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A CLASS OF BERNOULLI RANDOM MATRICES WITH CONTINUOUS SINGULAR STATIONARY MEASURES

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Assume that we have a measure μ on $Sl_2(\mathbf{R})$, the group of 2×2 real matrices of determinant 1. We look at measures μ on $Sl_2(\mathbf{R})$ supported on two points, the Bernoulli case. Let $\mathbf{P}^1$ be real projective one-space. We look at stationary measures for μ on $\mathbf{P}^1$. The major theorem that we prove here gives a general sufficiency condition in the Bernoulli case for the stationary measures to be singular with respect to Haar measure and nowhere atomic. Furthermore, this condition gives the first general examples we know about of continuous singular invariant (stationary) measures of $\mathbf{P}^1$ for measures on $Sl_2(\mathbf{R})$.

1. Introduction. Assume that we have a measure μ on $Sl_2(\mathbf{R})$, the group of 2×2 real matrices of determinant 1. Let $\mathbf{P}^1$ be real projective one-space, the quotient space of $\mathbf{R}^2\backslash\{0\}$ with the equivalence relation $v \sim kv$ for any $v \in \mathbf{R}^2\backslash\{0\}$ and $k \in \mathbf{R}\backslash\{0\}$. $Sl_2(\mathbf{R})$ acts on $\mathbf{P}^1$; hence, given $g \in Sl_2(\mathbf{R})$ and $x \in \mathbf{P}^1$, $gx \in \mathbf{P}^1$ in a well-defined way. Given μ, we say that v (a probability measure on $\mathbf{P}^1$) is a stationary measure for μ if for all continuous functions $f: \mathbf{P}^1 \to \mathbf{R}$,

$$\int f(x)\, dv(x) = \int \int f(gx)\, d\mu(g) \;\; dv(x).$$

Furstenberg (1963) proved that in this setting, there always exists at least one stationary measure for μ on $\mathbf{P}^1$. These stationary measures are useful objects: in this same paper (Furstenberg, 1963, Theorem 8.5, page 424), the stationary measures are used to give a formula for the almost sure limit of suitable norms of products of random matrices.

Here we look at measures μ on $Sl_2(\mathbf{R})$ supported on two points; i.e., $\mu(A) = p > 0$, $\mu(B) = 1 - p > 0$, where A, B are elements of $Sl_2(\mathbf{R})$. We call this the Bernoulli case. Harris (1956) proved a nice result assuring absolute continuity of stationary measures in a fairly general setting. Harris' theorem, however, assumes a technical condition not satisfied in the Bernoulli case.

The major theorem that we prove here gives a general sufficiency condition in the Bernoulli case for the stationary measure(s) to be singular (with respect to Haar measure) and nowhere atomic. Furthermore, this condition gives the first general examples we know about of continuous singular invariant (stationary) measures on $\mathbf{P}^1$ for measures on $Sl_2(\mathbf{R})$.

2. Notation.

DEFINITION 1. Let $a, b \in \mathbf{R}$. Define $a \vee b := \max(a, b)$, and $a \wedge b := \min(a, b)$.

We will define two parametrizations of $\mathbf{P}^1$. The first will give $\mathbf{P}^1$ in θ-coordinates, $\theta \in [0, \pi)$, while the second will give $\mathbf{P}^1$ in cotangent coordinates, i.e., we will associate to $\mathbf{P}^1$ the real line with infinity added.

Both of these parametrizations assume that we have chosen coordinates on $\mathbf{R}^2$, and depend on the particular choice of coordinates. Certainly, if we are working with matrices (as opposed to linear transformations), we have tacitly made such a choice.

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DEFINITION 2. We define a bijection z: $\mathbf{P}^1 \to [0, \pi)$. A point of $\mathbf{P}^1$, call it t, is the equivalence class of all vectors v such that $v = u\left[\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right]$ for some $u \in \mathbf{R}\backslash\{0\}$ and some fixed $\theta \in [0, \pi)$. Then define $z(t) := \theta$.

DEFINITION 3. We define a bijection y: $\mathbf{P}^1 \to \mathbf{R} \cup \{\infty\}$. A point of $\mathbf{P}^1$, call it t, is either the equivalence class of all vectors v such that $v = u\left(\begin{smallmatrix}\lambda\\ 1\end{smallmatrix}\right)$ for some $u \in \mathbf{R}\backslash\{0\}$ and some fixed $\lambda \in \mathbf{R}$, or is the equivalence class of all vectors of the form $u\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$ for some $u \in \mathbf{R}$. In the former case, define $y(t) := \lambda$; in the latter case, define $y(t) := \infty$.

