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# SPECIAL INVITED PAPER

## RANDOM WALKS ON DISCRETE GROUPS: BOUNDARY AND ENTROPY

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The paper is devoted to a study of the exit boundary of random walks on discrete groups and related topics. We give an entropic criterion for triviality of the boundary and prove an analogue of Shannon's theorem for entropy, obtain a boundary triviality criterion in terms of the limit behavior of convolutions and prove a conjecture of Furstenberg about existence of a nondegenerate measure with trivial boundary on any amenable group. We directly connect Kesten's and Fölner's amenability criteria by consideration of the spectral measure of the Markov transition operator. Finally we give various examples, some of which disprove some old conjectures.

**Introduction.** Probabilistic properties of random walks on groups are deeply intertwined with many essential algebraic characteristics of groups and their group algebras (amenability, exponential growth, etc.). On the other hand, random walks on groups regarded as a special class of Markov processes provide new simply describable examples of nontrivial probabilistic behavior. Both these aspects make the subject especially interesting and important.

Investigations of random walks on groups "in general" were started in the 1950's ([29, 41]), but no doubt the interest in this topic has grown during recent years. In fact, there is a great gap between the complete theory of "classic" walks on abelian groups ([60] etc.) and the state of the problem for sufficiently general groups (e.g. discrete). There are no answers for the simplest questions and the number of studied examples is very scanty in the second case.

The roughest question which does not appear in the classic theory is that about triviality (or nontriviality) of the exit boundary of random walk. The boundary itself gives valuable information about the group and the measure on it (see below), and the determination of the boundary (or deciding its triviality) should precede the study of more traditional problems such as laws of large numbers, central limit theorems, etc. (e.g. [34]) which lie beyond the scope of this paper.

In the present paper we solve to a certain extent the problem of boundary triviality for discrete groups, namely:

1) We give an entropic criterion for triviality of the boundary and prove an analogue of Shannon's theorem for entropy.

2) We obtain a boundary triviality criterion in terms of the limit behavior of convolutions and prove a conjecture of Furstenberg's about existence of a nondegenerate measure with trivial boundary on any amenable group.

3) We directly connect Kesten's and Fölner's amenability criteria by consideration of the spectral measure of the Markov transition operator.

4) We give various examples; some of these disprove certain old conjectures. Particularly, we give examples of solvable and locally finite (i.e., amenable) groups with symmetric measures whose boundaries are nontrivial and, conversely, an example of a solvable group of exponential growth whose boundary is trivial for every finitary symmetric measure.

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It is interesting to remark that at first glance it seems that the boundary for every amenable group with symmetric measure is trivial. The examples of Section 6 show that this conjecture is false. Moreover, a controversial statement that every group of exponential growth has nontrivial boundary for any measure is also false. In other words, the problem of boundary is more delicate than that of growth: the number of words with fixed length gives only rough estimates of entropy.

We emphasize that many concrete problems on boundaries are unsolved. For instance, the boundaries of semisimple (or free) groups with nonfinitary measures are still undetermined. It is worth mentioning that one can canonically define a series of unitary representations of $G$ in the space of square-integrable functions on the boundary. This series coincides with the principal series of representations for semi-simple Lie groups with spherical measures [23]. Interesting properties of these representations for the free group were discovered in the recent paper [70]. The authors intend to return to this question further.

The results of the present paper were announced in [65] without proofs. Later other proofs (close to ours) of Shannon's theorem and of Furstenberg's conjecture were given by Derrienic [17] and J. Rosenblatt [58], respectively. See Section 7 for further bibliographical comments.

There exists a good number of different definitions of boundary. Their interrelations in the case of discrete groups are considered in the authors' paper [66]. Briefly, all the boundaries coincide in a sense. In this paper we use only some coincidences (particularly, between the exit and stationary boundaries) specified in Section 0. In general the question of boundary triviality involves such different problems as existence of phase transitions and properties of characters of locally finite groups (e.g. see [64]). From the "general measure theory" point of view it may be reduced to the theory of decreasing sequences of measurable partitions (or $\sigma$-algebras) [63]. However, random walks on groups have their own peculiarity which is expressed, for instance, by the fact that the boundary is a $G$-space with quasi-invariant measure and carries some additional structures (see Section 3 and [66]).

Thus the subject can be dealt with by different approaches:

1) Since random walk on a group is a homogeneous (in time and space) Markov process with $\sigma$-finite stationary measure, the problem in question certainly belongs to the general theory of Markov processes.

2) Since the measure $\mu$ on a discrete group $G$ is an element of the group algebra $\ell^1(G)$, the transition operator assigned to $\mu$ (Markov operator) can be studied by means of harmonic analysis. In particular, the spectrum of the operator in the spaces $\ell^2(G)$ and $\ell^\infty(G)$ determines to a certain extent the probabilistic behavior of trajectories of the random walk.

3) Since the one-dimensional distribution of the random walk at a time $n$ is the $n$-fold convolution of the measure $\mu$, the problem is related to the asymptotic theory of convolutions, theory of equidistribution, etc.

4) Since the transition operator is a discrete analogue of the Laplace operator, the problem is connected with the theory of potential, the theory of harmonic functions, Martin boundaries, etc.

5) Lastly, the shift in the trajectory space of the random walk is a measure preserving transformation (dissipative, in general) with $\sigma$-finite stationary measure. Therefore one can apply here an arsenal of ergodic theory techniques (with due regard for $\sigma$-finiteness of measure); that is, one can use the theory of measurable partitions, entropic methods, etc. (these methods appear to be new for $\sigma$-finite invariant measure in general).

We underline that the examples naturally appearing in the theory of random walks turn out to be very sophisticated if the group structure is eliminated. For instance, the transition operator assigned to a random walk on a free group can scarcely be studied without use of the group properties.

The methods developed here (entropic, in the first place) can prove helpful for generalizations to broader classes of random processes such as random walks

1) on continuous groups,
2) with continuous time,
3) on semigroups and more general algebraic structures (e.g. formal grammars, graphs, etc.),
4) on trajectories of a $G$-space (particularly, measurable).

Connections with some of the questions listed above are also considered in the authors' paper [66] devoted to boundaries.

**0. Basic notions.** The object of this section is to give the definition of random walk, to introduce certain measures on the space of trajectories of the random walk and to give a list of different (equivalent) definitions of the boundary of random walk, $\Gamma(G, \mu)$, as a measurable $G$-space with $\mu$-stationary probability measure $\nu$.

0.1. *Preliminary definitions and notations.* Throughout this paper we shall use the following notations: $G$ is a *countable infinite discrete group* with the identity element $e$, $\mu$ – a *probability measure* on $G$. The pair $(G, \mu)$ will be called a *group with measure.* The *support* of the measure $\mu$ will be denoted by supp $\mu$:

(1) $$\operatorname{supp} \mu = \{g \in G : \mu(g) > 0\}.$$

The *reflection* of the measure $\mu$ will be denoted by $\check{\mu}$:

(2) $$\check{\mu}(g) = \mu(g^{-1}), \quad g \in G.$$

The measure $\mu$ will be called *nondegenerate* if the semigroup generated by its support is all of $G$, *finitary* if supp $\mu$ is finite, and *symmetric* if $\mu = \check{\mu}$. *In the rest of the paper the nondegeneracy condition for the measure $\mu$ will be assumed satisfied unless otherwise specified. We do not assume aperiodicity of the measure $\mu$.*

The infinite Cartesian product of $G$ with itself, $G^{\mathbb{Z}_+}$ ($\mathbb{Z}_+ = \{0, 1 \cdots\}$) will be denoted by $G^\infty$. In the sequel the elements of the product

(3) $$y = (y_0, y_1, y_2, \cdots) \in G^\infty$$

will be called *trajectories*, and the set $G^\infty$ is the *space of trajectories* (or the path space). The space $G^\infty$ carries the natural topology—the product of discrete ones on every factor. This topology is a metrizable separable complete one, thus $G^\infty$ can be regarded as a Polish space.

The *coordinate maps* from $G^\infty$ into $G$ will be denoted by $C^n$

(4) $$C^n : y = (y_0, y_1, \cdots) \mapsto y_n, \quad n \geq 0.$$

*Cylinder subsets of* $G^\infty$ (subsets consisting of trajectories hitting fixed points of $G$ at fixed times) will be denoted as follows:

(5) $$C_g^n = \{y \in G^\infty : C^n(y) = g\}$$

is the set of trajectories hitting a point $g$ at a fixed time $n$,

(6) $$C_{g_1, \cdots, g_k}^{n_1, \cdots, n_k} = \bigcap_{i=1}^k C_{g_i}^{n_i}$$

is the set of trajectories hitting a fixed sequence of points at fixed times, and

(7) $$C_y^{n_1, \cdots, n_k} = \bigcap_{i=1}^k C_{g_i}^{n_i}, \quad g_i = C^{n_i}(y),$$

is the set of trajectories hitting the same points as a fixed trajectory $y$ at fixed times $n_i$. In the sequel $C^n(y)$ (the $n$th coordinate of trajectory $y$) will frequently be denoted simply by $y_n$.

0.2. *Definition of random walk.*

DEFINITION 0.1. The *right random walk* $(G, \mu)$ on a group $G$ determined by a probability measure $\mu$ is the time homogeneous Markov chain with the state space $G$ and with the transition probabilities

$$p(g \mid h) = \mu(h^{-1}g), \quad g, h \in G, \tag{8}$$

which are invariant under the canonical left action of $G$ on itself (here and from now on $p(g \mid h)$ is the transition probability from $h$ to $g$).

In other words, the position of the random walk can be obtained from the preceding one by right multiplication with the independent random group element (increment of the random walk) which has the distribution $\mu$. Note that following [60] we define here the random walk only by its transition probabilities (without any fixed initial distribution), so that the random walk can be identified with the pair $(G, \mu)$. The *left* random walk can be defined similarly (with transition probabilities $\check{p}(g \mid h) = \mu(gh^{-1})$).

Evidently, $\check{p}(g^{-1} \mid h^{-1}) = \check{\mu}(h^{-1}g)$ and replacing the measure $\mu$ by the reflected one $\check{\mu}$ reduces the study of the left random walks to the study of the right ones. Everywhere below we shall consider only right random walks without stipulating that explicitly.

If the *initial distribution* of the random walk is concentrated on an element $g$ of $G$, then a probability Borel measure ${}_g\mathbb{P}^\mu$ arises on the trajectory space $G^\infty$ in the usual way—this measure is the image of the product-measure $\mu^\infty = \mu \times \mu \times \cdots$ on the infinite cartesian product $\prod_{i=1}^\infty G$ by the map

$$(x_1, x_2, \cdots,) \mapsto (g, gx_1, gx_1x_2, \cdots) \tag{9}$$

from the space of increments $\prod_{i=1}^\infty G$ into the trajectory space $G^\infty$. The measure ${}_e\mathbb{P}^\mu$ corresponding to the initial distribution $\delta_e$ is the most important and it will be denoted simply by $\mathbb{P}^\mu$. The measure $\mathbb{P}^\mu$ is concentrated on the subset

$$G_\mu^\infty = \{y \in G^\infty : y_0 = e, y_{k-1}^{-1}y_k \in \operatorname{supp} \mu, k \geq 1\} \tag{10}$$

of the trajectory space $G^\infty$.

An *arbitrary initial distribution* $\theta$ determines on the trajectory space the measure

$${}_\theta\mathbb{P}^\mu = \textstyle\sum_g \theta(g)_g\mathbb{P}^\mu. \tag{11}$$

Recall that the *convolution* $\mu * \mu'$ of two measures $\mu$ and $\mu'$ on a group $G$ is defined as the image of the product-measure $\mu \times \mu'$ on $G \times G$ by the canonical map $(x_1, x_2) \mapsto x_1x_2$ from $G \times G$ into $G$. Then $n$-fold convolution of the measure $\mu$ will be denoted by $\mu_n$. It is useful to regard the point measure $\delta_e$ as $\mu_0$. Thus the *one-dimensional distribution* of the measure ${}_\theta\mathbb{P}^\mu$ at a time $n \geq 0$ can be written as

$$C^n \circ {}_\theta\mathbb{P}^\mu \doteq \theta * \mu_n. \tag{12}$$

Let $\ell^1(G, \mathbb{C})$ be the *group algebra* of the group $G$ over the field $\mathbb{C}$. The measure $\mu = \sum \mu(g)\delta_g$ is an element of the group algebra, hence $\mu$ defines a linear operator on every representation space of $G$ (particularly on $\ell^1(G)$ itself, on $\ell^2(G)$, $\ell^\infty(G)$, etc.) This operator $P^\mu$ acts on any space of functions on $G$ as

$$P^\mu f(g) = \textstyle\sum_x f(gx)\mu(x) = \displaystyle\int f(y_1)\, d_g\mathbb{P}^\mu(y) \tag{13}$$

i.e., $P^\mu$ acting on the space $\ell^\infty(G)$ is the *Markov operator* of the random walk $(G, \mu)$. Since $P^\mu$ is Markov, it is positive (i.e., preserves the cone of positive functions on the group) and the constant function $\mathbb{I}$ is invariant under $P^\mu$.

As usual, the *conditional measure* $\mathbb{P}^\mu$ of the set $A_1$ given the condition $A_2$ will be denoted by $\mathbb{P}^\mu(A_1 \mid A_2)$. For example, $\mathbb{P}^\mu(C_g^n \mid C_h^k)$ is the conditional probability of hitting a point $g$ at time $n$ for trajectories which hit a point $h$ at time $k$. The following formula for the transition probabilities

(14) $$\mathbb{P}^{\mu}(C_g^{n+1} \mid C_h^n) = \mu(h^{-1}g)$$

evidently holds whenever the condition $C_h^n$ has non-zero measure. The measure space $(G^\infty, \mathbb{P}^\mu)$ and all of its quotient spaces are *Lebesgue spaces*, hence the standard technique of conditional measures and measurable partitions (e.g., [56]) is applicable to these spaces.

0.3. *Boundaries of random walks.* Given a random walk $(G, \mu)$, one can canonically define a measurable $G$-space $\Gamma$ with quasi-invariant probability measure $\nu$ which is *$\mu$-stationary*, i.e.,

(15) $$\nu(A) = \mu * \nu(A) = \textstyle\sum_g \nu(g^{-1}A)\mu(g), \quad A \subset \Gamma.$$

In the sequel the measure $G$-space $(\Gamma, \nu)$ will be called the *boundary* of the random walk $(G, \mu)$. The boundary naturally appears in very different situations [27, 64, 67] and it can be defined in rather different ways. Here we list only some of the possible definitions which will be used below. A detailed study of the correspondence between different boundaries of the pair $(G, \mu)$ is given in [66].

a) *Stationary boundary.* A measurable subset of the trajectory space $A \subset G^\infty$ is called *stationary* (mod 0) if it contains simultaneously with almost every trajectory $y$ also all trajectories $y'$ which can be obtained from $y$ by coordinate shifts and by replacing any finite number of coordinates, i.e., all $y'$ such that $y'_{n+k} = y_n$ for all sufficiently large $n$ and a fixed integer $k$. The $\sigma$-algebra $\mathcal{S}$ of classes of stationary sets ($\mathbb{P}^\mu$ - mod 0) is called the *stationary $\sigma$-algebra.*

A subset $a \subset G$ is called a $\mu$-trap (or: trap, if $\mu$ is fixed) if the limit $\lim_n \mathbb{I}_a(y_n)$ exists for $\mathbb{P}^\mu$ almost all trajectories $y \in G^\infty$, i.e., if almost every trajectory entirely belongs to $a$ for sufficiently large times or never hits it "at infinity." Define the natural equivalence of traps: traps $a$ and $a'$ are equivalent (or: mod 0 coincident) if their symmetric difference $a\Delta a'$ is a completely transitive set, i.e.,

(16) $$\mathbb{P}^\mu\{y \in G^\infty : \lim_n \mathbb{I}_{a\Delta a'}(y_n) = 0\} = 1.$$

The classes of traps form a Boolean algebra as can be easily seen.

