The Annals of Probability
1984, Vol. 12, No. 1, 120–132

EXTRAPOLATION AND MOVING AVERAGE REPRESENTATION FOR STATIONARY RANDOM FIELDS AND BEURLING'S THEOREM[1]

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Strong regularity for stationary discrete random fields is discussed. An extension of the classical Beurling's Theorem to functions of several variables is given. Necessary and sufficient conditions for the moving average representation of stationary random fields are obtained. A recipe formula for the best linear extrapolator is also given.

0. Introduction. In this article we consider a set of real or complex random variables $X(t)$, $t \in Z^n$, over a probability space $(\Omega, \mathscr{B}, P)$, where Z^n = the Cartesian product of Z (set of integers) with itself n-times. Such a family is called a (discrete) univariate random field. Let E denote the expectation with respect to P. We assume that $EX(t) = 0$ and $E|X(t)|^2 < \infty$ as elements of a Hilbert space $L^2(\Omega, \mathscr{B}, P)$ of random variables Y with $E|Y|^2 < \infty$. A random field $X(t)$, $t \in Z^n$, is called stationary if the corresponding covariance function $R(s, t) = EX(s)\bar{X}(t)$ depends only on $t - s$.

The theory of extrapolation which refers to analyzing the behavior of a process up to a certain point (present) and predicting its behavior from that point on (future), has achieved a great success in the case that the parameter t runs through integers. This is because of the elegant theory of functions in the unit circle. Indeed with Szegö's theorem an application of Beurling's theorem is the key to the moving average representation and ultimately prediction problems.

The theory of extrapolation for random fields i.e., $t \in Z^n$, is not well developed. Some of the important results in this area are included in the work of Helson and Lowdenslager [2], where a generalization of Szegö's theorem is given; Chiang Tse-Pei [1], where the regularity problem for half spaces is discussed; and Kallianpur and Mandrekar [3], where a generalization of the Wold-Halmos theorem in the time domain is given. In none of the work in this topic, to our best knowledge, has the theory of Hardy functions in polydiscs been employed. This (might be) is because some important theorems in the theory of harmonic analysis for the unit disc U which play the main role in the prediction theory (such as Beurling's theorem) fail for the case of U^n, $n > 1$.

Our aim in this article is to employ the function theory in polydiscs in order to develop the theory of extrapolation for random fields. For this we need a version of Beurling's theorem. Indeed Beurling's theorem gives the characteri-

Received May 1982; revised March 1983.

[1] This research was supported by AFOSR Grant No. F49620 82C 0009.

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AMS 1980 *subject classifications*. Primary, 60G60, 62M20; secondary, 32A35.

Key words and phrases. Stationary random fields, regularity, function theory on polydiscs, Beurling's theorem, moving average representation, linear extrapolator.

zation of every invariant subspace of $H^2(U)$, the Hardy functions of class H^2, and also states that $S(h) = H^2(U)$ if and only if h is outer, where $S(h)$ is the invariant subspace of $H^2(U)$ generated by $h \in H^2(U)$. One part of this theorem holds if $n > 1$ (h is outer if $S(h) = H^2(U^n)$) and the other part (the useful one) fails. Section 2 is devoted to obtaining necessary and sufficient conditions for which $S(h) = H^2(U^n)$. In Section 1 we state some of the known results of the function theory in polydiscs and give some preliminary lemmas for later use. Section 3 deals with the regularity problem. We define four notions of regularity: Strong regularity, regularity for the half spaces (vertical or horizontal) and weak regularity. Szegö's alternative theorem and the Wold-Cramer concordance theorem are given for strong regularity in this section. For different types of regularity in interpolation theory such as J_0, J_∞ and J_c regularity see [5], [9]. In Section 4 we give necessary and sufficient conditions for the moving average representation of a stationary random field. In this paper, for simplicity, we consider random fields $X(t)$ with parameter $t \in Z^2$. This is of course no restriction and the results can be stated for the case of $t \in Z^n$ in a similar way.

1. Preliminaries. Following [7] let C be the complex field, U the open unit disc in C with boundary T and C^2, U^2, T^2 be the Cartesian product of two copies of C, U, T respectively. The complex conjugate of a complex-valued function f is denoted by $\bar{f}$.