We've already noted that $Sl_2(\mathbf{R})$ acts on $\mathbf{P}^1$ in a well-defined way; i.e., gx makes sense for $g \in Sl_2(\mathbf{R})$ and $x \in \mathbf{P}^1$. Indeed, for any specific $g \in Sl_2(\mathbf{R})$, the map G: $\mathbf{P}^1 \to \mathbf{P}^1$ given by $G(x) := gx$ is a bijection. To see how a matrix in $Sl_2(\mathbf{R})$, g, acts on the parametrizations of $\mathbf{P}^1$ given by Definitions 2 and 3, we have

DEFINITION 4. For $g \in Sl_2(\mathbf{R})$, define a bijection $Z(g)$: $[0, \pi) \to [0, \pi)$ by $Z(g)(\theta) := zgz^{-1}(\theta)$.

DEFINITION 5. For $g \in Sl_2(\mathbf{R})$, define a bijection $Y(g)$: $\mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ by $Y(g)(\lambda) := ygy^{-1}(\lambda)$ for all $\lambda \in \mathbf{R} \cup \{\infty\}$.

For a given $g \in Sl_2(\mathbf{R})$, we can be more explicit about $Z(g)$ and $Y(g)$. Let ta^{-1}: $\mathbf{R} \cup \{\infty\} \to [0, \pi)$ be the inverse of the map tan: $[0, \pi) \to \mathbf{R} \cup \{\infty\}$. Let $\eta\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) := ta^{-1}(v/u)$, for $\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) \in \mathbf{R}^2\backslash\{0\}$. Then it is easy to check that $Z(g)(\theta) = \eta \circ g\left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$. Also, let $\sigma\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) := (u/v)$ if $v \neq 0$, and let $\sigma\left(\begin{smallmatrix}u\\ 0\end{smallmatrix}\right) := \infty$ (where $\sigma: \mathbf{R}^2\backslash\{0\} \to \mathbf{R} \cup \{\infty\}$). Then for $\lambda \in \mathbf{R}$, it is again easy to see that $Y(g)(\lambda) = \sigma \circ g\left(\begin{smallmatrix}\lambda\\ 1\end{smallmatrix}\right)$, and for $\{\infty\}$, $Y(g)(\{\infty\}) = \sigma \circ g\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$.

We will often (tacitly) use the trivial facts that for $g_1, g_2 \in Sl_2(\mathbf{R})$, $Z(g_1g_2) = Z(g_1)Z(g_2)$ and $Y(g_1g_2) = Y(g_1)Y(g_2)$.

We finish this section with two notation-saving definitions.

DEFINITION 6. For $\theta \in [0, \pi)$, let $e(\theta) := \left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$.

DEFINITION 7. Let $\operatorname{diag}(a, b) := \left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right)$.

3. Statement of the main theorem. We have a natural Haar measure $d\alpha$ of total mass 1 on $\mathbf{P}^1$. Any non-singular linear transformation takes $d\alpha$ into an equivalent measure. So any linear change of coordinates on $\mathbf{R}^2$ defines another Haar measure $d\beta$, which is also equivalent to $d\alpha$. Let C be the class of all measures equivalent to any of these measures.

Just as for elements of $Sl_2(\mathbf{R})$, non-singular linear transformations act in a well-defined way on $\mathbf{P}^1$. We will be considering linear transformations T: $\mathbf{R}^2 \to \mathbf{R}^2$ such that T has determinant 1 and has 2 distinct real eigenvalues. Each eigenvalue corresponds to a point of $\mathbf{P}^1$, namely, to each of the eigenvalues corresponds a 1-dimensional family in $\mathbf{R}^2$ of eigenvectors for T, and to each of these 1-dimensional families corresponds a point of $\mathbf{P}^1$. Call the point of $\mathbf{P}^1$ corresponding to a given eigenvalue the *eigenpoint* for that eigenvalue of T.

Given 2 points of $\mathbf{P}^1$, t and s, note that $\mathbf{P}^1\backslash\{t, s\}$ has 2 connected components. Finally, for a measure v on $\mathbf{P}^1$, we say that v is continuous if $v(t) = 0$ for all $t \in \mathbf{P}^1$.

We now have (a rather long)

ASSUMPTION 1. T_1 and T_2 are 2 linear transformations of $\mathbf{R}^2 \to \mathbf{R}^2$, each of determinant 1. T_1 and T_2 both have distinct real eigenvalues, and no eigenvector of T_1 is an eigenvector of T_2. Denote the eigenvalues of T_1 by $\lambda_{11}, \lambda_{12}$ and those of T_2 by $\lambda_{21}, \lambda_{22}$. For $i = 1, 2$ since $\lambda_{i1}\lambda_{i2} = 1$ and λ_{i1} and λ_{i2} are distinct, $|\lambda_{i1}| \neq |\lambda_{i2}|$. Hence without loss of generality we can assume that $|\lambda_{i1}| > |\lambda_{i2}|$ for $i = 1, 2$. For $i = 1, 2, j = 1, 2$, let $v_{ij} \in \mathbf{P}^1$ be the eigenpoint for λ_{ij} of T_i.