PROPOSITION 0.1. *The correspondence*

(17) $$a \mapsto A(a) = \{y \in G^\infty : \lim_n \mathbb{I}_a(y_n) = 1\}$$

*determines an isomorphism between the Boolean algebra of classes of traps and that of classes of stationary sets.*

This fact enables us to define a *measure type preserving left action* of $G$ on the stationary $\sigma$-algebra $\mathcal{S}$.

DEFINITION 0.2. The *stationary boundary* $\Gamma = \Gamma(G, \mu)$ of the random walk $(G, \mu)$ is the quotient space of the measure space $(G^\infty, \mathbb{P}^\mu)$ with respect to the measurable partition attached to the stationary $\sigma$-algebra $\mathcal{S}$.

The canonical factorizing map will be denoted by bnd:

(18) $$\mathrm{bnd} : G^\infty \to \Gamma.$$

This map determines on $\Gamma$ the probability measure $\nu = \mathrm{bnd} \circ \mathbb{P}^\mu$ which is $G$-quasi-invariant and $\mu$-stationary. The measure $\nu$ will be called the *exit measure* of the random walk.

One can define the *conditional measures* $\mathbb{P}^\mu_\gamma$ on the trajectory space for almost all points $\gamma \in \Gamma$, i.e.,

(19) $$\mathbb{P}^\mu(C \mid A) = \frac{1}{\mathbb{P}^\mu(A)} \int_A \mathbb{P}^\mu_\gamma(C)\, d\nu(\gamma)$$

for any nontrivial stationary $A \subset G^\infty$ and cylinder set $C \subset G^\infty$. These measures $\mathbb{P}^\mu_\gamma$ are

Markov ones for almost all $\gamma \in \Gamma$ and determine homogeneous (in time, but not in space!) Markov chains on $G$ (*conditional random walks*) with transition probabilities

$$\mathbb{P}^{\mu}_{\gamma}(C^{n+1}_{g} \mid C^{n}_{h}) = \mathbb{P}^{\mu}(C^{n+1}_{g} \mid C^{n}_{h}) \cdot \frac{dg\nu}{dh\nu}(\gamma). \tag{20}$$

b) *Poisson boundary.* A real-valued function $f$ on the group $G$ is called *$\mu$-harmonic* if the equality

$$f(g) = \textstyle\sum_x f(gx)\mu(x) \tag{21}$$

holds for every $g \in G$, i.e., if $f$ is an invariant function of the Markov operator $P^{\mu}$. The Banach space of all *$\mu$-harmonic bounded functions* on $G$ (with sup-norm) will be denoted by $H^{\infty}_{\mu}$. The space $H^{\infty}_{\mu}$ can be regarded as a commutative Banach algebra with the multiplication

$$(f_1 \times f_2)(g) = \lim\nolimits_n \textstyle\sum_x f_1(gx) f_2(gx) \mu_n(x). \tag{22}$$

Define the *Poisson space* $\Pi_{\mu}(G)$ of the pair $(G, \mu)$ as the spectrum of the Banach algebra $H^{\infty}_{\mu}$. One can define a probability measure $\tilde{\nu}$ (*Poisson kernel* of $\mu$) on the space $\Pi_{\mu}(G)$ as follows: let $f \mapsto \hat{f}$ be the Gelfand transform from $H^{\infty}_{\mu}$ onto $C(\Pi_{\mu})$; then

$$\int \hat{f}(x)\, d\tilde{\nu}(x) = f(e). \tag{23}$$

Every bounded $\mu$-harmonic function $f$ admits the following *Poisson representation*

$$f(g) = \int \hat{f}(x)\, dg\tilde{\nu}(x) = \int f(x) \frac{dg\tilde{\nu}}{d\tilde{\nu}}(x)\, d\tilde{\nu}(x), \tag{24}$$

and the *Poisson formula* (24) constitutes an explicit form of the Gelfand transform from $H^{\infty}_{\mu}$ onto $C(\Pi_{\mu}) = L^{\infty}(\Pi_{\mu}, \tilde{\nu})$.

The Poisson space $(\Pi_{\mu}, \tilde{\nu})$ as a measure $G$-space is canonically isomorphic to the boundary $(\Gamma, \nu)$.

c) *Martin boundary.* Let $\Delta(G, \mu)$ be the *Martin boundary* of the pair $(G, \mu)$, i.e., the compact closure of extremal positive $\mu$-harmonic functions on $G$ (normed by the condition $f(e) = 1$) in the topology of pointwise convergence. The Martin boundary is a topological $G$-space with naturally defined left action of $G$.

Given a positive $\mu$-harmonic function $f$, there exists a unique *representing measure* $\nu_f$ on the Martin boundary, i.e.,

$$f(g) = \int_{\Delta} h(g)\, d\nu_f(h), \quad g \in G. \tag{25}$$

Denote the representing measure for the constant function $\mathbb{I}$ ($\mathbb{I}(g) = 1, g \in G$), by $\nu_1$ and its support, supp $\nu_1$, by $\Delta_1$. The compact $\Delta_1$ is called the *active part* of the Martin boundary. The compact $\Delta_1$ as a $G$-space with measure $\nu_1$ is canonically isomorphic to the boundary $(\Gamma, \nu)$.

d) *Exit boundary.* Let $\mathscr{A}^{\infty}_{n}$ be the $\sigma$-algebra of measurable subsets of the trajectory space $(G^{\infty}, \mathbb{P}^{\mu})$ which are determined by the coordinates $y_n, y_{n+1}, \cdots$, of the trajectory $y$. The intersection

$$\mathscr{A}_{\infty} = \textstyle\bigcap_n \mathscr{A}^{\infty}_{n} \tag{26}$$

is called the *tail* (*residual, asymptotic*) $\sigma$-algebra of the random walk. A measurable subset $A \subset G^{\infty}$ belongs to $\mathscr{A}_{\infty}$ iff the fact that a trajectory $y \in G^{\infty}$ belongs to $A$ does not depend on any finite set of coordinates and is determined only by its behavior "at infinity." The tail $\sigma$-algebra of the random walk $(G, \mu)$ coincides ($\mathbb{P}^{\mu}$ - mod 0) with the stationary $\sigma$-algebra $\mathscr{S}$, so that the *exit boundary* of the random walk, $(G^{\infty}, \mathbb{P}^{\mu})/\mathscr{A}_{\infty}$, is canonically isomorphic to the boundary $(\Gamma, \nu)$ as a measure space.

**1. Entropic criterion of boundary triviality.** In this section we define the entropy $h(G, \mu)$ for random walk on a discrete group $G$ determined by a measure $\mu$, and state the following entropic criterion of boundary triviality: the boundary $\Gamma$ is trivial iff the equality $h(G, \mu) = 0$ holds (Theorem 1.1). Further we consider some applications of this criterion to groups of nonexponential growth.

1.1. *Definition of entropy.* Let $G$ be a countable discrete group. The *entropy* of a probability measure $\mu$ on $G$ will be denoted by $H(\mu)$

$$H(\mu) = -\textstyle\sum_{g\in\operatorname{supp}\mu} \mu(g)\log \mu(g). \tag{1}$$

The following simple proposition permits to estimate the entropy of the convolution of two probability measures on $G$.

PROPOSITION 1.1. *Given two probability measures $\mu'$ and $\mu''$ on a group $G$ with finite entropies, the entropy of their convolution is also finite and*

$$H(\mu' * \mu'') \le H(\mu') + H(\mu''). \tag{2}$$

PROOF. The canonical map $(g_1, g_2) \mapsto g_1 g_2$ from $G \times G$ onto $G$ translates the direct product $\mu' \times \mu''$ into the convolution $\mu' * \mu''$, hence by monotonicity of entropy [57] and by the evident relation $H(\mu' \times \mu'') = H(\mu') + H(\mu'')$ we obtain the desired result.

Now fix a probability measure $\mu$ on $G$ with finite entropy $H(\mu)$. Denote the entropies of its convolutions $H(\mu_n)$ by $h_n$ $(n \ge 0)$. Obviously, $h_0 = H(\mu_0) = H(\delta_0) = 0$. The following evident formula for $h_n$ will be useful in the sequel:

$$h_n = H(\mu_n) = -\textstyle\sum_{y_n} \mu_n(y_n)\log \mu_n(y_n) = -\displaystyle\int \log \mu_n(y_n)\, d\mathbb{P}^\mu(y). \tag{3}$$

Proposition 1.1 implies that all the $h_n$ are finite and the sequence $\{h_n\}$ is subadditive, i.e., $h_{n+m} \le h_n + h_m$ $(n, m \ge 0)$. Thus there exists a finite limit $\lim_n(h_n/n) \le H(\mu)$.

DEFINITION 1.1.

[1] Let $G$ be a discrete countable group, $\mu$ a probability measure on $G$ with finite entropy $H(\mu)$. Then the limit

$$h(G, \mu) = \lim_n \frac{H(\mu_n)}{n} \tag{4}$$

is called the *entropy of the pair* $(G, \mu)$.

This invariant has the following probabilistic meaning: asymptotically $h(G, \mu)$ is *the mean specific quantity of information on one factor contained in the product* $x_1 \cdot \cdots \cdot x_n = y_n$ of $n$ independent $G$-valued random variables $x_i$ with distribution $\mu$ (cf. below Proposition 1.2).

REMARK. One can easily show that if the entropy $H(\mu)$ is infinite, then all the $h_n = H(\mu_n)$ are also infinite.

1.2. *Criterion of boundary triviality.* Let $\{\eta_n\}$ and $\{\alpha_n\}$ $(n \ge 1)$ be the following two sequences of measurable partitions on the trajectory space $(G^\infty, \mathbb{P}^\mu)$

$$\begin{aligned} y &\overset{\eta_n}{\sim} y' \Leftrightarrow \forall k \ge n \quad y_k = y'_k, \\ y &\overset{\alpha_n}{\sim} y' \Leftrightarrow \forall k \le n \quad y_k = y'_k; \end{aligned} \tag{5}$$

i.e., two trajectories lie in the same element of the partition $\eta_n$ if their coordinates with indices $k \ge n$ coincide, and in the same element of the partition $\alpha_n$ if their initial segments (up to the $n$th coordinate) coincide. The partitions $\alpha_n$ increase and their join $\bigvee_{n=1}^{\infty} \alpha_n$ is equal to $\varepsilon$ (the partition of the trajectory space into points). The sequence $\{\eta_n\}$ is decreasing; the corresponding measurable intersection $\bigwedge_{n\geq 1}\eta_n$ will be denoted by $\eta$. The partition $\eta$ is called the *tail partition* of the random walk $(G, \mu)$ because the $\sigma$-algebra of $\eta$-measurable subsets coincides ($\mathbb{P}^\mu$ - mod 0) with the tail $\sigma$-algebra $\mathscr{A}_\infty$ of the random walk.

Recall that given two measurable partitions $\xi$ and $\zeta$ of a Lebesgue space $(X, m)$, the quantity

$$H(\xi \mid \zeta) = -\int \log m(x, \xi \mid \zeta)\, dm(x) \tag{6}$$

where $m(x, \xi \mid \zeta)$ is the conditional measure of $\xi(x)$ (the element of $\xi$ containing $x$) with respect to $\zeta$, is called the *mean conditional entropy* of the partition $\xi$ with respect to the partition $\zeta$ [57].

Now evaluate the mean conditional entropy of the partitions $\alpha_k$ with respect to the partitions $\eta_n$.

PROPOSITION 1.2. *If $0 < k \leq n$ then the mean conditional entropy of the partition $\alpha_k$ with respect to the partition $\eta_n$ equals*

$$H(\alpha_k \mid \eta_n) = kh_1 + h_{n-k} - h_n. \tag{7}$$

*In particular,*

$$H(\alpha_1 \mid \eta_n) = h_1 + h_{n-1} - h_n. \tag{8}$$

PROOF. Evaluate the conditional measure $\mathbb{P}^\mu(y, \alpha_k \mid \eta_n)$ when a trajectory $y \in G^\infty$ is given:

$$\begin{aligned}\mathbb{P}^\mu(y, \alpha_k \mid \eta_n) &= \mathbb{P}^\mu(C_y^{\overline{0,k}} \mid C_y^n) = \mathbb{P}^\mu(C_y^{\overline{0,k}} \cap C_y^n)/\mathbb{P}^\mu(C_y^n) \\ &= \mu(x_1)\cdot \;\cdots\; \cdot\mu(x_k)\cdot\mu_{n-k}(y_k^{-1}y_n)/\mu_n(y_n).\end{aligned} \tag{9}$$

Integrating now $\log \mathbb{P}^\mu(y, \alpha_k \mid \eta_n)$ by the measure $\mathbb{P}^\mu$ we obtain the desired result.

Now, because $\eta_n > \eta_{n+1}$ for all $n$, we get

$$H(\alpha_1 \mid \eta_n) < H(\alpha_1 \mid \eta_{n+1}), \tag{10}$$

hence

$$h_n - h_{n-1} > h_{n+1} - h_n \tag{11}$$

(here we used a well known monotonicity property of the mean conditional entropy [57]). Comparing (11) with the definition of $h(G, \mu)$ we obtain

PROPOSITION 1.3. *The sequence $\{h_{n+1} - h_n\}$ decreases monotonically to the entropy $h(G, \mu)$.*

We now proceed to the proof of the main theorem.

THEOREM 1.1. *Let $G$ be a discrete countable group, $\mu$ a probability measure on $G$ with finite entropy $H(\mu)$. Then the boundary $\Gamma(G, \mu)$ of the random walk $(G, \mu)$ is trivial iff $h(G, \mu) = 0$.*

PROOF. Since the decreasing sequence of partitions $\eta_n$ converges to the tail partition $\eta$, the sequence of mean conditional entropies $H(\alpha_k \mid \eta_n)$ converges to $H(\alpha_k \mid \eta)$ [57]. Passing in (7) to the limit on $n$ and applying Proposition 1.3 we obtain

$$H(\alpha_k \mid \eta) = kH(\mu) - kh(G, \mu) \tag{12}$$

and, in particular

$$H(\alpha_1 \mid \eta) = H(\mu) - h(G, \mu). \tag{13}$$

Since the equality $H(\xi \mid \zeta) = H(\xi)$ holds for partitions $\xi$ with finite entropy iff the partitions $\xi$ and $\zeta$ are independent, and since $H(\alpha_k) = kH(\mu)$, we obtain $h(G, \mu) = 0$ iff the partitions $\alpha_k$ and $\eta$ are independent for all $k$; but this means triviality of $\eta$ and of the boundary $\Gamma(G, \mu)$ by the Kolmogorov 0-1 law.

REMARK. Theorem 1.1 in the given form is not applicable to the case $H(\mu) = \infty$. For example, if $G$ is an abelian group and the entropy $H(\mu)$ is infinite then $\lim_n(h_n/n) = \infty$, but the boundary $\Gamma(G, \mu)$ is trivial by the classical Choquet-Deny theorem [13].

1.3. *Applications of entropic criterion. Problem of growth.* The criterion stated in Theorem 1.1 enables us to state triviality for boundaries of some classes of groups with measure.

