \rho/3)z/|z| \in \text{the convex hull of } C_y \cup \{w\}\}.$$

Note that the area of $F \cap \partial B(0, 1)$ (with respect to surface measure on $\partial B(0, 1)$) exceeds $c(\sigma)\rho^{n-1}$.

Using the estimate $|y - z| \leq c(\sigma)\rho$ for $z \in F$, we have

$$P^y[X_T \in E] \geq P^y[X_T \in F] = A(\alpha, n) \int_F \left[\frac{1 - |y|^2}{|z|^2 - 1}\right]^{\alpha/2} |y - z|^{-n}\, dz$$

$$\geq c(\sigma, \alpha, n)\, \rho^{-n} \int_1^{1+\rho/3} \int_{F \cap \partial B(0,1)} \left[\frac{1 - |y|}{r^2 - 1}\right]^{\alpha/2} r^{n-1}\, dr\, d\theta$$

$$\geq c(\sigma, \alpha, n) \rho^{\alpha/2 - n} \rho^{n-1} \int_1^{1+\rho/3} \frac{r}{[r^2 - 1]^{\alpha/2}}\, dr \geq c_1(\sigma, \alpha, n) > 0. \quad \square$$

For z with $|z| > 1$ we introduce the measures P_z^x on the paths commencing at x and conditioned to be at z on exiting $B(0, 1)$. These are the h-paths of Doob (1957) with $h(x) = \hat{P}_1(x, z)$. The transition density on P_z^x paths is given by

$$p^z(t, x, y) = p(t, x, y)\,\hat{P}_1(y, z)/\hat{P}_1(x, z),$$

where $p(t, x, y)$ is the transition density for the symmetric stable process killed on exiting $B(0, 1)$.

LEMMA 4.3. *There exists a constant* $c_2 = c_2(\sigma, \alpha, n) > 0$ *such that if* $|z| > 1$ *and* $1 > \delta > \sigma/(1-\sigma)$ *then*

$$\inf_{x \in \Gamma_\sigma(z)} P_z^0(X_t \in B(x, \delta(1-|x|)), \text{ for some } t \le T) \ge c_2.$$

PROOF OF LEMMA 4.3. If $x \in \overline{B(0, \sigma)}$ then

$$\delta(1-|x|) > \delta(1-\sigma) > \sigma, \quad \text{so} \quad 0 \in B(x, \delta(1-|x|)),$$

and there is nothing to prove. So let $\sigma < |x| = \rho < 1$. By the definition of $\Gamma_\sigma(z)$, $1 < |z| < 2 - \sigma$. $B(x, \delta(1-\rho)) \cap B(0, \rho)^c$ contains the set

$$B = \{y : \rho \le |y| \le \rho + \delta(1-\rho)/2,\ (\rho + \delta(1-\rho)/2)y/|y| \in B(x, \delta(1-\rho))\}.$$

If $C_x = \{y/|y| : y \in B\}$, the area of C_x (with respect to surface measure on $\partial B(0, 1)$) exceeds $c(\sigma)(1-\rho)^{n-1}$. When $y \in B$, recalling that $w(z)$ is the tip of $\Gamma_\sigma(z)$ and $1 - |w(z)| = |z| - 1$,

$$\begin{aligned}
|y - z| &\le |y - x| + |x - w(z)| + |w(z) - z| \\
&\le \delta(1-\rho) + c(\sigma)(1-\rho) + 2(1 - |w(z)|), \quad \text{since} \quad x \in \Gamma_\sigma(z), \\
&\le c(\sigma)(1-\rho), \quad \text{using} \quad 1 - \rho = 1 - |x| \ge 1 - |w(z)|.
\end{aligned}$$