We can now state the main theorem of this paper.

THEOREM 1. *Let T_1, T_2 be 2 linear transformations of $\mathbf{R}^2 \to \mathbf{R}^2$ satisfying Assumption 1. Let μ be a probability measure on the space of linear transformations of $\mathbf{R}^2 \to \mathbf{R}^2$, such that $\mu(T_1) = p$, $\mu(T_2) = q := 1 - p$, where $0 < p < 1$. Let U and V be the 2 connected components of $\mathbf{P}^1 \backslash \{v_{11}, v_{21}\}$. Assume that v_{12}, v_{22} are both in U, or both in V. Without loss of generality, assume that both are in U. Also assume that $T_1(V \cup \{v_{11}\} \cup \{v_{21}\}) \cap T_2(V \cup \{v_{11}\} \cup \{v_{21}\}) = \emptyset$. Pick v on $\mathbf{P}^1$, a stationary measure for μ. Then*

(i) *$v(V \cup \{v_{11}\} \cup \{v_{21}\}) = 1$ and*

(ii) *v is continuous singular with respect to any $c \in C$.*

4. Main lemmas. The following two lemmas are easily proved; the proofs are omitted.

LEMMA 1. *Let T be a linear transformation of $\mathbf{R}^2 \to \mathbf{R}^2$ such that* $\det T = 1$ *and T has distinct real eigenvalues λ_1 and λ_2. Let v_i, $i = 1, 2$ be the eigenpoints ($\in \mathbf{P}^1$) for λ_i of T. Then the flow goes from the smaller eigenpoint towards the larger eigenpoint. More precisely, we can assume (without loss of generality) that $|\lambda_i| > |\lambda_2|$. Pick $x \in \mathbf{P}^1$ such that $x \notin \{v_1, v_2\}$. Then $\mathbf{P}^1 \backslash \{x, v_1\}$ has 2 connected components—call them U, V. Assume (without loss of generality) that $v_2 \in U$. Then*

(i) *$Tx \in V$ and*

(ii) *as $n \to \infty$, $T^n x \to v_1$ (in the topology of $\mathbf{P}^1$).*

LEMMA 2. *Assume that T_1, T_2 satisfy the conditions of Theorem 1. Then there exist $\pi/4 < \theta_1, \theta_2 < \pi$ and a coordinate system on $\mathbf{R}^2$ such that for these coordinates*

(i) *the parametrization $z\colon \mathbf{P}^1 \to [0, \pi)$ satisfies $z(v_{11}) = 0$, $z(v_{21}) = \pi/4$, $z(v_{12}) = \theta_1$, $z(v_{22}) = \theta_2$, and*

(ii) *$Z(g_1)([0, \pi/4]) \cap Z(g_2)([0, \pi/4]) = \emptyset$, where g_i is the matrix in this coordinate system for T_i, $i = 1, 2$.*

We break the proof of Theorem 1 into manageable chunks. First we see that v is continuous, after which we show (i) of Theorem 1. Next, we find a set that enables us to prove Theorem 1 (ii), and finally, we prove (ii).

5. Proof that v is continuous. Assume that v has an atom. Then there exists $t \in \mathbf{P}^1$ such that $v(\{t\}) > 0$. So we pick t_0 such that $v(\{t_0\})$ is a maximum; i.e., for all $t \in \mathbf{P}^1$, $v(\{t\}) \le v(\{t_0\})$. Since v is stationary for μ, $v(\{t_0\}) = pv(\{T_1^{-1}t_0\}) + qv(\{T_2^{-1}t_0\})$. But since $v(\{T_1^{-1}t_0\}) \le v(\{t_0\})$, and $v(\{T_2^{-1}t_0\}) \le v(\{t_0\})$, it follows that $v(\{t_0\}) = v(\{T_1^{-1}t_0\}) = v(\{T_2^{-1}t_0\})$. Upon applying this procedure to $v(\{T_1^{-1}t_0\})$ and $v(\{T_2^{-1}t_0\})$, we see (inductively) that for all $n \in Z_+$, $v(\{T_1^{-n}t_0\}) = v(\{T_2^{-n}t_0\}) = v(\{t_0\})$. But since only finitely many distinct t_i can satisfy $v(\{t_0\}) = v(\{t_i\})$ (where $t_i \in \mathbf{P}^1$), it follows that there exist $i, j, k, m \in Z_+$ such that $T_1^{-i}t_0 = T_1^{-j}t_0$ and $T_2^{-k}t_0 = T_2^{-m}t_0$, where $i \ne j$, $k \ne m$. Assume (without loss of generality) that $i > j$, $k > m$. Then $t_0 = T_1^{i-j}t_0$, $t_0 = T_2^{k-m}t_0$.