Let $G$ be a finitely generated group, $T$ a generating finite set. Consider the sequence of powers of $T$:

$$T^n = \{g = g_1 \cdot \cdots \cdot g_n : g_i \in T\}. \tag{14}$$

DEFINITION 1.2.
[52] The group $G$ is said to be of *exponential growth* if

$$\lim_n | T^n |^{1/n} \tag{15}$$

and of *nonexponential growth* otherwise. If

$$| T^n | \leq C \cdot n^d \tag{16}$$

then $G$ is said to be a group of *polynomial growth* with degree not more than $d$.

It can be easily seen that this definition does not depend on the choice of $T$.

PROPOSITION 1.4.
[2] *Let $G$ be a group of nonexponential growth and $\mu$ a finitary measure on $G$, then the boundary $\Gamma(G, \mu)$ is trivial.*

PROOF. Denote the support of $\mu$ by $T$. Since supp $\mu_n = T^n$ we obtain $H(\mu_n) \leq \log | T^n |$. Nonexponentionality of growth of $G$ implies that $\log | T^n | = o(n)$, hence $h(G, \mu) = 0$. Thus, $\Gamma(G, \mu)$ is trivial by Theorem 1.1.

Triviality of the boundary for some classes of exponential groups (see 6.3) and for certain nonfinitary measures on nonexponential groups also can be obtained by estimating the entropy $H(\mu_n)$.

Let $\mu$ be a probability measure on a finitely generated group $G$ with finite generating set $T$ (containing $e$). Define the probability measure $\kappa$ on $\mathbb{Z}$ in the following way:

$$\kappa(k) = \mu(T^k \backslash T^{k-1}) \tag{17}$$

is the measure $\mu$ of the words whose length is exactly $k$. Then one can easily obtain the following estimate:

$$H(\mu_n) \leq -\textstyle\sum_k \kappa_n(k) \log \kappa_n(k) \cdot \log | T^k | \tag{18}$$

where $\kappa_n$ is the $n$th convolution of $\kappa$. The inequality (18) reduces the problem of boundary triviality to that of the asymptotic behavior of convolutions on the group of integers and permits to prove triviality of the boundary for some classes of nonfinitary measures on nonexponential groups.

Triviality of the boundary for groups of polynomial growth with arbitrary measure can be easily deduced from structure theory. All the polynomial groups are finite extensions of nilpotent groups (Gromov's theorem [30]), and the boundary for nilpotent groups is trivial [21, 49]. Hence the boundary is trivial for any measure on a polynomial group (cf. Lemma 4.2 of [25]). A recent paper by Grigorchuk (in print in Dokl. Akad. Nauk. SSSR) contains the statement that there exists a finitely generated periodic group with intermediate growth (between polynomial and exponential). The problem of the existence of such groups was raised in [52].

CONJECTURE. Given an exponential group $G$, there exists a symmetric (nonfinitary, in general) measure with nontrivial boundary.

It also seems plausible that such a measure can be chosen finitary (but nonsymmetric). On the other hand, there exists an exponential group whose boundary is trivial for all finitary symmetric measures (see 6.3) in spite of a conjecture from [1].

The notion of growth for countably generated groups is more delicate. It does not seem right to regard all the locally finite groups as those of polynomial growth for no other reason than the boundedness of the sequence $\{|T^n|\}$ for all finite $T$ [47]. The notion of *uniform polynomial growth* defined in [11] appears to be more natural.

DEFINITION 1.3. A discrete group $G$ has *uniform polynomial growth* if for each positive $k$ there exists a polynomial $p_k$ such that

$$|T^n| \leq p_k(n) \tag{19}$$

for each $T$ consisting of $k$ elements.

In other words, $p_k$ gives a uniform estimate of growth for $k$-generated subgroups of $G$. The degrees of the polynomials $p_k$ are bounded for groups which are polynomial in the sense of Definition 1.2, but in general the degrees of $p_k$ are unbounded (e.g. consider the group $\sum_{-\infty}^{+\infty} \mathbb{Z}$).

By analogy with Definition 1.3, one can give the following.

DEFINITION 1.4. A discrete group $G$ has *weak exponential growth* if

$$\lim_n \sup_{|T|=k} \frac{\log |T^n|}{n} = c_k > 0 \tag{20}$$

for a certain natural $k$.

It is easy to show that the symmetric group $\mathfrak{S}_\infty$ (which is locally finite) has weakly exponential growth. This fact is in keeping with the existence of a random walk with nontrivial boundary on $\mathfrak{S}_\infty$ (see 6.7).

**2. Shannon Theorem for random walks.** Random walks on countable infinite groups have no finite stationary measure. However, an analogue of the Shannon-McMillan-Breiman theorem can be stated for this class of Markov processes (Theorem 2.1). The theorem implies many interesting corollaries.

2.1. *Random walks and endomorphisms with σ-finite measure.* Let $\theta$ be a positive measure (not necessarily probability) on a group $G$. Consider the corresponding measure

$$_\theta\mathbb{P}^\mu = \textstyle\sum_g \theta(g)\, _g\mathbb{P}^\mu \tag{1}$$

on the trajectory space $G^\infty$. The measure $_\theta\mathbb{P}^\mu$ is finite (or σ-finite) simultaneously with the initial measure $\theta$. Denote the *shift* on the trajectory space $G^\infty$ by $T\colon (Ty)_n = y_{n+1}$. Generally speaking, $T$ does not preserve the measure $_\theta\mathbb{P}^\mu$ and even the type of measure. The measure $_\theta\mathbb{P}^\mu$ is $T$-invariant iff the measure $\theta$ (regarded as a function on $G$) is $\check{\mu}$-harmonic. There exist no non-zero summable harmonic functions on infinite groups, so that every $T$-

invariant measure for infinite $G$ is only $\sigma$-finite. The $T$-invariant $\sigma$-finite measure ${}_m\mathbb{P}^\mu = \sum_g {}_g\mathbb{P}^\mu$ corresponding to the Haar measure $m$ on $G$, is of most interest. Many properties of random walk can be expressed in terms of the *endomorphism $T$ with $\sigma$-finite measure* ${}_m\mathbb{P}^\mu$. For instance, *conservativity* of $T$ is equivalent to *recurrence* of the random walk, *ergodicity* of $T$ to *triviality of the exit boundary*, and *mixing* of $T$ to *triviality of the stationary boundary* of the random walk $(G, \mu)$ (details see in [58, 66]).

Thus the entropy $h(G, \mu)$ defined in Section 1 is a *measure of nonergodicity* of the endomorphism $T$, so that our entropy is quite different from the entropy of conservative transformations with $\sigma$-finite invariant measure [42, 43].

2.2. *Shannon's Theorem.* The entropy defined in Section 1 can be also determined individually (i.e., along trajectories of the random walk).

THEOREM 2.1. *Suppose the entropy $H(\mu)$ of a probability measure $\mu$ on a countable group $G$ is finite. Then the equality*

$$\lim_n (1/n) \log \mu_n(y_n) = -h(G, \mu) \tag{2}$$

*holds for $\mathbb{P}^\mu$ almost all trajectories $y \in G^\infty$.*

PROOF. Define the following sequence of measurable functions on the trajectory space

$$\begin{aligned}\phi_k(y) &= \log \mathbb{P}^\mu(C_y^1 \mid \eta_{k+1}) = \log \mathbb{P}^\mu(C_y^1 \mid C_y^{k+1}) \\ &= \log \frac{\mu(y_1)\mu_k(y_1^{-1}y_{k+1})}{\mu_{k+1}(y_{k+1})}, \qquad k \ge 0,\end{aligned} \tag{3}$$

and define

$$\phi(y) = \log \mathbb{P}^\mu(C_y^1 \mid \eta). \tag{4}$$

The functions $\phi_k$ and $\phi$ are logarithms of conditional probabilities of the first coordinate $y_1$ of a trajectory $y$ with respect to the $\sigma$-algebras $\mathscr{A}_{k+1}$ and $\mathscr{A}_\infty$. By the convergence theorem for conditional probabilities we obtain $\phi_k(y) \to \phi(y)$ $\mathbb{P}^\mu$ a.e. The convergence $\phi_k \to \phi$ can be also stated in $L^1(\mathbb{P}^\mu)$. Since

$$\int \phi_k(y)\, d\mathbb{P}^\mu(y) = -H(\alpha_1 \mid \eta_{k+1}), \qquad \int \phi(y)\, d\mathbb{P}^\mu(y) = -H(\alpha_1 \mid \eta) \tag{5}$$

(see Proposition 1.2), and because $H(\alpha_1 \mid \eta_k) \to H(\alpha_1 \mid \eta)$ by properties of conditional entropy, we get

$$\int \phi_k(y)\, d\mathbb{P}^\mu(y) \to \int \phi(y)\, d\mathbb{P}^\mu(y). \tag{6}$$

The functions $\phi_k$ are nonpositive, hence the sequence $\phi_k$ converges to $\phi$ in $L^1(\mathbb{P}^\mu)$ (e.g., see [51]).

Expanding now $\mu_n(y_n)$ as

$$\mu_n(y_n) = \frac{\mu_n(y_n)}{\mu_{n-1}(y_1^{-1}y_n)} \cdot \frac{\mu_{n-1}(y_1^{-1}y_n)}{\mu_{n-2}(y_2^{-1}y_n)} \cdot \ldots \cdot \frac{\mu_2(y_{n-2}^{-1}y_n)}{\mu(y_{n-1}^{-1}y_n)} \cdot \mu(y_{n-1}^{-1}y_n), \tag{7}$$

we obtain

$$\log \mu_n(y_n) = \textstyle\sum_{i=1}^n \log \mu(y_{i-1}^{-1}y_i) - \sum_{i=1}^n \phi_{n-i}(U^{i-1}y), \tag{8}$$

where $U$ is the ergodic measure preserving transformation on the trajectory space $(G^\infty, \mathbb{P}^\mu)$ induced by the Bernoulli shift on the space of increments of random walk:

$$(Uy)_k = y_1^{-1}y_{k+1}, \qquad k \ge 0. \tag{9}$$

Now it is easy to demonstrate that the relation

$$\frac{1}{n}\sum_{i=1}^{n}\phi_{n-i}(U^{i-1}y)\to\int\phi(y)\,dP^{\mu}(y)=h(G,\mu)-H(\mu) \tag{10}$$

holds for almost all trajectories $y$ (this technique is standard cf. [8], Theorem 13.1). Since

$$(1/n)\sum_{i=1}^{n}\log\mu(y_{i-1}^{-1}y_i)\to -H(\mu) \tag{11}$$

evidently holds for almost all trajectories $y$ we obtain the desired equality (2).

REMARK. In fact, we also have proved convergence of the sequence $(1/n)\log\mu_n(y_n)$ to $-h(G,\mu)$ in the space $L^1(P^{\mu})$.

2.3. *Corollaries of Shannon's Theorem.* Here we suppose that the entropy $H(\mu)$ is finite. The entropy of the pair $(G,\mu)$ is denoted by $h(G,\mu)$.

COROLLARY 1. *Triviality of the boundary* $\Gamma(G,\mu)$ *is equivalent to the condition*:

$$\lim_n(1/n)\log\mu_n(y_n)=0 \tag{12}$$

*holds for* $P^{\mu}$ *almost all trajectories* $y\in G^{\infty}$.

Kesten [39, 40] introduced the condition

$$\lim_n(1/n)\log\mu_n(e)=0 \tag{13}$$

on the pair $(G,\mu)$ which is equivalent (if $\mu$ is nondegenerate and symmetric) to amenability of $G$. Examples from Section 6 and the Corollary of Theorem 4.2 show that the condition (12) is stronger than Kesten's condition. Thus the *probability of returning to e turns out to be an atypical even in the logarithmic scale* for certain amenable groups ($\log\mu_n(e)=o(n)$, but $\log\mu_n(y_n)=O(n)$ a.e.).

COROLLARY 2. *The equality*

$$\lim_n(1/n)\sum_{k=1}^{n}\log P^{\mu}(C_y^k\,|\,C_y^{k+1})=-H(\alpha_1\,|\,\eta)=h(G,\mu)-H(\mu) \tag{16}$$

*holds for* $P^{\mu}$ *almost all trajectories y—the mean value of the logarithms of the cotransition probabilities* $P^{\mu}(C_y^k\,|\,C_y^{k+1})$ *converges to* $h(G,\mu)-H(\mu)$ *along almost all trajectories.*

PROOF. Since

$$P^{\mu}(C_y^k\,|\,C_y^{k+1})=\frac{P^{\mu}(C_y^k)}{P^{\mu}(C_y^{k+1})}\mu(y_k^{-1}y_{k+1}), \tag{15}$$

we get

$$\begin{aligned}&(1/n)\sum_{k=1}^{n}\log P^{\mu}(C_y^k\,|\,C_y^{k+1})\\ &\qquad=(1/n)\Big(\sum_{k=1}^{n}\log\mu(y_k^{-1}y_{k+1})+\log\mu_1(y_1)-\log\mu_{n+1}(y_{n+1})\Big)\\ &\qquad\to -H(\mu)+h(G,\mu).\end{aligned} \tag{16}$$

COROLLARY 3. *The equality*

$$\lim_n(1/n)\sum_{k=1}^{n}\log P^{\mu}(C_y^k\,|\,C_y^{k-1},\eta)=-H(\alpha_1\,|\,\eta) \tag{17}$$

*holds for* $P^{\mu}$ *almost all trajectories y of the random walk.*

PROOF. Since the transition probabilities of the random walk are invariant with respect to the left action of $G$ on itself, we get

$$P^{\mu}(C_y^k\,|\,C_y^{k-1},\eta)=P^{\mu}(C_{y'}^1\,|\,\eta) \tag{18}$$

where $y' = U^{k-1}y$. Thus, applying the ergodic theorem to the shift $U$, we obtain

$$\lim_n \frac{1}{n} \sum_{k=1}^n \log \mathbb{P}^\mu(C_y^k \mid C_y^{k-1}, \eta) = \int \log \mathbb{P}^\mu(C_y^1 \mid \eta)\, d\mathbb{P}^\mu(y) = -H(\alpha_1 \mid \eta). \tag{19}$$

REMARK. Denoting the exit point of the trajectory $y$ by $y_\infty - y_\infty = \mathrm{bnd}(y)$, we can rewrite (17) as

$$\lim_n (1/n) \sum_{k=1}^n \log \mathbb{P}^\mu(y_k \mid y_{k-1}, y_\infty) = -H(\alpha_1 \mid \eta) = h(G, \mu) - H(\mu), \tag{20}$$

i.e., the mean value (along trajectories) of logarithms of transition probabilities for almost all conditional walks (provided the exit point is fixed) equals $h(G, \mu) - H(\mu)$.

COROLLARY 4. *The equality*

$$\lim_n (1/n) \sum_{k=1}^n \log \mathbb{P}^\mu(y_k \mid y_{k+1}) = \lim_n (1/n) \sum_{k=1}^n \log \mathbb{P}^\mu(y_k \mid y_{k-1}, y_\infty) \tag{21}$$

*holds for almost all trajectories y of the random walk, i.e., the mean value of logarithms of cotransition probabilities and the mean value of logarithms of transition probabilities for conditional walks coincide for almost all trajectories.*

COROLLARY 5. *The equality*

$$\lim_n \frac{1}{n} \log \frac{dy_n\nu}{d\nu}(y_\infty) = h(G, \mu), \tag{22}$$

*where* $y_\infty = \mathrm{bnd}(y)$ *and* $\nu$ *is the exit measure on* $\Gamma$, *holds for almost all trajectories of the random walk.*

PROOF. By the formula (20) from Section 0 we get

$$\lim_n \frac{1}{n} \sum_{k=1}^n \log \mathbb{P}^\mu(C_y^k \mid C_y^{k-1}, \eta) = \lim_n \frac{1}{n} \left\{ \sum_{k=1}^n \log \mathbb{P}^\mu(C_y^k \mid C_y^{k-1}) + \log \frac{dy_n\nu}{d\nu}(y_\infty) \right\}, \tag{23}$$

hence by Corollary 3 we simply obtain the desired result.