Thus, since $|z| \ge |y|$ and $\rho = |x| > \sigma$,

$$\begin{aligned}
&P_z^0(X_t \in B(x, \delta(1-\rho)), \text{ some } t < T) \\
&\quad \ge P_z^0(X_t \in B, \text{ some } t < T) \ge A(\alpha, n) \int_B \hat{P}_\rho(0, y) \frac{\hat{P}_1(y, z)}{\hat{P}_1(0, z)}\, dy \\
&\quad = A(\alpha, n) \int_B \frac{\rho^\alpha}{[y^2 - \rho^2]^{\alpha/2}} \left[\frac{|z|}{|y|}\right]^n [1 - |y|^2]^{\alpha/2} |y - z|^{-n}\, dy \\
&\quad \ge A(\alpha, n)\rho^\alpha \int_B \left(\frac{1 - |y|}{|y|^2 - \rho^2}\right)^{\alpha/2} |y - z|^{-n}\, dy \\
&\quad \ge c(\sigma, \alpha, n)(1-\rho)^{-n} \int_{C_x} \int_\rho^{\rho + \delta(1-\rho)/2} (r^2 - \rho^2)^{-\alpha/2}(1 - r)^{\alpha/2} r^{n-1}\, dr\, d\theta \\
&\quad \ge c(\sigma, \alpha, n)(1-\rho)^{-1} \int_\rho^{\rho + \delta(1-\rho)/2} (r^2 - \rho^2)^{-\alpha/2}(1 - r)^{\alpha/2} r^{n-1}\, dr
\end{aligned}$$

and since $1 - r \ge (1 - (\delta/2))(1 - \rho)$, this last expression is

$$\begin{aligned}
&\ge c(\sigma, \alpha, n)(1-\rho)^{\alpha/2 - 1} \int_\rho^{\rho + \delta(1-\rho)/2} (r^2 - \rho^2)^{-\alpha/2}\, r\, dr \\
&\ge c(\sigma, \alpha, n). \quad \square
\end{aligned}$$

PROOF OF THEOREM 4.1. Take $\lambda > 0$ and define E as in Lemma 4.2. Choose $y \in E$ and observe that $u(x) > \lambda$ for some $x = x(y) \in \Gamma_{\sigma(y)}$. Let $\delta = \frac{1}{2}(1-\sigma)^{-1}$, and using Harnack's

Inequality (Lemma 2.1) for the ball $B(x, 1 - |x|)$,

$$u(w) \geq c(\sigma, \alpha, n)u(x) > c(\sigma, \alpha, n)\lambda$$

for $w \in B(x, \delta(1 - |x|))$, and $c(\sigma, \alpha, n) > 0$. Thus

$$\begin{aligned} P^0(u^* > c(\sigma, \alpha, n)\lambda) &= \int_{R^n} P^0_y(u^* > c(\sigma, \alpha, n)\lambda)\, P^0(X_T \in dy) \\ &\geq \int_E P^0_y(u^* > c(\sigma, \alpha, n)\lambda) m(dy) \\ &\geq \int_E P^0_y(X_t \in B(x(y), \delta(1 - |x(y)|)), \text{some } t \leq T) m(dy) \\ &\geq c_2(\sigma, \alpha, n) m(E), \quad \text{by Lemma 4.3.} \end{aligned}$$

Integrating $p\lambda^{p-1}$ against both sides of this inequality yields the left-hand side of Theorem 4.1.

For the other inequality consider G as in Lemma 4.2 and the stopping time $S = \inf\{t > 0 : X_t \notin G\}$. Then

$$\begin{aligned} m(E) = P^0(X_T \in E) &\geq P^0(X_T \in E, S < T) = E^0(P^{X_S}(X_T \in E); S < T) \\ &\geq c_1(\sigma, \alpha, n)\, P^0(S < T), \quad \text{by Lemma 4.2} \\ &\geq c_1 P^0(u^* > \lambda). \end{aligned}$$

The last inequality follows since for u^* to exceed λ, the path X_t must enter $G^c \cap B(0, 1)$ before T. Now multiply by $p\lambda^{p-1}$ and integrate to obtain the right-hand inequality. □

REFERENCES

BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19–42.

BURKHOLDER, D. L. (1977). Exit times of Brownian motion, harmonic majorization, and Hardy spaces. *Adv. in Math.* **26** 182–205.

BURKHOLDER, D. L., GUNDY, R. F., and SILVERSTEIN, M. L. (1971). A maximal function characterization of the class H^p. *Trans. Amer. Math. Soc.* **157** 137–153.

DOOB, J. L. (1957). Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France.* **85** 431–458.

LANDKOF, N. S. (1972). *Foundations of Modern Potential Theory.* Springer-Verlag, New York.

MEYER, P. A. (1976). *Un Cours sur les Integrales Stochastiques.* Séminaire de Probabilités X. Springer, New York.

OBERHETTINGER, F. (1974). *Tables of Mellin Transforms.* Springer, New York.

RIESZ, M. (1938). Intégrales de Riemann-Liouville et potentiels. *Acta Szeged* **9** 1–42.

WATANABE, S. (1962). On stable processes with boundary conditions. *J. Math. Soc. Japan* **14** 170–198.

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