Recall from linear algebra, that if a nonsingular linear transformation $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ has 2 distinct real eigenvalues, then there are only 2 distinct 1-dimensional families of eigenvectors in $\mathbf{R}^2$ for S. Thus, $t_0 = T_1^{i-j}t_0$ implies $t_0 = v_{11}$ or v_{12}. (Take $S = T_1^{i-j}$.)

Similarly, we see that $t_0 = T_2^{k-m}t_0$ implies that $t_0 = v_{21}$ or v_{22}. But now we've reached a contradiction, since by assumption, the four points v_{11}, v_{21}, v_{12}, v_{22} are distinct. Hence v is continuous.

6. Proof of (i). It suffices to do this proof in a coordinate system. Pick $\pi/4 < \theta_1, \theta_2 < \pi$ and a coordinate system on $\mathbf{R}^2$ such that (i) and (ii) of Lemma 2 hold. Then it suffices to show that $v(z^{-1}[0, \pi/4]) = 1$.

Let g_i be the matrix for T_i in this coordinate system, $i = 1, 2$. We first must see that $Z(g_i^{-1})([0, \pi/4]) \supset [0, \pi/4]$, for $i = 1, 2$. This follows from Lemma 1.

As an immediate consequence of what we've just noted, we get (inductively) the following: Let h_i, $i = 1, \cdots, n$ be such that $h_i = g_1$ or g_2 for all i, $i = 1, \cdots, n$. Then $Z(h_n^{-1} \cdots h_1^{-1})([0, \pi/4]) \supset [0, \pi/4]$.

Next, we'll see that given $0 \leq \theta < \pi$, there exists $n \in Z_+$ and $h_1, \cdots, h_n$, $h_i = g_1$ or g_2 for all i, such that $Z(h_1 \cdots h_n)\ (\theta) \in [0, \pi/4]$. We've already seen this for $0 \leq \theta \leq \pi/4$; namely, let $h_1 = g_1$ and $n = 1$. Now pick $\pi/4 < \theta < \pi$.

CASE I. $\theta_1 < \theta_2$. (Recall that $z(v_{12}) = \theta_1$, $z(v_{22}) = \theta_2$.) For $\theta < \theta_2$, Lemma 1 implies that $Z(g_2^n)(\theta) := \phi_n$, where $\phi_n \to \pi/4$. In particular, there exists N such that $\phi_n < \theta_1$. Again by Lemma 1, $Z(g_1^n)(\phi_N) := \psi_n$, where $\psi_n \to 0$ (for all n, $\psi_n \in (0, \theta_1)$), so there exists M such that $\psi_M \in [0, \pi/4]$. Thus $Z(g_1^M g_2^N) \in [0, \pi/4]$.

For $\theta > \theta_2$, $Z(g_2^n)\ (\theta) := \phi_n$, where $\phi_n \to \pi/4$ (from below, by Lemma 1. That is, $\phi_n \in (\theta_2, \pi) \cup [0, \pi/4)$.). Thus there exists N such that $Z(g_2^N) := \phi_N \in [0, \pi/4]$. And if $\theta = \theta_2$, by Lemma 1, $Z(g_1)\ (\theta) := \phi \in (\theta_2, \pi)$. So apply the above argument to $Z(g_1)(\theta)$, and get that there exists N such that $Z(g_2^N g_1)\ (\theta) \in [0, \pi/4]$.

CASE II. This argument is similar to that for Case I, and is omitted.

We now prove

LEMMA 3. *Given $n \in Z_+$, and h_i, $i = 1, \cdots, n$, where $h_i = g_1$ or g_2 for all i, then*

$$v(h_n^{-1} \cdots h_1^{-1}(z^{-1}[0, \pi/4]) \backslash z^{-1}[0, \pi/4]) = 0.$$

PROOF. The proof is by induction on n. Since v is stationary for μ, $v(z^{-1}[0, \pi/4]) = pv(g_1^{-1}(z^{-1}[0, \pi/4])) + qv(g_2^{-1}(z^{-1}[0, \pi/4]))$. Since $Z(g_i^{-1})([0, \pi/4]) \supset [0, \pi/4]$ for $i = 1, 2$, we can rewrite the above equation (using that $gz^{-1}\theta = z^{-1}Z(g)(\theta)$) as

$$v(z^{-1}[0, \pi/4]) = p(v(z^{-1}Z(g_1^{-1})[0, \pi/4] \backslash z^{-1}[0, \pi/4]) + v(z^{-1}[0, \pi/4]))$$
$$+ q(v(z^{-1}Z(g_2^{-1})[0, \pi/4] \backslash z^{-1}[0, \pi/4]) + v(z^{-1}[0, \pi/4])).$$