REMARK. It is of interest to extend the results of this Section (particularly the formula (21) which seems to be sufficiently general) to a greater class of Markov processes with a reasonable definition of entropy (similar to our $h(G, \mu)$).

## 3. Differential entropy of boundary and Radon-Nikodym transform.

This section is devoted to the description of the entropy $h(G, \mu)$ in terms of the differential entropy of shifts of the exit measure $\nu$ on the boundary $\Gamma$. We define a $\mu$-entropy $E(B, \lambda, \mu)$ of an arbitrary measurable $G$-space $B$ with $\mu$-stationary probability measure $\lambda$ and state the inequality $E(B, \lambda, \mu) \le h(G, \mu)$ which holds with equality iff $(B, \lambda)$ is a covering space of the boundary $(\Gamma, \nu)$. As a corollary we obtain a simple formula for the entropy $h(G, \mu')$ of a measure $\mu'$ expanded in powers of $\mu$.

3.1. *Kullback-Leibler distance and entropy $h(G, \mu)$.* Recall that the quantity

$$I(\nu_1 \mid \nu_2) = -\int \log \frac{d\nu_1}{d\nu_2}(x)\, d\nu_2(x) \tag{1}$$

is called the *Kullback-Leibler distance* (or *informational deviation*) between two equivalent probability measures $\nu_1$ and $\nu_2$ on a measurable space $X$ [44]. The distance takes nonnegative values (including $+\infty$) and equals zero iff the measures $\nu_1$ and $\nu_2$ coincide. In general, the Kullback-Leibler distance is nonsymmetric, i.e., $I(\nu_1 \mid \nu_2) \ne I(\nu_2 \mid \nu_1)$. If $X$ is a $G$-space, then the natural action of $G$ on the space of measures on $X$ preserves the distance $I$, i.e. $I(g\nu_1 \mid g\nu_2) = I(\nu_1 \mid \nu_2)$.

**THEOREM 3.1.** *If the entropy $H(\mu)$ of a measure $\mu$ on a group $G$ is finite then*

$$h(G, \mu) = -\sum_g \mu(g) \int \log \frac{dg^{-1}\nu}{d\nu}(\gamma)\, d\nu(\gamma) = \sum_g \mu(g) I(g^{-1}\nu \mid \nu). \tag{2}$$

*In other words, $h(G, \mu)$ is the mean (with respect to $g$) distance between the measures $g^{-1}\nu$ and $\nu$.*

PROOF. Using the formulas (20) from Section 0 and (13) from Section 1 we obtain

$$\begin{aligned} h(G, \mu) &= H(\mu) - H(\alpha_1 \mid \mu) = H(\mu) + \int \log \mathbb{P}^\mu(C_y^1 \mid \eta) \mathbb{P}^\mu(y) \\ &= H(\mu) + \int \log\left(\mathbb{P}^\mu(C_y^1) \cdot \frac{dy_1\nu}{d\nu}(\mathrm{bnd}(y))\right) d\mathbb{P}^\mu(y) \\ &= H(\mu) + \sum_g \mu(g) \int \log\left(\mu(g) \cdot \frac{d\nu}{dg^{-1}\nu}(\gamma)\right) d\nu(\gamma) \\ &= -\sum_g \mu(g) \int \log \frac{dg^{-1}\nu}{d\nu}(\gamma)\, d\nu(\gamma) \end{aligned} \tag{3}$$

(in the last step we used the substitution $y_1 = g$, $\mathrm{bnd}(y) = g\gamma$)

3.2. *$\mu$-entropy of a G-space.* Based upon Theorem 3.1, we can give the following

**DEFINITION 3.1.**
[37] Let $B$ be a measurable $G$-space with a $\mu$-stationary probability measure $\lambda$. The quantity

$$E(B, \lambda, \mu) = \sum_g \mu(g) I(g^{-1}\lambda \mid \lambda) \tag{4}$$

is called the *entropy of the space $(B, \lambda)$ with respect to $\mu$* ($\mu$-entropy of the pair $(B, \lambda)$).

Evidently, $E(B, \lambda, \mu)$ equals to zero iff $\lambda$ is $G$-invariant. On the other hand, $\lambda$ is $\mu$-stationary, hence

$$\frac{1}{\mu(g^{-1})} \geq \frac{dg^{-1}\lambda}{d\lambda} \geq \mu(g), \tag{5}$$

and we get

**PROPOSITION 3.1.** *The $\mu$-entropy $E(B, \lambda, \mu)$ satisfies the inequality*

$$E(B, \lambda, \mu) \leq -\sum_g \mu(g) \log \mu(g) = H(\mu). \tag{6}$$

Let $\mu'$ and $\mu''$ be two probability measures on $G$ and let the measure $\lambda$ be $\mu'$- and $\mu''$-stationary simultaneously. Then $\lambda$ is evidently stationary with respect to the convolution $\mu' * \mu''$ and to all convex combinations $\alpha'\mu' + \alpha''\mu''$ of the measures $\mu'$ and $\mu''$.

**PROPOSITION 3.2.** *Given two measures $\mu'$ and $\mu''$ on $G$ and a measurable $G$-space $B$ with probability measure $\lambda$ such that $\mu' * \lambda = \mu'' * \lambda = \lambda$, the following equalities hold:*

$$\begin{aligned} E(B, \lambda, \mu' * \mu'') &= E(B, \lambda, \mu') + E(B, \lambda, \mu''), \\ E(B, \lambda, \alpha'\mu' + \alpha''\mu'') &= \alpha' E(B, \lambda, \mu') + \alpha'' E(B, \lambda, \mu''). \end{aligned} \tag{7}$$

PROOF. It immediately follows from the definition of $\mu$-entropy. For example,

$$E(B, \lambda, \mu' * \mu'') = \textstyle\sum_g \mu' * \mu''(g) I(g^{-1}\lambda \mid \lambda)$$

$$\begin{aligned} &= -\textstyle\sum_{g_1, g_2} \mu'(g_1)\mu''(g_2) \displaystyle\int \log \frac{d\lambda(g_1 g_2 b)}{d\lambda(b)}\, d\lambda(b) \\ &= -\textstyle\sum_{g_1, g_2} \mu'(g_1)\mu''(g_2) \displaystyle\int \log \frac{d\lambda(g_1 g_2 b)}{d\lambda(g_2 b)}\, d\lambda(b) + E(B, \lambda, \mu'') \\ &= E(B, \lambda, \mu') + E(B, \lambda, \mu''). \end{aligned} \tag{8}$$

COROLLARY 1. *Let $\lambda$ be a $\mu$-stationary probability measure, $\mu' = \sum_{k\geq 0} \alpha_k \mu_k$ ($\sum_{k\geq 0} \alpha_k = 1$, $\alpha_k \geq 0$), then*

$$E(B, \lambda, \mu') = E(B, \lambda, \mu) \cdot \textstyle\sum_{k\geq 0} k\alpha_k. \tag{9}$$

COROLLARY 2. *Let $\mu$ be a probability measure on a group $G$, $\mu' = \sum_{k\geq 0} \alpha_k \mu_k$ ($\sum_{k\geq 0} \alpha_k = 1$, $\alpha_k \geq 0$), then*

$$h(G, \mu') = h(G, \mu) \cdot \textstyle\sum_{k\geq 0} k\alpha_k. \tag{10}$$

PROOF. Coincidence of boundaries $\Gamma(G, \mu)$ and $\Gamma(G, \mu')$ [37, 66] and Corollary 1 imply the desired result.

COROLLARY 3. *If the entropy $H(\mu)$ is finite then*

$$E(B, \lambda, \mu) \leq h(G, \mu) \tag{11}$$

*for every measurable $G$-space $B$ with $\mu$-stationary probability measure $\lambda$.*

PROOF. This statement is a consequence of Corollary 1 for the case $\mu' = \mu_k$, Proposition 3.1 and the definition of $h(G, \mu)$.

3.3. *Radon-Nikodym transform.* Now we shall give a necessary and sufficient condition of the equality $E(B, \lambda, \mu) = h(G, \mu)$.

DEFINITION 3.2. Let $B$ be a measurable $G$-space with $\mu$-stationary probability measure $\lambda$. The Radon-Nikodym transform $rn$ of the space $(B, \lambda)$ is the measurable map from $B$ into $\mathbb{R}^G$ (space of real-valued functions on $G$ with the topology of pointwise convergence) defined (mod 0) by the formula

$$(rn(b))(g) = \frac{dg\lambda}{d\lambda}(b). \tag{12}$$

Evidently, $rn$ is a homomorphism of the measure space $(B, \lambda)$ onto its image. The support of the measure $rn \circ \lambda$ (compact by inequalities (5)) will be denoted by $RN(B, \lambda)$ and called the *Radon-Nikodym compact* of the space $(B, \lambda)$. Almost all (with respect to the measure $rn \circ \lambda$) elements of $RN$ are positive $\mu$-harmonic functions on $G$, since $\lambda$ is $\mu$-stationary. The compact $RN$ consists of only the point $\mathbb{1}$ iff the measure $\lambda$ is invariant. The action of $G$ on $RN$ (induced by the action on $B$) is given by the following formula:

$$(h\gamma)(g) = \frac{dg(rn \circ \lambda)}{drn \circ \lambda}(h\gamma) = \frac{dh^{-1}g(rn \circ \lambda)}{dh^{-1}(rn \circ \lambda)}(\gamma) = \frac{\gamma(h^{-1}g)}{\gamma(h^{-1})}. \tag{13}$$

Obviously, the measure $rn \circ \lambda$ is $\mu$-stationary with respect to this action.

As a measure space, $(RN, rn \circ \lambda)$ is isomorphic to the quotient of the space $(B, \lambda)$ with respect to the measurable partition generated by the countable set of Radon-Nikodym derivatives on $B$:

$$\Delta_g(b) = \frac{dg\lambda}{d\lambda}(b). \tag{14}$$

Thus we obtain:

PROPOSITION 3.3. *For almost all $b \in B$*

$$\frac{dg\lambda}{d\lambda}(b) = \frac{dg(rn\circ\lambda)}{drn\circ\lambda}(rn(b)). \tag{15}$$

PROPOSITION 3.4. *The Radon-Nikodym transform does not change the value of $\mu$-entropy, i.e.*

$$E(RN, rn\circ\lambda, \mu) = E(B, \lambda, \mu). \tag{16}$$

Remark now that by Proposition 3.3 the Radon-Nikodym compact determines a decomposition of $\mathbb{I}$ into positive $\mu$-harmonic functions (elements of $RN$)

$$\mathbb{I} = \int \gamma \, drn\circ\lambda(\gamma), \tag{17}$$

since for every $g \in G$

$$\int \gamma(g)\, drn\circ\lambda(\gamma) = \int \frac{dg(rn\circ\lambda)}{drn\circ\lambda}(\gamma)\, drn\circ\lambda(\gamma) = 1. \tag{18}$$

Consider the Radon-Nikodym transform of the boundary $\Gamma(G, \mu)$ of the random walk. To begin with, recall that the boundary $(\Gamma, \nu)$ is isomorphic (as a measure $G$-space) to the active part of the Martin boundary $\Delta_1(G, \mu)$ with the measure $\nu_1$ (representing measure of $\mathbb{I}$), the action of $G$ on the Martin boundary being the same as the action (13) of $G$ on $RN$—see 0.3. By the relation

$$\mathbb{I} = \int (g^{-1}\gamma)(x)\, d\nu_1(g^{-1}\gamma) = \int \frac{\gamma(gx)}{\gamma(g)}\, dg\nu_1(\gamma), \quad g, x \in G, \tag{19}$$

we obtain two decompositions of $\mathbb{I}$ into extreme $\mu$-harmonic functions:

$$\mathbb{I} = \int \gamma\, d\nu_1(\gamma) = \int \gamma \frac{dg\nu_1(\gamma)}{\gamma(g)}. \tag{20}$$

By uniqueness of the representing measure on the Martin boundary, it follows that for almost all $\gamma \in \Delta_1(G, \mu)$

$$\frac{dg\nu_1}{d\nu_1}(\gamma) = \gamma(g). \tag{21}$$

Thus we get

PROPOSITION 3.5. *The Radon-Nikodym transform of the boundary $(\Gamma, \nu)$ is an isomorphism of the measure spaces and the corresponding Radon-Nikodym compact $RN$ with the measure $rn\circ\lambda$ coincides with the active part of the Martin boundary $\Delta_1(G, \mu)$ with the measure $\nu_1$—the representing measure of $\mathbb{I}$.*

Since every decomposition of $\mathbb{I}$ into $\mu$-harmonic functions can be obtained by integration from the decomposition into extreme $\mu$-harmonic functions, we obtain by Proposition 3.5:

PROPOSITION 3.6. *Given a measurable $G$-space $B$ with $\mu$-stationary probability measure $\lambda$, the Radon-Nikodym compact $RN(B, \lambda)$ as a $G$-space with measure $rn\circ\lambda$ is a quotient space of the boundary $(\Gamma, \nu)$.*

THEOREM 3.2. *Let $\mu$ be a probability measure on $G$ with finite entropy $H(\mu)$, and $B$ a measurable $G$-space with $\mu$-stationary probability measure $\lambda$. Then*

$$E(B, \lambda, \mu) \leq h(G, \mu), \tag{22}$$

*and equality holds iff the Radon-Nikodym compact $(RN, rn \circ \lambda)$ of the space $(B, \lambda)$ is isomorphic as a measure $G$-space to the boundary $(\Gamma, \nu)$ of the random walk $(G, \mu)$. In other words, the equality holds iff the Radon-Nikodym transform as a homomorphism of measure $G$-spaces can be passed through $\Gamma$:*

$$\begin{tikzcd} B \arrow[rr, "rn"] \arrow[dr, dashed] & & \mathbb{R}^G \\ & \Gamma \arrow[ur] & \end{tikzcd} \tag{23}$$

*here $\Gamma \to \mathbb{R}^G$ is the canonical imbedding defined in Proposition* 3.5.

PROOF. We base it upon the following simple property of the Kullback-Leibler distance: if $\nu_1$ and $\nu_2$ are two equivalent measures on a space $X$, $p$ a measurable map of $X$ onto $X'$, $\nu_i' = p \circ \nu_i (i = 1, 2)$, then $I(\nu_1' \mid \nu_2') \leq I(\nu_1 \mid \nu_2)$ and equality holds iff $(d\nu_1/d\nu_2)(x) = (d\nu_1'/d\nu_2')(p(x))$ a.e.

Now let $(RN, rn \circ \lambda)$ be the Radon-Nikodym compact of the space $(B, \lambda)$. By Proposition 3.6 there exists a factorizing map $p: \Gamma \to RN$. By the given property of the Kullback-Leibler distance and by Proposition 3.4 we immediately obtain

$$\begin{aligned} E(B, \lambda, \mu) &= E(RN, rn \circ \lambda, \mu) = \textstyle\sum_g \mu(g) I(g^{-1}(rn \circ \lambda) \mid rn \circ \lambda) \\ &\leq \textstyle\sum_g \mu(g) I(g^{-1}\nu \mid \nu) = h(G, \mu). \end{aligned} \tag{24}$$

The equality $E(B, \lambda, \mu) = h(G, \mu)$ is equivalent to the equality

$$\frac{dg\nu}{dg}(\gamma) = \frac{dg(rn \circ \lambda)}{drn \circ \lambda}(p(\gamma)) \tag{25}$$

for all $g \in \operatorname{supp} \mu$ and almost all $\gamma \in \Gamma$. By the nondegeneracy of $\mu$ we obtain that (25) holds for all $g \in G$; but by Proposition 3.5 this means that $p$ is an isomorphism of the measure $G$-spaces. The theorem is proved.