The Hardy space of functions $H^p(U^2)$, $p > 0$, is the class of all complex-valued analytic functions f in U^2 for which

$$\sup_{0 \le r < 1} \int_{T^2} |f_r(w)|^p \, dm_2 < \infty,$$

where $f_r(w) = f(rw)$, $w = (w_1, w_2) \in T^2$ and m_2 is the Lebesgue measure on T^2.

Here we state some known results of the $H^p(U^2)$ theory. We refer the readers to [7] and [10]. Throughout this paper $L^p(T^2)$ stands for $L^p(T^2, m_2)$ and for $g \in L^1(T^2)$, $g^\vee$ stands for the inverse Fourier transform of g, i.e., $g^\vee(t) = \int_{T^2} \bar{w}^t g(w) \, dm_2$, $t \in Z^2$.

1.1 THEOREM. *Let $f \in H^p(U^2)$, $1 \le p < \infty$. Then*

(a) *f has a nontangential limit $f^*(w)$ in each variable at almost every $w \in T^2$. In particular $f^*(w) = \lim_{r \to 1} f(rw)$ exists for almost every $w \in T^2$. Also f_r tends to f^* in L^p norm as $r \to 1$, i.e., $\int_{T^2} |f_r - f^*|^p \, dm_2 \to 0$ as $r \to 1$. Moreover* $\log |f^*| \in L^1(T^2)$.

(b) *$f^{*\vee}(t) = \int_{T^2} \bar{w}^t f^*(w) \, dm_2 = 0$ for $t \notin Z^{2+}$ and $f(z) = \sum_{t \in Z^{2+}} f^{*\vee}(t) z^t$, where $\bar{w}^t = \bar{w}_1^{t_1} \bar{w}_2^{t_2}$, $t = (t_1, t_2)$, $Z^{2+} = \{t \in Z^2 : t_i \ge 0, i = 1, 2\}$ and the convergence of the summation is in L^p norm. Moreover if $g \in L^p(T^2)$ with $g^\vee(t) = 0$, $t \notin Z^{2+}$, the function $g(z) = \sum_{t \in Z^{2+}} g^\vee(t) z^t$, $z \in U^2$, is well defined and belongs to H^p.*

(c) *$f = P[f^*] = C[f^*]$, where*

$$P[f^*](z) = \int_{T^2} P(z, w) f^*(w) \, dm_2$$

and

$$C[f^*](z) = \int_{T^2} C(z, w) f^*(w)\, dm_2, \quad z \in U^2$$

are the Poisson integral and Cauchy integral of f^* *respectively, with* $P(z, w) = P_{r_1}(\theta_1 - \phi_1) P_{r_2}(\theta_2 - \phi_2)$ *the Poisson kernel, where* $z = (z_1, z_2)$, $z_j = re^{i\theta_j}$, $w_j = e^{i\phi_j}$: $j = 1, 2$ *and* $P_r(\theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)$, *and* $C(z, w) = 1/(1 - \bar{w}_1 z_1) \cdot 1/(1 - \bar{w}_2 z_2)$ *the Cauchy kernel.*

It is well known that every real-valued function $u \in L^p(T)$ is the real part of the boundary value of an $f = u + iv \in H^p(U)$. Such property is no more satisfied for the case that $n > 1$. The following theorem gives the details. This theorem is stated in [10] page 129 for $p = 2$. With the help of the above theorem, the proof can be carried out similarly for any $1 \le p < \infty$.

1.2 THEOREM. *Let* $f \in H^p(U^2)$, $1 \le p < \infty$, *with* $f(z) = u(z) + iv(z)$. *Then* $f^*(w) = u(w) + iv(w)$ *exists almost everywhere and*

(a) $f(z) = \int_{T^2} 2C(z, w) u(w)\, dm_2$

(b) $v(z) = \int_{T^2} Q(z, w) u(w)\, dm_2$

the boundary values u and v satisfying the condition that $v^\vee(t) = (-i \operatorname{sign}^* t) u^\vee(t)$, *where* $\operatorname{sign}^* t = 1$ *if* $t \in Z^{2+}$ *and* -1 *if* $t \notin Z^{2+}$, *where* $Q(z, w)$ *is the imaginary part of* $C(z, w)$.