Since $p + q = 1$ and $0 < p < 1$, we see that

$$v(z^{-1}Z(g_2^{-1})[0, \pi/4] \backslash z^{-1}[0, \pi/4]) = v(z^{-1}Z(g_1^{-1})[0, \pi/4] \backslash z^{-1}[0, \pi/4]) = 0,$$

and hence the statement of the lemma is true for $n = 1$.

The general induction step is similar to the $n = 1$ case and is left to the reader.

Let D be the set of all sequences $h_1, \cdots, h_n$ such that $h_i = g_1$ or g_2 for all $i = 1, \cdots, n$, and such that $n \in Z_+$. D is countable. Let

$$M := h_n^{-1} \cdots h_1^{-1}(z^{-1}[0, \pi/4]), \text{ and let } Q := z^{-1}[0, \pi/4].$$

By Lemma 3, $\sum_D v(M \backslash Q) = 0$, and hence $v(\cup_D(M \backslash Q)) = 0$. Thus (since Q is fixed) $v((\cup_D M) \backslash Q) = 0$. But we claim that $\cup_D M = \mathbf{P}^1$.

To see this, pick $t \in \mathbf{P}^1$ and $h_1, \cdots, h_n$ (where $n \in Z_+$ and $h_i = g_1$ or g_2 for all i) such that $Z(h_1 \cdots h_n)z(t) \in [0, \pi/4]$. For this sequence $(h_1, \cdots, h_n)$, we see that $h_1 \cdots h_n(t) = z^{-1}Z(h_1 \cdots h_n)z(t) \in z^{-1}[0, \pi/4]$, and hence $t \in h_n^{-1} \cdots h_1^{-1}(z^{-1}[0, \pi/4]) = M$, as desired.

Hence $v(\mathbf{P}^1 \backslash Q) = v((\cup_D M) \backslash Q) = 0$, and hence $v(z^{-1}[0, \pi/4]) = 1$, as desired.

7. An important set. We now start to prove Theorem 1 (ii). Pick $\theta_3 \in (\theta_2, \pi) \cap (\theta_1, \pi)$ and $\theta_4 \in (\pi/4, \theta_1) \cap (\pi/4, \theta_2)$,. By Lemma 1, $Z(g_i)([\theta_3, \pi) \cup [0, \theta_4]) \subset [\theta_3, \pi) \cup [0, \theta_4]$, for $i = 1, 2$. Let $x_i := z^{-1}(\theta_i)$, for $i = 3, 4$.

Now make a linear coordinate change (i.e., pick new coordinates on $\mathbf{P}^1$) such that in these coordinates

$$z(x_3) = 0,\ z(x_4) = \pi/2,\ z(v_{11}) := \theta_5,\ z(v_{21}) := \theta_6,$$

where

$$0 < \theta_5 < \theta_6 < \pi/2,$$

and

$$\pi/2 < z(v_{12}), z(v_{22}) < \pi.$$

In these coordinates,

$$v(z^{-1}[\theta_5, \theta_6]) = 1.$$

Let T_i be given by the matrix h_i in this coordinate system, $i = 1, 2$.

Let γ be Haar measure for this coordinate system; that is, γ is a probability measure on $\mathbf{P}^1$ such that

$$\gamma(z^{-1}[\theta_7, \theta_8]) = (\theta_8 - \theta_7)/\pi \quad \text{for} \quad 0 \leq \theta_7 \leq \theta_8 < \pi.$$

To prove Theorem 1(ii), it suffices to find a compact set A such that $v(A) = 1$ and $\gamma(A) = 0$.

Since in the previous parametrization

$$Z(g_i)([\theta_3, \pi) \cup [0, \theta_4]) \subset [\theta_3, \pi) \cup [0, \theta_4] \quad \text{for} \quad i = 1, 2,$$

we have that in this parameterization,

$$Z(h_i)[0, \pi/2] \subset [0, \pi/2], \quad \text{for} \quad i = 1, 2.$$

CLAIM. For any Borel set, $v(A) = 1$ implies that $v(h_1A \cup h_2A) = 1$.