**4. Convolutions and boundary triviality. Proof of Furstenberg's conjecture.** In this section we state the following criterion for boundary triviality in terms of convolutions of the measure $\mu: \Gamma(G, \mu)$ is trivial iff the sequence $\mu_n$ converges to a left-invariant mean on $G$ (Theorem 4.2). On the basis of this criterion we give a proof of Furstenberg's conjecture: for every amenable $G$ there exists a probability measure $\mu$ with $\operatorname{supp} \mu = G$ for which the boundary $\Gamma(G, \mu)$ is trivial (Theorem 4.4).

4.1. *Boundary triviality and uniform distribution.* The following theorem is obtained by arguments usual in the theory of Markov processes.

THEOREM 4.1. *Let $\mu$ be a probability measure on $G$. The boundary $\Gamma(G, \mu)$ of the corresponding random walk is trivial iff the following strengthened condition on uniformity of distribution holds for almost all trajectories $y \in G^\infty$ and each $g \in$* supp $\mu$:

$$\lim_n \frac{\mu_{n-1}(g^{-1}y_n)}{\mu_n(y_n)} = 1. \tag{1}$$

PROOF. The conditional probability of the cylinder set $C_g^1$ (when the $n$th coordinate of the trajectory is fixed) equals

$$\mathbb{P}^\mu(C_g^1 \mid C_y^n) = \mu(g) \frac{\mu_{n-1}(g^{-1}y_n)}{\mu_n(y_n)}. \tag{2}$$

If the boundary $\Gamma(G, \mu)$ is trivial, then the convergence theorem for conditional probabilities gives

$$\mathbb{P}^\mu(C_g^1 \mid C_y^n) \to_n \mathbb{P}^\mu(C_g^1 \mid \eta) = \mathbb{P}^\mu(C_g^1) = \mu(g), \tag{3}$$

hence (1) holds.

Conversely, if $\mu_{n-1}(g^{-1}y_n)/\mu_n(y_n) \to_n 1$ for each $g \in \operatorname{supp} \mu$, then $\mu_{n-k}(g^{-1}y_n)/\mu_n(y_n) \to_n 1$ for each $g \in \operatorname{supp} \mu_k$ and almost all trajectories $y$. But the latter means exactly that all cylinder sets $C_g^k$ (hence all cylinder sets $C^{\overline{0,k}}_{g_0,\dots,g_k}$) are independent of the tail $\sigma$-algebra $\mathscr{A}_\infty$. Therefore $\mathscr{A}_\infty$, $\eta$ and $\Gamma\phi(G, \mu)$ are trivial by the Kolmogorov 0-1 law.

4.2. *Convergence of convolutions to the invariant mean.* Recall (e.g., see [28]) that a sequence $\{\phi_n\}$ of probability measures on $G$ is called *weakly convergent to a left-invariant mean* on $G$ if the sequence $\{g\phi_n - \phi_n\}$ tends to zero weakly in $\ell^\infty(G)^*$ for every $g \in G$, and *strongly convergent to a left-invariant mean* if $\{g\phi_n - \phi_n\}$ tends to zero in the norm topology of $\ell^1(G)$ (here $g\phi(A) = \phi(g^{-1}A)$ is the left shift of $\phi$ by $g$).

Also recall that the *period* of $\mu$ is the greatest common divisor of $\{K \geq 0 : \mu_K(e) \leq 0\}$, $\mu$ is called *aperiodic* if it is of period 1.

THEOREM 4.2. *Given a nondegenerate and aperiodic probability measure $\mu$ on $G$, the following conditions are equivalent*:

i) *The boundary $\Gamma(G, \mu)$ is trivial,*

ii) *The sequence $\{\mu_n\}$ of convolutions of the measure $\mu$ converges strongly to a left-invariant mean on $G$,*

iii) *The sequence $\{\mu_n\}$ converges weakly to a left-invariant mean on $G$.*

PROOF. By the aperiodicity of $\mu$ we can without loss of generality assume that the identity element of the group is charged by $\mu$

(i) $\Rightarrow$ (ii). Since $e \in \operatorname{supp} \mu$, by Theorem 4.1 we get

$$\lim_n \frac{\mu_{n-1}(g^{-1}y_n)}{\mu_n(y_n)} = \lim_n \frac{\mu_{n-1}(g^{-1}y_n)}{\mu_{n-1}(y_n)} = \lim_n \frac{\mu_n(g^{-1}y_n)}{\mu_n(y_n)} = 1 \tag{4}$$

for each $g \in \operatorname{supp} \mu$ and almost all $y \in G^\infty$, hence

$$\mu_n\left\{x \in G : \left| 1 - \frac{\mu_n(g^{-1}x)}{\mu_n(x)} \right| > \varepsilon\right\} \to_n 0 \tag{5}$$

for all $g \in \operatorname{supp} \mu$ and every $\varepsilon > 0$. Since $\operatorname{supp} \mu$ generates $G$ by the nondegeneracy of $\mu$, this completes the proof.

(ii) $\Rightarrow$ (iii). This implication is trivial.

(iii) $\Rightarrow$ (i). Apply the fact that triviality of the boundary $\Gamma(G, \mu)$ is equivalent to the absence of nontrivial bounded $\mu$-harmonic functions on $G$ (see 0.3). Let $f$ be such a function, then for every $x \in G$, one can write

$$f(x) = \textstyle\sum_g f(g)\mu_n(x^{-1}g) \tag{6}$$

and, in particular,

$$f(e) = \textstyle\sum_g f(g)\mu_n(g). \tag{7}$$

Subtracting (7) from (6) we get

$$f(x) - f(e) = \textstyle\sum_g f(g)(\mu_n(x^{-1}g) - \mu_n(g)). \tag{8}$$

By weak convergence of $\{\mu_n\}$ to a left-invariant mean and by boundedness of $f$ the right-hand side of (8) tends to zero if $n \to \infty$. Hence $f(x) = f(e)$ for every $x \in G$, i.e., $f$ is constant. Therefore, every bounded $\mu$-harmonic function on $G$ is constant and $\Gamma(G, \mu)$ is trivial.

COROLLARY. *If $G$ is nonamenable (i.e., there are no invariant means on $G$) then the boundary $\Gamma(G, \mu)$ is nontrivial for every nondegenerate measure $\mu$ on $G$.*

REMARK 1. If $\mu$ is periodic with period $d > 1$ then an analogous assertion about the sequence of measures $\bar{\mu}_n = (1/d)(\mu_n + \mu_{n+1} + \cdots + \mu_{n+d-1})$ holds true.

REMARK 2. Existence of a sequence of measures weakly converging to an invariant mean on a group is well known [28] to be equivalent to existence of a sequence strongly converging to an invariant mean. Theorem 4.2 shows that these types of convergence coincide for sequences of convolutions.

REMARK 3. The *right* random walk and the *left* one (see 0.2) differ as Markov processes, but their one-dimensional distributions $\mu_n$ coincide. The criterion of boundary triviality from Theorem 1.1 depends only on one-dimensional distributions of the random walk, and does not depend on whether right or left walk is considered. Therefore, given a measure $\mu$ with finite entropy $H(\mu)$ the boundaries of the left and right random walks are trivial (or non-trivial) together, and *convergence (strong or weak) of convolutions $\mu_n$ to a left invariant mean is equivalent to convergence (strong or weak) of the sequence $\mu_n$ to a right-invariant mean on $G$.* However, an example from 6.5 shows that this equivalence does not hold for measures $\mu$ with infinite entropy $H(\mu)$.

4.3. *Convolutions and amenability.* *Reiter's condition* is one of the conditions equivalent to amenability of a group $G$ (see Section 5): a group $G$ is amenable iff for any finite subset $K \subset G$ and $\varepsilon > 0$ there exists a probability measure $\phi$ on $G$ such that $\| \phi - g\phi \| < \varepsilon$ for every $g \in K$. Evidently, Reiter's condition can be reformulated in the following way: $G$ is amenable iff there exists a sequence of probability measures $\phi_n$ on $G$ such that $\lim_n \| \phi_n - g\phi_n \| = 0$ for all $g \in G$. Can one choose the sequence $\{\phi_n\}$ with some special properties? For instance, can the sequence $\{\phi_n\}$ be assumed to consist of convolutions of a measure? The following criterion answers the question.

THEOREM 4.3. *In order that a countable group $G$ be amenable, it is necessary and sufficient that there exists a nondegenerate probability measure $\mu$ on $G$ such that*

$$\lim_n \| \mu_n - g\mu_n \| = 0 \tag{9}$$

*for all $g \in G$ ($\mu_n$ is the $n$-fold convolution of $\mu$).*

PROOF. Sufficiency obviously follows from Reiter's condition.

The proof of necessity is of main interest and consists in a direct construction of the desired measure using Reiter's condition. This construction is not trivial; for instance, the measure in question cannot be chosen finitary for certain groups (see 6.2).

Let $e \in K_0 \subset K_1 \subset \cdots$ be an increasing sequence of finite sets exhausting $G$, $\{t_i\}_{i=1}^{\infty}$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ two sequences of positive real numbers such that $\sum_{i=1}^{\infty} t_i = 1$ and $\varepsilon_i$ decreases to zero. Let $\{n_i\}$ be a sequence of integers such that

$$(t_1 + \cdots + t_{i-1})^{n_i} < \varepsilon_i. \tag{10}$$

The group $G$ is amenable, hence by Reiter's condition there exists a sequence of probability measures $\alpha_m$ on $G$ with finite supports $A_m = \operatorname{supp} \alpha_m$ such that the following condition is satisfied:

$$\| \alpha_m - g\alpha_m \| < \varepsilon_m, \ \forall g \in B_m = K_m \cup (A_{m-1})^{n_m}. \tag{11}$$

Obviously, the measures $\alpha_m$ can be chosen to satisfy the condition $A_m = \operatorname{supp} \alpha_m \supset B_m$. Now let

$$\mu = \sum_{m=1}^{\infty} t_m \alpha_m \tag{12}$$

and show that $\mu$ is the desired measure.

Let $g$ be an element of $G$, then $g$ belongs to $A_{m-1}$ for a certain natural $m$. Denote the element $n_m$ of the sequence $\{n_i\}$ by $n$. Consider the $n$-fold convolution of $\mu$:

$$\mu_n = \sum_k t_{k_1} \cdots t_{k_m} \alpha_{k_1} * \cdots * \alpha_{k_n} \tag{13}$$

where the sum is taken over all multi-indices $k = (k_1, \cdots, k_n)$ with nonnegative $k_i$. Subdivide the sum (13) into two summands:

$$\nu_1 = \sum_{|k|<m} t_{k_1} \cdots t_{k_n} \alpha_{k_1} * \cdots * \alpha_{k_n}, \tag{14}$$
$$\nu_2 = \mu_n - \nu_1 \; (|k| = \max_i k_i).$$

Evidently

$$\| \nu_1 \| = \sum_{|k|<m} t_{k_1} \cdots t_{k_n} = (t_1 + \cdots + t_{m-1})^{n_m} < \varepsilon_m. \tag{15}$$

Consider now the measure $\nu_2$:

$$\nu_2 = \sum_{|k|\geq m} t_{k_1} \cdots t_{k_n} \alpha_{k_1} * \cdots * \alpha_{k_n}. \tag{16}$$

Fix a multi-index $k = (k_1, \cdots, k_n)$ such that $|k| \geq m$. Let $j$ be the lowest index such that the inequality $k_j \geq m$ holds; then we can rewrite $\theta = \alpha_{k_1} * \cdots * \alpha_{k_n}$ in the form

$$\theta = \theta_1 * \alpha_{k_j} * \theta_2. \tag{17}$$

Since the inequality $k_i < m$ holds for every $i < j$ by the choice of $j$, the inclusion supp $\alpha_{k_i} \subset A_{m-1}$ also holds for every $i < j$ (the sets $A_m$ increase). Since $j \leq n$ we get the inclusion

$$\operatorname{supp} \theta_1 \subset (A_{m-1})^{n-1}. \tag{18}$$

Besides that, $g \in A_{m-1}$ and therefore supp $g\theta_1 \subset (A_{m-1})^n$; hence by (11) the inequalities

$$\| \alpha_{k_j} - g\theta_1 * \alpha_{k_j} \| < \varepsilon_m, \quad \| \alpha_{k_j} - \theta_1 * \alpha_{k_j} \| < \varepsilon_m \tag{19}$$

hold. Consequently

$$\| g\theta_1 * \alpha_{k_j} - \theta_1 * \alpha_{k_j} \| < 2\varepsilon_m. \tag{20}$$

Hence

$$\| g\theta_1 * \alpha_{k_j} * \theta_2 - \theta_1 * \alpha_{k_j} * \theta_2 \| < 2\varepsilon_m, \tag{21}$$

i.e.,

$$\| g\theta - \theta \| < 2\varepsilon_m. \tag{22}$$

The latter inequality implies

$$\| g\nu_2 - \nu_2 \| < 2\varepsilon_m. \tag{23}$$

Since $\| \nu_1 \| < \varepsilon_m$ (hence $\| g\nu_1 \| < \varepsilon_m$) we finally obtain

$$\| g\mu_n - \mu_n \| < 4\varepsilon_m. \tag{24}$$

The proof is completed because the sequence of norms $\{\| g\mu_n - \mu_n \|\}_{n=1}^{\infty}$ is monotone non-increasing and the sets $A_m$ exhaust $G$ (hence every $g$ belongs to all $A_m$ with sufficiently large $m$ and $\| g\mu_n - \mu_n \| \to_n 0$).

REMARK 1. The case of an arbitrary *locally compact σ-compact group* can be dealt with in the same way by replacing finite sets $K_1$ by compacts, etc.

REMARK 2. The measures $\alpha_i$ obtained by Reiter's condition can be chosen symmetric. Then the constructed measure $\mu$ is also *symmetric* and the sequence of its convolutions converges to a *bi-invariant mean* on $G$.

4.4. *Furstenberg's conjecture.* The proven criterion and Theorem 4.2 imply the following:

THEOREM 4.4. *For every countable amenable group $G$ there exists a nondegenerate symmetric probability measure $\mu$ such that the boundary $\Gamma(G, \mu)$ of the corresponding random walk is trivial.*

Passing (if necessary) from the measure $\mu$ to a convex combination of its convolutions (this replacement does not change the boundary $\Gamma$—see [37, 66]), one can assume the support of the measure $\mu$ to be all of $G$. Thus Theorem 4.4 proves the following conjecture due to Furstenberg [26]: *A group $G$ possesses a measure $\mu$ whose support is all of $G$ and for which the boundary $\Gamma(G, \mu)$ is trivial iff $G$ is amenable.*

**5. Amenability. Fölner's and Kesten's conditions.** For the sake of completeness, we consider here another important quantitative invariant connected with the random walk—the spectral radius $\lambda(G, \mu)$ of the corresponding Markov operator. Below we give a simple direct proof of the equivalence of Fölner's and Kesten's conditions and on the basis of this proof we obtain estimates for the probability $\mu_n(e)$ of returning to the identity element $e$ in $n$ steps by means of the speed of growth of Fölner sets on the group. Further, we discuss some conjectures and examples connected with the latter characteristic.