PROOF. For any Borel set T, $v(T) = pv(h_1^{-1}T) + qv(h_2^{-1}T)$. Let $T := h_1A \cup h_2A$. Then

$$pv(h_1^{-1}(h_1A \cup h_2A)) + qv(h_2^{-1}(h_1A \cup h_2A)) = v(h_1A \cup h_2A).$$

The left side of this equation $\geq pv(A) + qv(A) = v(A) = 1$, and hence $v(h_1A \cup h_2A) = 1$.

Let F be the set of all sequences $k_n, \cdots, k_1$ such that $k_i = h_1$ or h_2 for all i. Define

$$A_n := \cup_F k_n \cdots k_1(z^{-1}[\theta_5, \theta_6]);$$

A_n is compact. Note that $A_{n+1} = h_1A_n \cup h_2A_n$. The preceding claim implies that $v(A_n) = 1$ for all $n \in Z_+$.

Let $A := \cap_{n=1}^{\infty} A_n$; A is compact. Take any sequence $k_1, \cdots, k_{n+1}$ with $k_i = h_1$ or h_2 for all i. Then

$$Z(k_1)[\theta_5, \theta_6] \subset [\theta_5, \theta_6],$$

and hence

$$Z(k_{n+1} \cdots k_1)([\theta_5, \theta_6]) \subset Z(k_{n+1} \cdots k_2)([\theta_5, \theta_6]).$$

Since $z^{-1}Z(k_{n+1} \cdots k_2)([\theta_5, \theta_6]) \subset A_n$, we have that $A_{n+1} \subset A_n$ for all $n \in Z_+$. Since the A_n form a decreasing sequence of sets, $v(\cap_{n=1}^{\infty} A_n) = v(A) = 1$.

8. Proof of (ii). Using A as in Section 7, in order to prove (ii), it remains to see that $\gamma(A) = 0$. In order to do this, we use cotangent coordinates. (Recall Definitions 3 and 5.) We have two parametrizations of $\mathbf{P}^1$, z and y. Hence we have a map δ: $[0, \pi) \to \mathbf{R} \cup \{\infty\}$ given by $\delta(\theta) := yz^{-1}(\theta)$; by unraveling definitions, we see that $\delta = \cot$. As a consequence, we have that for $g \in Sl_2(\mathbf{R})$, and $\theta \in [0, \pi)$, $\cot(Z(g)(\theta)) = Y(g)(\cot\theta)$ (since both equal $ygz^{-1}(\theta)$). We now have another easily proved lemma, whose proof is omitted.

LEMMA 4. *Take* $0 \leq a < b \leq \infty$. *Then*

$$\delta^{-1}([a, b]) = [\delta^{-1}(b), \delta^{-1}(a)];$$

furthermore,

$$0 \leq \delta^{-1}(b) < \delta^{-1}(a) \leq \pi/2, \quad \text{and} \quad |\delta^{-1}(b) - \delta^{-1}(a)| \leq |b - a|.$$

To see that $\gamma(A) = 0$, it suffices to show (given $\varepsilon > 0$, arbitrary) that $\gamma(A) < \varepsilon$, and so it suffices to find N such that $\gamma(A_N) < \varepsilon$.

Given N, let $B :=$ the set of all sequences $(k_N, \cdots, k_1)$ with $k_i = h_1$ or h_2 for all i. Let $k := k_N \cdots k_1$.

Since $Z(k)$ is continuous and bijective from $[\theta_5, \theta_6]$ into itself, we have

FORMULA 1. $Z(k)[\theta_5, \theta_6] = [\alpha, \beta]$, where

$$\alpha := Z(k)(\theta_6) \wedge Z(k)(\theta_5) \quad \text{and} \quad \beta := Z(k)(\theta_6) \vee Z(k)(\theta_5).$$

It follows immediately that

$$\gamma(A_N) \leq (\textstyle\sum_B |Z(k)(\theta_6) - Z(k)(\theta_5)|)/\pi.$$

Next, we claim that

$$\textstyle\sum_B |Z(k)(\theta_6) - Z(k)(\theta_5)| \leq \sum_B |Y(k)(\cot\theta_6) - Y(k)(\cot\theta_5)|.$$

Note that $Y(k)([\cot\theta_6, \cot\theta_5]) \subset [\cot\theta_6, \cot\theta_5]$, and that $\delta^{-1}Y(k)(\cot\theta_5) = Z(k)(\theta_5)$, $i = 5, 6$. This now follows at once from the second part of Lemma 4.

So define $F_N := (\sum_B |Y(k)(\cot\theta_6) - Y(k)(\cot\theta_5))|/\pi$. We'll be done with Theorem 1 if we can find $M \in Z_+$ such that $F_M < \varepsilon$.

By the choice of the new coordinate system, and Lemma 1, we see that $Y(h_1)(\cot\theta_5) = \cot\theta_5$, $Y(h_2)(\cot\theta_5) \in (\cot\theta_6, \cot\theta_5)$, $Y(h_1)(\cot\theta_6) \in (\cot\theta_6, \cot\theta_5)$, and $Y(h_2)(\cot\theta_6) = \cot\theta_6$.

Recall that in the previous coordinate system, $Z(g_1)[0, \pi/4] \cap Z(g_2)[0, \pi/4] = \emptyset$

Thus in the new coordinate system, $Z(h_1)[\theta_5, \theta_6] \cap Z(h_2)[\theta_5, \theta_6] = \emptyset$. If $\alpha \in Z(h_1)[\theta_5, \theta_6]$, and $\beta \in Z(h_2)[\theta_5, \theta_6]$, then $\alpha < \beta$. Thus, given

$$\alpha \in Y(h_1)[\cot\theta_6, \cot\theta_5], \beta \in Y(h_2)[\cot\theta_6, \cot\theta_5],$$

we see that $\alpha > \beta$.

Let $r := \cot\theta_6$, $s := Y(h_2)(\cot\theta_5)$, $t := Y(h_1)(\cot\theta_6)$, $u := \cot\theta_5$. We thus see that $r < s < t < u$, and hence F_1, which $= |s - r| + |u - t|, < \infty$.

We claim that

$$F_{n+1} \leq (1 - r(t - s)/s(u - r))F_n$$

for all n. Let $\lambda := r(t - s)/s(u - r)$. Clearly $0 < \lambda < 1$. Once we show the claim, we deduce that $F_{n+1} \leq (1 - \lambda)^n F_1$, and since $F_1 < \infty$ we will have that there exists M such that $F_M < \varepsilon$.

Thus it suffices to see that $F_{n+1} \leq (1 - \lambda)F_n$ for all $n \in Z_+$.

Pick $r \leq \tau < \psi \leq u$. We claim that $Y(h_1)(\tau) < Y(h_1)(\psi)$. If $\psi = u$, $Y(h_1)(\psi) = u$, and hence this follows from Lemma 1. If $\psi < u$, just as in the verification of Formula 1, we see that $Y(h_1)[\psi, u] = [Y(h_1)(\psi), u]$. Since $Y(h_1)$ is a bijection from $\mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$, $Y(h_1)(\psi) \neq Y(h_1)(\tau)$. If $Y(h_1)(\psi) < Y(h_1)(\tau)$, then

$$(\text{since} \quad Y(h_1)[r, u] \subset [r, u]), \quad Y(h_1)(\tau) \in [Y(h_1)(\psi), u]$$

and thus $Y(h_1)(\tau) = Y(h_1)(\rho)$ for some $\rho \in [\psi, u]$, a contradiction; hence $Y(h_1)(\tau) < Y(h_1)(\psi)$.

Similarly, for $r \leq \tau < \psi \leq u$, $Y(h_2)(\tau) < Y(h_2)(\psi)$. Pick a sequence $k_1, \cdots, k_N$ with $k_i = h_1$ or h_2 for all i. It is immediate by induction and the above that for $r \leq \tau < \psi \leq u$, $Y(k)(\tau) < Y(k)(\psi)$.

To see that $F_{N+1} \leq (1 - \lambda)F_N$, it suffices to see that

$$|Y(kh_1)(r) - Y(kh_1)(u)| + |Y(kh_2)(r) - Y(kh_2)(u)| \leq (1 - \lambda)|Y(k)(r) - Y(k)(u)|$$

for any sequence in B. Thus it suffices to see that

$$(Y(k)(u) - Y(k)(t)) + (Y(k)(s) - Y(k)(r)) \leq (l - \lambda)(Y(k)(u) - Y(k)(r)),$$

or that

$$\lambda \leq (Y(k)(t) - Y(k)(s))/(Y(k)(u) - Y(k)(r))$$

for any sequence in B.

So pick $(k_N, \cdots, k_1) \in B$, and let $k = k_N \cdots k_1$. Write k as $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, where we know that $ad - bc = 1$ (since h_1, h_2 have determinant one). We also know that $Y(k)[0, \infty] \subset (0, \infty)$ (by Lemma 1), and thus $Y(k)(0) \neq \infty$, $Y(k)(\infty) \neq \infty$. Since $k(0, 1)' = (b, d)'$, and $k(1, 0)' = (a, c)'$, we deduce that $d \neq 0$, $c \neq 0$.

Claim. $d/c > 0$.