5.1. *Criteria of amenability.* At the moment, a good number of amenability criteria are known (see [28, 69]), but *Fölner's condition* seems to be the simplest and most profound of them. Fölner's condition for discrete groups can be formulated as follows: Given a finite subset $K \subset G$ and $\varepsilon > 0$. A finite subset $A \subset G$ is called a *right Fölner set* for the pair $(K, \varepsilon)$ if

$$|Ag \,\Delta\, A| < \varepsilon |A|, \quad \forall g \in K. \tag{1}$$

Then existence of a right-hand invariant mean on $G$ (i.e., *amenability* of $G$) is equivalent to existence of Fölner sets for every pair $(K, \varepsilon)$. (In this Section we consider (for technical reasons) right-invariant means on $G$. Their existence is well known to be equivalent to existence of left-invariant (or bi-invariant) ones [28], which were dealt with in the previous Section.)

Other amenability conditions can be easily deduced from Fölner's condition. Reiter's condition used in Section 4 is among them. Various modifications of Fölner's condition are important in applications, particularly, in the theory of $G$-dynamical systems (e.g. [55]).

Another important amenability criterion is the condition stated by Kesten [39, 40] which connects the existence of an invariant mean on a group with the rate of decrease of the probabilities of returning to the unit element of the group $e$, and with spectral properties of the Markov operator of a random walk in the space $\ell^2(G)$. By the criterion a group $G$ is amenable iff the equality

$$\lambda(G, \mu) = \limsup_n (\mu_n(e))^{1/n} = 1 \tag{2}$$

holds for every symmetric probability measure $\mu$ on $G$ (here $\lambda(G, \mu)$ is the spectral radius of the Markov operator $P^\mu$ attached to the measure $\mu$ (see 0.2) in $\ell^2(G)$).

5.2. *Correspondence between Kesten's and Fölner's conditions.* Now we shall state a direct connection between Kesten's and Fölner's conditions. Let $P$ be the *Markov operator* in $\ell^2(G)$ assigned to the random walk $(G, \mu)$:

$$Pf(g) = \textstyle\sum_x f(gx)\mu(x). \tag{3}$$

Then the following equality can be easily deduced

$$\mu_n(g) = \langle \delta_e, P^n \delta_g \rangle, \quad g \in G, \tag{4}$$

(here $\langle f_1, f_2 \rangle = \sum_g f_1(g) f_2(g)$ is the scalar product in $\ell^2(G)$). Note that the operator $P$ is selfadjoint iff the measure $\mu$ is symmetric. If $P$ is selfadjoint then its square $Q = P^2$ is a positive operator in $\ell^2(G)$. Obviously $\| Q \| \leq \| P \| \leq 1$. Consider the *spectral representation* of $Q$:

(5) $$Q = \int_0^1 t \, dE(t),$$

where $E$ is a *projection-valued* measure on $\ell^2(G)$. Then

(6) $$\mu_{2n}(e) = \langle \delta_e, P^{2n} \delta_e \rangle = \langle Q^n \delta_e, \delta_e \rangle = \int_0^1 t^n \, d\gamma(t)$$

where

(7) $$\gamma(\Delta) = \langle E_\Delta \delta_e, \delta_e \rangle, \ \Delta \subset [0, 1],$$

is the corresponding *spectral measure.*

Since the spectral radius of $Q$ is completely determined by its diagonal spectral measures and $\langle E_\Delta \delta_g, \delta_g \rangle = \langle E_\Delta \delta_e, \delta_e \rangle$ for every $g \in G$, we get

(8) $$r(Q) = \sup \operatorname{supp} \gamma = \lim_n (\mu_{2n}(e))^{1/n}$$

and

(9) $$r(P) = \sqrt{r(Q)} = \lambda(G, \mu) = \limsup_n (\mu_n(e))^{1/n}.$$

Note that $\mu_n(e) > 0$ for every even $n$ by the symmetry $\mu$. Hence the limit $\lim_n (\mu_n(e))^{1/n}$ exists iff the period of $\mu$ is one (i.e., iff $\mu_n(e) > 0$ for all sufficiently large $n$).

THEOREM 5. *If a countable group $G$ is amenable then the spectral radius $\lambda(G, \mu)$ equals 1 for every symmetric probability measure $\mu$ on $G$. Conversely, if $\lambda(G, \mu) = 1$ for some symmetric measure $\mu$ whose support generates $G$, then $G$ is amenable.*

PROOF. 1) Let $G$ be an amenable group. If the support of $\mu$ is finite then by Fölner's condition there exists a nonzero $\varepsilon$-invariant function for $P$ in $\ell^2(G)$ for every $\varepsilon > 0$ (i.e., $\| Pf - f \| \leq \varepsilon \| f \|$). For instance, one can take the characteristic function of a Fölner set for supp $\mu$. Hence the norm of $P$ (and the spectral radius) is equal to one, i.e., $\lambda(G, \mu) = 1$. If the support of $\mu$ is infinite then for every $\varepsilon > 0$ there exists a finite subset $T_\varepsilon \subset \operatorname{supp} \mu$ such that $\mu(T_\varepsilon) > 1 - \varepsilon$, and we can take for almost invariant function the characteristic function of a Fölner set for $T_\varepsilon$ with sufficiently small $\varepsilon$.

2) Let $\lambda(G, \mu) = 1$ for a symmetric non-degenerate measure $\mu$ on $G$. Fix a finite subset $K \subset G$. Passing (if necessary) to a $n$-fold convolution of the measure $\mu$, we can suppose that supp $\mu$ contains $K$. Consider a decomposition of the measure $\mu$ into a convex combination of two symmetric probability measures $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$, i.e., $P = \alpha P_1 + (1 - \alpha) P_2$ where $P_i$ are selfadjoint and $\| P_i \| \leq 1$. Then $\| P \| \leq \alpha \| P_1 \| + (1 - \alpha) \| P_2 \|$, hence $\| P_i \| = 1$. Thus we can assume that the measure $\mu$ is finitary and supp $\mu$ contains $K$. Now, there exists an almost $P$-invariant function $f \in \ell^2(G)$, i.e., $\| f - Pf \| \leq \varepsilon \| f \|$. We can take for $f$ the characteristic function of a finite subset $A \subset G$ (see proof of Theorem 3.6.3. in [28]). By the finiteness of supp $\mu$,

$$\min\{\mu(g) : g \in \operatorname{supp} \mu\} = \delta > 0,$$

hence either $P\mathbb{1}_A(g) = 1$ or $P\mathbb{1}_A(g) \leq 1 - \delta$ for every $g \in G$. Thus

(10) $$\delta \sqrt{|A \Delta Ag|/2} = \delta \sqrt{|A \setminus Ag|} \leq \| f - Pf \| \leq \varepsilon \| f \| = \varepsilon \sqrt{|A|}, \quad g \in K.$$

Hence $A$ is a $2\varepsilon^2/\delta^2$-Fölner set for $K$. The theorem is proved.

5.3. *Estimation of spectral measure.* The first part of the proof of Theorem 5.1 can be made constructive to give an estimate of the spectral measure through the growth of Fölner sets. Thus we can obtain *effective* estimates of the spectral measure $\gamma$ and probabilities $\mu_{2n}(e)$.

THEOREM 5.2.
[36] *Let $A_\varepsilon \subset G$ be an $\varepsilon$-Fölner set for $T = \operatorname{supp} \mu$. Then for every $h < 1$*

$$\gamma([1-h, 1]) \geq \frac{1 - 2\varepsilon/h^2}{|A_\varepsilon|} \tag{11}$$

*and*

$$\mu_{2n}(e) \geq (1-h)^n \frac{1 - 2\varepsilon/h^2}{|A_\varepsilon|}. \tag{12}$$

The final estimate of $\mu_{2n}(e)$ for given $G$ and $\mu$ now can be obtained by maximization of the right-hand side of (12) as a function of $h$ and $\varepsilon$.

EXAMPLE. Consider the group $G_k$ (see 6.1) with a symmetric measure $\mu$ concentrated on the generators given in 6.1. Using the method from [62] we obtain the following $\varepsilon$-Fölner set for supp $\mu$:

$$A_\varepsilon = \{(x, f): x \in C, f|_{\mathbb{Z}^k \setminus C} = 0\} \tag{13}$$

where

$$C = \left\{ x = (x_1, \cdots, x_k) \in \mathbb{Z}^k : 0 \leq x_i \leq \frac{1}{\varepsilon} \right\} \tag{14}$$

is the cube with edge $1/\varepsilon$ in $\mathbb{Z}^k$. Thus

$$|A_\varepsilon| \leq 2^{\varepsilon^{-k}} \cdot \varepsilon^{-k} \tag{15}$$

and by simple computations we obtain the estimate

$$\mu_{2n}(e) \geq \exp(-c_k n^{2k/(2k+1)}) \tag{16}$$

where $c_k$ is a constant depending only on $k$.

REMARK 1. It is interesting to compare the estimate (16) with the asymptotic behavior of the *Golod-Shafarevich series* (e.g. [14]) for the same groups given in [7]:

$$\operatorname{gosha}_n(G_k) \sim \exp(n^{k/(k+1)}) \tag{17}$$

REMARK 2. Note that the estimate given in Theorem 5.2 depends on supp $\mu$ only and thus it seems to be rather imprecise. So, for the abelian group $\mathbb{Z}^k$ with a symmetric measure concentrated on generators and Fölner sets

$$A_\varepsilon = \left\{ x = (x_1, \cdots, x_k): 0 \leq x_i \leq \frac{1}{\varepsilon \cdot} \right\} \tag{18}$$

we get $\mu_{2n}(e) \geq c_k/n^{2k}$, but in fact $\mu_{2n}(e) \sim c_k/n^{k/2}$ (see [60]).

REMARK 3. Finally we emphasize that the problem of interrelations between spectral properties of $P$ and algebraic characteristics of $G$ is an interesting and complicated one. For instance, it is even unknown when the spectral measure $\gamma$ is absolutely continuous with respect to the Lebesgue measure.

5.4. *Growth of Fölner sets; localizing conjecture.* The problem of ascertaining the least possible growth of Fölner sets for a given amenable group which was first outlined in [62] becomes more interesting in connection with Theorem 5.2. The "*localizing conjecture*" proposed in [28] that given an amenable group $G$ with finite generating set $U$, the powers $U^n$ become for all sufficiently large $n$ $(K, \varepsilon)$-Fölner sets for any given pair $(K, \varepsilon)$ is false. In fact, if $G$ is of exponential growth and $|U^n g \Delta U^n| < \varepsilon |U^n|$ for all sufficiently large $n$ and $g \in U$, then

$$|U^{n+1}| \leq |U^n|(1 + \varepsilon |U|). \tag{19}$$

Hence $\lim_n |U^n|^{1/n} = 1$, which contradicts the exponentiality of $G$.

(For examples of amenable groups with exponential growth see Sections 6.1, 6.6. It should be mentioned here that Olshanskij [71] recently constructed an example of a finitely generated periodic non-amenable group. This example disproves the well-known conjecture [28].)

Recall that *superexponential* growth of Fölner sets for groups of exponential growth follows from very fast decrease of spectral measure $\gamma([1 - h, 1])$ as $h$ tends to 0 (see 5.3). Several interesting questions appear in this connection.

1. Does there exist an exponential amenable group $G$ such that the cardinalities of $(K, \varepsilon)$-Fölner sets grow not faster than $\exp(1/\varepsilon)$ for a given subset $K \subset G$?

2. Given a fixed growth of Fölner sets (as a function of $\varepsilon$) does there exist a finitely generated countable group with this growth?

3. It was proved in [28] (Theorem 3.6.6) that given a non-exponential group $G$ with finite generating set $U$, the sequence $U^n$ contains an infinite number of $(K, \varepsilon)$-Fölner sets for any pair $(K, \varepsilon)$. It is not clear whether all the $U^n$ are $(K, \varepsilon)$-Fölner sets for sufficiently large $n$ in this case. This assertion was proved for abelian groups ([28], Theorem 3.6.5) but the proof essentially depends on special properties of vector groups. Remark that Gromov's theorem [30] reduces the problem for polynomial groups to the case of nilpotent ones.

**6. Examples.** This section is devoted to the determination of the boundaries for several interesting classes of groups with measure. Random walks on groups $G_k = \mathbb{Z}^k \leftthreetimes \sum_{\mathbb{Z}^k} \mathbb{Z}_2$ (cross product of $\mathbb{Z}^k$ and $\sum_{\mathbb{Z}^k}\mathbb{Z}_2$) are considered in Sections 6.1–6.5. The study of this class of groups with measure enables us to construct some new nontrivial examples. Further we consider random walks on the affine group of the dyadic-rational line (6.6), on the infinite symmetric group (6.7) and on the free group (6.8).

6.1. *The groups $G_k$ and random walks on them.* Let $\mathbb{Z}^k = \sum_{i=1}^k \mathbb{Z}$ be the *$k$-dimensional integer lattice* and $\sum_{\mathbb{Z}^k}\mathbb{Z}_2$ be the direct sum of isomorphic copies of the group $\mathbb{Z}_2 = \{0, 1\}$ which are indexed by elements of $\mathbb{Z}^k$. It is useful to regard the group $\sum_{\mathbb{Z}^k}\mathbb{Z}_2 = F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ as the *additive group of finite configurations* on $\mathbb{Z}^k$ (with the operation of pointwise addition mod 2). The value of a configuration $f \in F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ on an element $x \in \mathbb{Z}^k$ will be denoted by $f(x)$ and the *support* of $f$ by supp $f$:

$$\operatorname{supp} f = \{x \in \mathbb{Z}^k : f(x) \neq 0\}. \tag{1}$$

Let $G_k = \mathbb{Z}^k \leftthreetimes F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ be the *cross* product of the groups $\mathbb{Z}^k$ and $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ (the lattice $\mathbb{Z}^k$ naturally acts on $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ by shifts). The group $G_k$ as a set consists of the ordered pairs $g = (x, f)$ where $x \in \mathbb{Z}^k$ and $f \in F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ and has the following *group operation*

$$(x_1, f_1)(x_2, f_2) = (x_1 + x_2, f_1 + T_{x_1} f_2) \tag{2}$$

where $T$ is the action of $\mathbb{Z}^k$ on $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ by *shifts*:

$$(T_x f)(y) = f(y - x), \quad x, y \in \mathbb{Z}^k, f \in F_0(\mathbb{Z}^k, \mathbb{Z}_2). \tag{3}$$

The groups $G_k$ are *solvable* of length 2, they are *finitely generated* with generators $a_{\pm 1}, \cdots, a_{\pm k}$ and $\delta_0$ (the $a_i$ are the generators of $\mathbb{Z}^k$, $\delta_0$ is the configuration on $\mathbb{Z}^k$ consisting of only one point—the zero of $\mathbb{Z}^k$) and they have *exponential growth*.