Assume that $d/c < 0$. Then $k(m, 1)' = (am + b, cm + d)'$. For $m = -d/c$, $(-d/c)a + b \neq 0$, for otherwise $bc - ad = 0$, a contradiction. Since $-d/c > 0$, and since $am + b$ is continuous (in m), there exists $\varepsilon_0 > 0$ such that both $(-d/c) + \varepsilon_0$ and $(-d/c) - \varepsilon_0$ are > 0, and both $a((-d/c) + \varepsilon_0) + b$, $a((-d/c) - \varepsilon_0) + b$ are nonzero and of the same sign as $a(-d/c) + b$.

$$\begin{aligned} k((-d/c) + \varepsilon_0, 1)' &= (a((-d/c) + \varepsilon_0) + b, c((-d/c) + \varepsilon_0) + d)' \\ &= (a((-d/c) + \varepsilon_0) + b, c\,\varepsilon_0)', \end{aligned}$$

and

$$\begin{aligned} k((-d/c) - \varepsilon_0, 1)' &= (a((-d/c) - \varepsilon_0) + b, c((-d/c) - \varepsilon_0) + d)' \\ &= (a((-d/c) - \varepsilon_0) + b, -c\,\varepsilon_0)'. \end{aligned}$$

Hence

$$Y(k)((-d/c) + \varepsilon_0) = (a((-d/c) + \varepsilon_0) + b)/c\,\varepsilon_0$$

and

$$Y(k)((-d/c) - \varepsilon_0) = (a((-d/c) - \varepsilon_0) + b)/(-c\,\varepsilon_0).$$

However, $(a((-d/c) + \varepsilon_0) + b)/c\,\varepsilon_0$ and $(a((-d/c) - \varepsilon_0) + b)/(-c\,\varepsilon_0)$ are of opposite sign, contradicting the assumption that both are in $(0, \infty)$. Thus $d/c > 0$.

$k(m, 1)' = (am + b, cm + d)'$, for any $m \in (0, \infty)$. Thus $(Y(k)(t) - Y(k)(s))/(Y(k)(u) - Y(k)(r)) = J/L$, where $J := ((at + b)/(ct + d)) - ((as + b)/(cs + d))$ and $L := ((au + b)/(cu + d)) - ((ar + b)/(cr + d))$. After some cancellation (and expansion of products) we see that $J = (ad - bc)(t - s)/(ct + d)(cs + d) = (t - s)/(ct + d)(cs + d)$, and (similarly) $L = (u - r)/(cr + d)(cu + d)$. Thus $J/L = (t - s)(cr + d)(cu + d)/(u - r) \cdot (ct + d)(cs + d)$.

To finish the proof, we want to see that $J/L \geq \lambda = r(t - s)/s(u - r)$. Thus, to finish the proof of (ii) it suffices to see that $(cr + d)(cu + d)/(ct + d)(cs + d) \geq r/s$, which is equivalent to seeing that $(r + (d/c))(u + (d/c))/(t + (d/c))(s + (d/c)) \geq r/s$. However, since $d/c > 0$, we have that $(u + (d/c))/(t + (d/c)) \geq 1$ (since $u > t$) and $(r + (d/c))/(s + (d/c)) \geq r/s$. Hence J/L is indeed $\geq \lambda$, and thus $F_{n+1} \leq (1 - \lambda)F_n$ for all $n \in Z_+$. The proof of Theorem 1 is now completed.

9. Notes. Call V the subset of $Sl_2(\mathbf{R}) \times Sl_2(\mathbf{R})$ such that $(A, B) \in V$ if and only if for all p, q with $p + q = 1$, $pq > 0$, if $\mu(A) = p$, $\mu(B) = q$, $\mu(Sl_2(\mathbf{R})/A, B) = 0$, then any induced invariant measure on $\mathbf{P}^1$ is continuous singular. Then V contains an open set. Thus our class of binomial cases giving singular invariant measures is fairly large.

The major restrictions in Theorem 1 are that we assumed v_{12}, v_{22} were both in U, or both in V(a betweenness condition on the eigenpoints), as well as

$$T_1(V \cup \{v_{11}\} \cup \{v_{21}\}) \cap T_2(V \cup \{v_{11}\} \cup \{v_{21}\}) = \emptyset.$$

(a disjointness assumption). It would be quite interesting to investigate analogous theorems to Theorem 1 in the cases where these restrictions weren't satisfied. In particular, in Theorem 1, the stationary measures turned out to resemble Cantor measure. However, if we don't have the disjointness assumption, it seems easy to prove that any stationary measure must be supported on an interval of $\mathbf{P}^1$.

Finally, the techniques of this paper should allow us to estimate suitable almost sure limits for products of certain Bernoulli random matrices, using the previously referred to formula in Furstenberg (1963). The key difficulty with the use of that formula is in integrating against a stationary measure, while the proof of Theorem 1(ii) enables us to get our hands on a useful singular set.

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