Now let $\mu$ be a probability measure on $G_k$. The increments of the random walk $(G, \mu)$ will be denoted by $(x_i, f_i)$ (i.e. $(x_i, f_i)$ are i.i.d. random $G$-valued variables with distribution $\mu$). Then

(4) $$(y_n, \phi_n) = (x_n, f_1) \cdots (x_n, f_n)$$

is the position of the random walk $(G, \mu)$ at time $n$. The definition of the group operation on $G_k$ implies the following formula expressing the subsequent state of the random walk in terms of the preceding one and the corresponding increment:

(5) $$y_{n+1} = y_n + x_{n+1}, \quad \phi_{n+1} = \phi_n + T_{y_n} f_{n+1}.$$

6.2. *A solvable group for which any finitary nondegenerate measure has nontrivial boundary.* Consider a finitary nondegenerate measure $\mu$ on $G_k (k \geq 3)$. Since every nondegenerate random walk on $\mathbb{Z}^k$ is transient for $k \geq 3$ [60], the first coordinate $y_n$ tends to infinity for almost all trajectories $\{(y_n, \phi_n)\}_{n=0}^{\infty}$ of the random walk $(G_k, \mu)$. The measure $\mu$ is finitary, hence the support of the configuration $T_{y_n} f_{n+1}$ and any fixed finite subset of $\mathbb{Z}^k$ are mutually disjoint for sufficiently large $|y_n|$ (here $|z| = \max_i |z_i|$ for any $z = (z_1, \cdots, z_k) \in \mathbb{Z}^k$). Thus the sequence of values $\phi_n(z)$ becomes constant for large $n$ a.s. for any fixed $z \in \mathbb{Z}^k$. In particular, the subset

(6) $$a = \{(x, f) \in G_k : f(0) = 0\}$$

of $G_k$ is a trap, and the corresponding set of trajectories

(7) $$A = \{\{(y_n, \phi_n)\}_{n=0}^{\infty} : \lim_n \phi_n(0) = 0\}$$

is a tail set. The set $A$ is nontrivial by nondegeneracy of $\mu$, hence the boundary $\Gamma(G_k, \mu)$ is nontrivial. Thus we proved:

PROPOSITION 6.1. *Let $\mu$ be a finitary nondegenerate probability measure on the group $G_k (k \geq 3)$. Then the boundary $\Gamma(G_k, \mu)$ is nontrivial.*

COROLLARY. *The entropy $h(G_k, \mu)$ is positive.*

REMARK. Since the groups $G_k$ are solvable and, by the same token, amenable, there exists a nondegenerate probability measure $\mu$ with trivial boundary $\Gamma(G_k, \mu)$ (Theorem 4.4). This measure $\mu$ cannot be finitary for $k \geq 3$ as follows from the above. Moreover, the entropy $H(\mu)$ for such a measure appears to be infinite.

6.3. *A group of exponential growth for which any finitary symmetric measure has trivial boundary.* Consider the group $G_1$ with symmetric measure $\mu$ whose support consists only of the elements $(0, \delta_0)$, $(z, 0)$, $(-z, 0)$, where $z$ is the generator of $\mathbb{Z}$ and $\delta_0$ is the configuration on $\mathbb{Z}$ charging the zero only. Consider the projection $\{y_n\}_{n=0}^{\infty}$ of a trajectory $\{(y_n, \phi_n)\}_{n=0}^{\infty}$ of the random walk $(G_1, \mu)$ from $G_1$ onto $\mathbb{Z}$. Since $y_n = x_1 + \cdots + x_n$ where $x_i$ are i.i.d. $\mathbb{Z}$-valued random variables with a finitary symmetric distribution (projection of $\mu$ onto $\mathbb{Z}$), Kolmogorov's classical inequality for sums of independent random variables implies the inequality

(8) $$\mathbb{P}^{\mu}\{\{(y_n, \phi_n)\}_{n=0}^{\infty} : \forall k \leq n \quad |y_k| \leq n^{3/4}\} \geq 1 - D/n^{1/2},$$

where $D$ is the variance of the increments $x_i$.

Thus the random walk hits the set

(9) $$a_n = \{(x, f) \in G : |x| \leq n^{3/4}, \operatorname{supp} f \subset [-n^{3/4}, n^{3/4}]\}$$

in $n$ steps with probability close to 1. The cardinality of the sets $a_n$ grows subexponentially, i.e. $\log|a_n| = o(n)$. The measure $\mu$ is finitary, so that

(10) $$\min\{\mu(g) : g \in \operatorname{supp} \mu\} = \delta > 0$$

and

$$\min\{\mu_n(g) : g \in \operatorname{supp} \mu_n\} \geq \delta^n. \tag{11}$$

Finally we get

$$H(\mu_n) \leq \log|a_n| - D/n^{1/2}\log \delta^n, \tag{12}$$

i.e., $H(\mu_n) = o(n)$, hence $h(G_1, \mu) = 0$ and $\Gamma(G_1, \mu)$ is trivial.

We have considered here, for the sake of simplicity, only measures $\mu$ concentrated on generators of $G_1$, but our considerations are also applicable to any finitary symmetric measure $\mu$ on $G_1$. Thus we obtained:

PROPOSITION 6.2. *The entropy $h(G_1, \mu)$ equals to zero (and the boundary $\Gamma(G_1, \mu)$ is trivial) for any finitary symmetric measure $\mu$ on $G_1$.*

REMARK. This example disproves the conjecture from [1] that every nondegenerate finitary measure on a group of exponential growth has nonzero entropy. The following problem is still open: *whether a measure with nonzero entropy exists on any group of exponential growth* (see 1.3).

6.4. *A criterion for boundary triviality for groups $G_k$.* Stabilization of the sequence $\{\phi_n(0)\}$ for almost all trajectories $\{(y_n, \phi_n)\}_{n=0}^{\infty}$ in the example from 6.2 enabled us to obtain a nontrivial tail set. One can also give examples of more complicated "tail behavior" when the sequence $\{\phi_n(z)\}$ has no limit, but the difference $\phi_n(z_1) - \phi_n(z_2)$ becomes stable a.s. for all $z_1, z_2 \in \mathbb{Z}^k$ [38]. At the moment we have no complete description of the boundary $\Gamma(G_k, \mu)$ for arbitrary $\mu$ just as for the symmetric group—6.7 and the free group—6.8). It is unknown whether the tail sets determined by the behavior of the configurations $\phi_n$ on finite subsets $A_1 \subset A_2 \subset \cdots \subset \mathbb{Z}^k$ $(\bigcup_{m=1}^{\infty} A_m = \mathbb{Z}^k)$ form a base for the whole tail $\sigma$-algebra. However, we can give the following simple sufficient condition for boundary triviality.

PROPOSITION 6.3 [38]. *If the induced random walk on $\mathbb{Z}^k$ is recurrent then the boundary $\Gamma(G_k, \mu)$ is trivial.*

PROOF. The subgroup $G_k^0 = \{(x, f) \in G_k : x = 0\}$ is a recurrence set for the random walk $(G_k,\mu)$, hence the boundary $\Gamma(G_k, \mu)$ is canonically isomorphic to the boundary of $G_k^0$ with an appropriate measure (cf. Lemma 4.2 from [25]). The latter boundary is trivial since $G_k^0$ is abelian. Hence the boundary $\Gamma(G_k, \mu)$ is also trivial.

The proven Proposition together with Proposition 6.1 gives a necessary and sufficient condition for boundary triviality for finitary measures, generalizing Proposition 6.2.

PROPOSITION 6.4. *The random walk on the group $G_k$ determined by a finitary measure $\mu$ has trivial boundary iff the projection of the random walk onto $\mathbb{Z}^k$ is recurrent. In particular, the boundary $\Gamma(G_k, \mu)$ for a symmetric finitary measure $\mu$ is trivial for $k = 1$, 2 and nontrivial for $k \geq 3$.*

In 6.5 we shall give an example of a measure $\mu$ such that the boundary $\Gamma(G_1, \mu)$ is trivial in spite of transience of the corresponding random walk on $\mathbb{Z}^1$.

6.5. *A probability measure on a solvable group for which the sequence of convolutions converges to a right-invariant but not left-invariant mean.* Convergence of the convolutions of a measure $\mu$ to a left-invariant mean on $G$ is equivalent (by Theorem 4.2) to triviality of the corresponding (right) random walk. Since the one-dimensional distributions coincide for right and left random walks, the entropic criterions of boundary triviality (Theorem 1.1) implies that convergence of the convolutions to a right-invariant mean is equivalent to convergence to a left-invariant mean for measures $\mu$ with finite entropy $H(\mu)$. The following example demonstrates that this equivalence does not hold if the entropy $H(\mu)$ is infinite. (This example was partly influenced by an example from M. Rosenblatt's book [59]. The authors are obliged to B.A. Rubshtein, who pointed this paper out to them.)

Evidently, the left random walk determined by $\mu$ is canonically isomorphic to the right one determined by the reflected measure $\check{\mu}$ (see 0.2). Thus the problem is reducible to construction of a measure $\mu$ with trivial boundary $\Gamma(G, \check{\mu})$ and nontrivial boundary $\Gamma(G, \mu)$.

Consider the following measure $\mu$ on the group $G_1$:

$$\begin{aligned}
&\mu(1, 0) = \frac{3}{8}, \quad \mu(-1, 0) = \frac{1}{8}, \quad \mu(0, \delta_0) = \varepsilon_0, \\
&\mu(0, \delta_1) = \mu(0, \delta_0 + \delta_1) = \frac{\varepsilon_1}{2}, \\
&\mu(0, \delta_2) = \mu(0, \delta_1 + \delta_2) = \mu(0, \delta_0 + \delta_2) = \mu(0, \delta_0 + \delta_1 + \delta_2) = \frac{\varepsilon_2}{4}, \\
&\text{- - - - - - - - - - - - - -} \\
&\mu(0, \delta_n) = \cdots = \mu(0, \delta_0 + \delta_1 + \cdots + \delta_n) = \frac{\varepsilon_n}{2^n}, \\
&\text{- - - - - - - - - - - - - -} \\
&\left(\varepsilon_n > 0, \textstyle\sum_{n=0}^{\infty} \varepsilon_n = \frac{1}{2}, \sum_{n=0}^{\infty} n\varepsilon_n = \infty\right).
\end{aligned} \tag{13}$$

The measure $\mu$ is nondegenerate and $H(\mu) = \infty$.

The inverse element of an element $(x, f) \in G_1$ is

$$(x, f)^{-1} = (-x, T_{-x}f). \tag{14}$$

In particular,

$$(0, f)^{-1} = (0, f), \quad (x, 0)^{-1} = (-x, 0), \tag{15}$$

hence

$$\check{\mu}(0, f) = \mu(0, f), \quad \check{\mu}(x, 0) = \mu(-x, 0).$$

It is easy to show that $\Gamma(G, \mu)$ is nontrivial (the induced random walk on $\mathbb{Z}$ is nonsymmetric and the measure $\mu$ is concentrated on the configurations charging the positive semi-axis of $\mathbb{Z}$ only). The proof of the triviality of $\Gamma(G, \check{\mu})$ is more complicated (for details see [38]). The definition of $\check{\mu}$ implies that $\check{\mu}_n(y_n, f + \phi_n)/\check{\mu}_n(y_n, \phi_n) \to_n 1$ for almost all trajectories $\{(y_n, \phi_n)\}_{n=0}^{\infty}$ of the random walk $(G, \check{\mu})$ and all configurations $f$. Hence every bounded $\check{\mu}$-harmonic function on $G$ is trivial and $\Gamma(G, \check{\mu})$ is also trivial (cf. Theorem 4.1). Thus we obtain:

PROPOSITION 6.5. *There exists a solvable group $G$ and a nondegenerate probability measure $\mu$ on $G$ such that the boundary $\Gamma(G, \check{\mu})$ is trivial and the boundary $\Gamma(G, \mu)$ is nontrivial, i.e., the sequence of convolutions of $\mu_n$ converges to a right-invariant mean on $G$, but not to a left-invariant mean.*

6.6. *Nontrivial boundary for random walk on the affine group.* Consider the group $G = \mathrm{Aff}(\mathbb{Z}[½])$ of matrices $\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}$ where $p = 2^k$, $q = m/2^n$ $(k, m, n \in \mathbb{Z})$ with the operation of matrix multiplication—*the affine group of the dyadic-rational line.* The group is *solvable* of length 2, has *exponential growth* and can be determined by the generators $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the relation $b^2a = ab$. The group $G$ is the homomorphic image of the group $\tilde{G} = Z \ltimes F_0(\mathbb{Z}, \mathbb{Z})$ under the canonical homomorphism $\pi: \tilde{G} \to G$

$$\pi(x, f) = \begin{pmatrix} 2^x & \Sigma\, 2^k f(k) \\ 0 & 1 \end{pmatrix} \tag{16}$$

Thus, the study of the boundary $\Gamma$ for random walks on $G$ can be reduced to the study of the boundary $\Gamma$ for random walks on $\tilde{G}$ and its behavior under the action of $\pi$. Namely, if $\tilde{\mu}$ is a measure on $\tilde{G}$ with $\mu = \pi \circ \tilde{\mu}$ then the boundary $\Gamma(G, \mu)$ is the quotient of the boundary $\Gamma(\tilde{G}, \tilde{\mu})$ with respect to the measurable partition into ergodic components corresponding to the action of the kernel of $\pi$ (see [66]):

$$\ker \pi = \{(x, f) \in \tilde{G}: x = 0; \textstyle\sum_k f(k) 2^k = 0\}. \tag{17}$$

In particular, if $\Gamma(\tilde{G}, \tilde{\mu})$ is trivial then $\Gamma(G, \mu)$ is also trivial.

The theory of random walks on the group $\tilde{G}$ is analogous to that of random walks on the group $G_1 = \mathbb{Z} \ltimes F_0(\mathbb{Z}, \mathbb{Z}_2)$ (cf. 6.4). Let $\{(y_n, \phi_n)\}_{n=0}^{\infty}$ be a trajectory of the random walk $(\tilde{G}, \tilde{\mu})$. If the configurations $\phi_n$ converge (a.s.) to a configuration $\phi_\infty$ on $\mathbb{Z}$ (hence $\Gamma(\tilde{G}, \tilde{\mu})$ is nontrivial) and the sum $\sum_{k=0} \phi_\infty(k) 2^k$ is finite a.s., then the action of $\ker \pi$ on the configuration space does not change the value $[\sum_{k=0} \phi_\infty(k) 2^k]$ ($[x]$ is the integer part of $x$). Thus there exists a nontrivial $\ker \pi$-invariant measurable function on the configuration space and $\Gamma(G, \mu)$ is nontrivial. Note that the measure $\mu$ can be chosen symmetric (but nonfinitary). On the other hand, it is easy to show that the boundary $\Gamma(G, \mu)$ is trivial for every finitary symmetric measure $\mu$ on $G$ (cf. 6.3).

6.7. *Nontrivial boundary for random walk on infinite symmetric group.* Consider the *symmetric group* $\mathfrak{S}_\infty$ of finite permutations of a countable set. The group $\mathfrak{S}_\infty$ is countable and locally finite. Every finitary measure $\mu$ on $\mathfrak{S}_\infty$ is contained in a finite subgroup by its local finiteness, thus the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ is trivial for these measures. Nevertheless, the following is true:

PROPOSITION 6.6. *There exists a symmetric probability measure $\mu$ with finite entropy $H(\mu)$ on the group $\mathfrak{S}_\infty$ such that the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ is nontrivial.*

PROOF. Let $V$ be a countable set. Consider the "natural" right-hand action of $\mathfrak{S}_\infty$ on $V$ by finite substitutions:

$$v \cdot g = g^{-1}(v), \quad v \in V, g \in \mathfrak{S}_\infty. \tag{18}$$

The main idea of the proof consists in the construction of a measure $\mu$ on $\mathfrak{S}_\infty$ such that the homogeneous Markov chain on $V$ with transition probabilities

$$p(x_1 \mid x_0) = \mu\{g \in \mathfrak{S}_\infty : g^{-1}(x_0) = x_1\} \tag{19}$$

(which is induced by the natural action of $\mathfrak{S}_\infty$) has a nontrivial exit boundary. Evidently, the inverse image of a tail set of the induced chain is a tail set of the random walk. Thus, nontriviality of the exit boundary of the induced chain implies nontriviality of the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$.

It will be useful to regard $V$ as the set of sequences $v = (v_1, \cdots, v_n)$ of finite length $0 \le |v| = n < \infty$ $(v_i = 0, 1)$. Define two sequences of elements of $\mathfrak{S}_\infty$:

$$\begin{aligned} a_i(v_1, \cdots, v_n) &= \begin{cases} (v_1, \cdots, v_n, 0), & i = n \\ (v_1, \cdots, v_{n-1}), & i = n-1, v_n = 0 \\ (v_1, \cdots, v_n) & \text{otherwise} \end{cases} \\ b_i(v_i, \cdots, v_n) &= \begin{cases} (v_1, \cdots, v_n, 1), & i = n \\ (v_1, \cdots, v_{n-1}), & i = n-1, v_n = 1 \\ (v_1, \cdots, v_n) & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

In other words $a_n$ exchanges the elements of $V_n = \{v:|v| = n\}$ and of $V'_{n+1} = \{v:|v| = n + 1, v_{n+1} = 0\}$, and $b_n$ the elements of $V_n$ and $V''_{n+1} = \{v:v = n + 1, v_{n+1} = 1\}$. Now it is easy to demonstrate that the measure $\mu$ defined by

$$\mu(a_i) = \mu(b_i) = \frac{\alpha_i}{2}, \tag{21}$$

where $\sum_{i=0}^{\infty} \alpha_i = 1$, is as desired for an appropriate choice of the sequence $\{\alpha_i\}$ (for details see [38]).

REMARK 1. The measure $\mu$ charges only the elements of $\mathfrak{S}_\infty$ of second order ($a_i^2 = b_i^2 = e$), hence $\mu$ is symmetric.

REMARK 2. The support of the constructed measure $\mu$ does not generate all of $\mathfrak{S}_\infty$, but a simple modification of the given construction provides an example of a nondegenerate symmetric measure $\mu'$ with nontrivial boundary.

REMARK 3. The measure $\mu$ with nontrivial boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ can be chosen to have the entropy $H(\mu)$ finite. On the other hand, the entropy $h(\mathfrak{S}_\infty, \mu')$ equals zero for any finitary measure $\mu'$ by the local finiteness of $\mathfrak{S}_\infty$. This fact demonstrates that in general *there exists no approximation of a measure $\mu$ in a countable group $G$ by finitary measures $\mu^{(k)}$ such that the entropies $h(G, \mu^{(k)})$ converge to $h(G, \mu)$.*

REMARK 4. It is unknown, whether the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ admits a complete description in terms of the Markov chain induced by the natural action of $\mathfrak{S}_\infty$ on a countable set $V$, and, in particular, whether triviality of the exit boundary of the induced chain implies triviality of the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$.

REMARK 5. The problem of describing all Markov chains on a countable set, induced by a random walk on the group of its finite substitutions is, evidently, equivalent to that of describing all convex combinations (not only finite) of finitary bistochastic matrices.

6.8. *Random walks on the free group.* Let $\mathscr{F}_k$ be the free group of rank $k$ with the set of generators $\mathscr{D}_k = \{a_{\pm 1}, \cdots, a_{\pm k}\}$ ($a_i^{-1} = a_{-1}$). The Martin boundary of random walks on $\mathscr{F}_k$ for some classes of measures was investigated in a number of papers. The case when $\mu$ is concentrated on the set of generators $\mathscr{D}_k$ was dealt with in [21, 48] (see also [20]). In this case the initial segments of the words $y_n$ coincide for sufficiently large $n$, i.e. the words $y_n$ converge to an infinite noncancellable word $y_\infty$ for almost all trajectories $\{y_n\}_{n=0}^{\infty}$ of the random walk. The Martin boundary (coinciding here with its active part) can be naturally identified with the compact of infinite noncancellable words (*ends* of $\mathscr{F}_k$—see [61])

$$\tilde{\mathscr{F}}_k = \{a_{i_1} a_{i_2} \cdots : a_{i_n} \in \mathscr{D}_k, i_n + i_{n+1} \neq 0\}. \tag{22}$$

Thus the boundary $\Gamma(\mathscr{F}_k, \mu)$ (coinciding as a measure $\mathscr{F}_k$-space with the active part of Martin boundary) can be identified with $\tilde{\mathscr{F}}_k$ and the exit measure $\nu$ on $\tilde{\mathscr{F}}_k$ can be completely computed. The Martin boundary for an arbitrary finitary measure on $\mathscr{F}_k$ also can be identified with $\tilde{\mathscr{F}}_k$ [15].

The problem of description of the stationary boundary (or the Martin boundary) for nonfinitary measures is still open. The words $y_n$ as before converge a.s. to an infinite word (G. A. Margulis oral communication), but it is unknown whether the tail sets determined by the infinite words exhaust the whole tail $\sigma$-algebra of the random walk. It should be noticed that the set $\tilde{\mathscr{F}}_\infty$ of infinite words for the free group $\mathscr{F}_\infty$ of infinite rank is not compact in the natural topology and, hence, cannot serve as the Martin boundary of a random walk.

**7. Comments and complementary notes.** In this Section we give bibliographic comments (of course, not complete) and complementary remarks to the main text.

0. The investigation of boundaries of nonabelian random walks started with the Martin boundary—it was determined in [20, 21] for free and nilpotent groups (for general Martin theory see [12, 18, 19, 54]; the Martin theory for random walks on locally compact groups is given in [6]).

The Poisson boundary for semisimple Lie groups was defined by Furstenberg in [23, 24]. This was motivated by the fact that the Poisson boundary for the group SL(2, $\mathbb{R}$) of the motions of the Poincaré disc coincides with the boundary circle, and the Poisson representation in this case coincides with the classic Poisson formula connecting the values of a harmonic function inside the unit disc with its boundary values.

The stationary $\sigma$-algebra (and the stationary boundary) can be similarly defined for arbitrary homogeneous Markov chain with countable state space [33, 54]. The traps were called in [9] and [54] "almost closed sets" and "regular sets"; they also were used in [22] for the definition of the Feller boundary which coincides in fact with our stationary boundary. The term "trap" seems to be the most adequate for this notion. The connection between the notion of trap and the notion of the "end of group" used in topology [61] makes itself conspicious; this connection was also pointed out in [41]. These notions coincide for the free group (see 6.8), but they differ in general and the problem of their interrelations is worth further investigation.

We underline that it is the nondegeneracy of $\mu$ that permits us to define the quasi-invariant action of $G$ on the stationary boundary. Nevertheless, the results of the Sections 1, 2 hold true also for adapted measures $\mu$ ($\mu$ is adapted if the group generated by supp $\mu$ is all the $G$).

The definitions of the exit boundary and of the tail $\sigma$-algebra are also taken from the general theory of Markov processes [19, 33]. Equality of the tail and the stationary $\sigma$-algebras ($\mathbb{P}^{\mu}$ - mod 0) can be easily deduced from Derrienic's 0–2 law for homogeneous Markov processes [16]. The authors came to know the paper [16] only after [65] had been published and the first proof of the equality was obtained by more direct considerations. If the measure $\mu$ is aperiodic then the $\sigma$-algebras $\mathscr{A}_\infty$ and $\mathscr{S}$ coincide ${}_\theta\mathbb{P}^{\mu}$-mod 0 for any initial distribution $\theta$. If $d \geq 1$ is the period of $\mu$ then any stationary set $A$ can be canonically subdivided into $d$ mutually disjoint tail sets. It is the fact that the tail and stationary $\sigma$-algebras for random walks coincide that enables us to use the entropic technique for investigations of the stationary boundary $\Gamma(G, \mu)$, since the tail partition can be canonically represented as the measurable intersection of a decreasing sequence of coordinate partitions.

1. The entropy $h(G, \mu)$ for discrete groups $G$ was defined by Avez in the paper [1]. In his next paper [2] he proved (by rather complicated means) that the equality $h(G, \mu) = 0$ implies triviality of the boundary $\Gamma(G, \mu)$ for finitary $\mu$ (see also [3]). A definition of $h(G, \mu)$ (in terms of differential entropy) for continuous groups $G$ with measure $\mu$ absolutely continuous with respect to the Haar measure on $G$ was given in [4] and there it was proved that $h(G, \mu) = 0$ implies triviality of all bounded $\mu$-harmonic functions on $G$. This theorem applied to discrete groups means that $h(G, \mu) = 0$ implies that $\Gamma(G, \mu) = \{\cdot\}$ (i.e., $\Gamma$ is trivial) for arbitrary $\mu$. Equivalence of the conditions $h(G, \mu) = 0$ and $\Gamma(G, \mu) = \{\cdot\}$ for discrete groups had been first shown in [65]. Here we give this proof. Another proof based upon Kingman's subadditive ergodic theorem was later independently given by Derrienic [17]. Generalization of the criterion from Theorem 1.1 to continuous groups is an interesting problem. Absence of continuity properties for differential entropy of continuous partitions similar to these of ordinary conditional entropy (see [57]) seems to be one of the main obstacles to this generalization.

2. Theorem 2.1 also had been published in [65]. Another proof was given independently in [17].

3. Although the entropy $h(G, \mu)$ was formally defined in [1], the differential entropy of the boundary $E(\Gamma, \nu, \mu)$ (coinciding with $h(G, \mu)$ by Theorem 3.1) had been used earlier by

Furstenberg [25] in investigations of boundaries for $SL(n, \mathbb{Z})$, $n = 2, 3$. In fact, he used statements close to our Theorem 3.2 (also see [32]). Note that the proof of inequality (22) from Theorem 3.2 is also valid for continuous groups and it permits one to obtain for these groups another proof of the implication $h(G, \mu) = 0 \Rightarrow \Gamma(G, \mu) = \{\cdot\}$ (cf. [4]). It is interesting to find out when the inequality (22) holds with equality for continuous groups. This problem is equivalent to that of finding necessary and sufficient conditions for triviality of boundary in entropic terms (see above comments to Section 1).

The Radon-Nikodym transform is a special case of a well known construction from the ergodic theory of dynamical systems: a $G$-dynamical system $(X, \nu)$ with quasi-invariant measure $\nu$ and generator $\psi$ can be realized by shifts in the space of functions on $G$ by means of the map

$$\psi : x \to \left\{ \frac{dg\nu}{d\nu}(x)\psi(gx) \right\}_{g \in G} \in \mathbb{R}^G. \tag{1}$$

In our case the generator $\psi$ is the unit function $\mathbb{I}$ and important properties of the Radon-Nikodyn transform (e.g. compactness of supp $rn \circ \lambda$) are due to peculiarities of the random walk and its boundary.

4. This section is also devoted to a detailed exposition of the results announced in [65]. The proof of Theorem 4.2 uses Nelson's proof of the classic Liouville theorem [53] (also see [4]). Another proof of Theorems 4.3 and 4.4 similar to ours was given independently by J. Rosenblatt [58] after the paper [65] had been published.

5. Theorem 5.1 is well known but the given simple proof is new. The proof of Theorem 5.2 is given in [36]. Remark that the indubitable connection between analytic characteristics of the random walk on $G$ and spectral properties of the corresponding Markov operator $P^\mu$ in $\ell^2(G)$ and $\ell^\infty(G)$ is still poorly understood. The spectrum of $P^\mu$ in $\ell^\infty(G)$ seems to depend on properties of $\mu$ to a larger degree then the spectrum in $\ell^2(G)$. For instance, nontriviality of the boundary $\Gamma(G, \mu)$ is equivalent to existence of nontrivial invariant functions for $P^\mu$ in $\ell^\infty(G)$ (see 0.3). The problem of finding conditions for symmetric $\mu$ (i.e. when $P^\mu$ is selfadjoint as an operator in $\ell^2(G)$) which imply reality of the spectrum of $P^\mu$ in $\ell^\infty(G)$ is of special interest. The spectrum of $P^\mu$ is not real for a symmetric finitary nondegenerate measure $\mu$ on any group $G_k$ [46]. Thus interrelations of boundary triviality and reality of the spectrum of $P^\mu$ are nontrivial (cf. 6.4). This problem is also connected with the problem of completely symmetrical (= Hermitian) group algebras. The investigated examples are also very close. Recall that the group algebra $\ell^1(G)$ is Hermitian if every selfadjoint element of it has real spectrum as an operator in $\ell^1(G)$ (or in $\ell^\infty(G)$). This definition differs from the usual one [10] only in formulation. The group algebra can be called positive Hermitian if every positive (as a measure on $G$) selfadjoint element of $\ell^1(G)$ has real spectrum in $\ell^1(G)$ (or in $\ell^\infty(G)$). The coincidence of these two notions is still an open question (also see [47]).

6. The first examples of determination of boundaries of random walk were given in [13, 21]. Triviality of boundary for random walks on abelian groups was proved in [13] and on nilpotent groups in [21] (also see [49, 51]). The first example of nontrivial boundary (for free group) was given in [21].

The problem of finding conditions for nontriviality of the boundary for sufficiently general groups with measure (even for solvable groups) is difficult and all examples known to the authors are listed in the present paper.

Sections 6.1–6.7 are the summary of the paper [38]. The group-theoretic construction used in the definition of the groups $G_k$ is well-known as the "direct wreath of groups". For instance, $G_k$ is the direct wreath of the passive group $\mathbb{Z}_2$ with the active group $\mathbb{Z}^k$ (notation: $\mathbb{Z}_2 \wr \mathbb{Z}^k$)—see e.g. [45]. Vershik in [62] proposed to use the groups $G_k$ as examples in the study of analytic characteristics of groups (and random walks on them). The groups $G_k$

provide us with another series of examples: their group algebras are non-Hermitian [46]. Remark that a group similar to our $G_k$ (but more sophisticated) was used in [35] also for the construction of an example of a non-Hermitian group algebra. This Hulanicki's group also can provide an example of nontrivial boundary on a locally finite solvable group [38] (cf. 6.7).

The following interesting algorithmic problem is connected with investigations of boundaries for the free group: to find an algorithm which solves the problem of boundary triviality for groups determined by one relation (the measure $\mu$ can be assumed to be equidistributed on generators) [64].

The problem of a complete description of the boundary is more difficult than that of finding a nontrivial stationary set. For example, it is unknown whether infinite words exhaust the whole boundary of random walks on the free group. The analogous questions for random walks on the symmetric group and on the groups $G_k$ are also open (cf. Section 6). The following plausible *conjecture* is worth mentioning in this connection: every $G$-invariant measurable partition $\xi$ of the boundary $\Gamma(G, \mu)$ such that the action of $G$ on the factor $\Gamma/\xi$ is effective, is trivial. The single example of an invariant partition of the boundary is the partition $\xi_H$ of $\Gamma$ into ergodic components with respect to the action of a normal subgroup $H$—in this case the action of $G$ on $\Gamma/\xi_H = \Gamma(G/H, \mu_H)$ is evidently noneffective [66]. Note that for groups $SL(n, \mathbb{Z})$, $n \geq 3$, there exist no nontrivial invariant partitions at all [50]. It should be noticed here that the action of the free group $\mathscr{F}$ on the space $\tilde{\mathscr{F}}$ with any quasi-invariant measure is tame (= approximable) [68, 72].

Continuous groups are beyond the scope of the present paper. Boundaries of semisimple Lie groups were treated in [5, 26]. An example of nonsymmetric measure on the affine group "$ax + b$" with nontrivial boundary was given in [5]. Our example from 6.2 seems to be the first example of a symmetric measure on a solvable group with nontrivial boundary. An analogous example for continuous groups can hardly be constructed, because every symmetric measure with compact support absolutely continuous with respect to the Haar measure on a connected amenable Lie group has trivial boundary as follows from [31]. The proof of the fact is based upon the structure theory of Lie groups and an entropic criterion from [4]. The very intriguing problem of interrelations between boundaries of a continuous group and of its lattices is still solved only for certain measures on $SL(n, \mathbb{R})$ and $SL(n, \mathbb{Z})$ [25, 32].

**Added in proof.** In a recent paper [73] Ledrappier (using equality of the entropy $h(G, \mu)$ and the differential entropy $E(\Gamma, \nu, \mu)$-see Section 3-proves that for a large class of random walks on discrete subgroups of $SL$ $(Z, \mathbb{C})$ the boundary is the Riemannian sphere $S^2$.

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