(c) *A real-valued function* $u \in L^p(T^2)$ *is the real part of the boundary value* f^* *of an* $f \in H^p(U^2)$ *if and only if* $u^\vee(t) = 0$ *for* $t \notin Z^{2+} \cup (-Z^{2+})$, *where* $-Z^{2+} = \{-t : t \in Z^{2+}\}$.

Recall from [7] page 73 that a function h of class H^p is said to be outer if

$$\log |h| = P[\log |h^*|].$$

The following lemma gives some of the properties of outer functions. For more on outer functions see [7] page 72.

1.3 LEMMA. *Let h be an outer function. Then*

(a) *h has no zero on* U^2.

(b) $\log |h|$ *is the real part of an analytic function,*

(c) $(\log |h^*|)^\vee(t) = 0$ *for* $t \notin Z^{2+} \cup (-Z^{2+})$,

(d) *if a real-valued function* $\phi(w)$, $w \in T^2$, *satisfies the condition that* $\phi^\vee(t) = 0$ *for* $t \notin Z^{2+} \cup (-Z^{2+})$, *then the function*

$$h(z) = \exp\left\{\int_{T^2} 2C(z, w) \phi(w)\, dm_2\right\} \tag{1.4}$$

is an outer function with $\log |h| = P[\phi]$ *and* $\log |h^*| = \phi$, a.e. m_2.

(e) *If $h^*(w)$, $w = (w_1, w_2) \in T^2$, is the boundary-value of an outer function, then the functions $h^*_{w_1}(\cdot) = h^*(w_1, \cdot)$, $h^*_{w_2}(\cdot) = h^*(\cdot, w_2)$ which are defined on T are outer for almost all w_1 and w_2 respectively.*

PROOF. (a)–(c) are self-evident. (d) follows from Theorem 1.2 and the definition of outer function. For (e) let

$$h_{w_2}(z_1) = \int P_{r_1}(\theta_1 - \phi_1) h^*(w_1, w_2)\, dm_1(w_1).$$

Then for $w_2 \notin A_{z_1}$, $h_{w_2}(z_1) = \lim_{z_2 \to w_2} h(z_1, z_2)$, where $A_{z_1} \subset T$, depending on z_1 with zero Lebesgue measure. This implies that $\log | h_{w_2}(z_1) | = \lim_{z_2 \to w_2} \log | h(z_1, z_2) |$, $w_2 \notin A_{z_1}$. Since h is outer

$$\log | h(z_1, z_2) | = \int_T P_{r_2}(\theta_2 - \phi_2) \int_T P_{r_1}(\theta_1 - \phi_1) \log | h^*(w_1, w_2) |\, dm_2.$$

Therefore for $w_2 \notin B_{z_1}$

$$\lim_{z_2 \to w_2} \log | h(z_1, z_2) | = \int_T P_{r_1}(\theta_1 - \phi_1) \log | h^*(w_1, w_2) |\, dm_1,$$

where B_{z_1} is a set of measure zero depending on z_1. Thus for $w_2 \notin D_{z_1} = A_{z_1} \cup B_{z_1}$,

$$\log | h_{w_2}(z_1) | = \int_T P_{r_1}(\theta_1 - \phi_1) \log | h^*(w_1, w_2) |\, dm_1.$$

Now let $\{z_j\}$ be a dense set in U and let $D = \cup_j D_{z_j}$. Then D has Lebesgue measure zero and for $w_2 \notin D$

$$\log | h_{w_2}(z_1) | = \int_T P_{r_1}(\theta_1 - \phi_1) \log | h^*(w_1, w_2) |\, dm_1 \quad \text{for} \quad \forall z_1 \in U,$$

giving the proof.

The following theorem gives one part of the Beurling's theorem [7] page 74.

1.5 THEOREM. *If $f \in H^p(U^2)$ and $\vee\{z^t f : t \in Z^{2+}\} = H^p(U^2)$ then f is outer, where $\vee\{z^t f : t \in Z^{2+}\}$ is the smallest invariant subspace of H^p containing f.*

2. Canonical representation and Beurling's Theorem.

NOTATIONS. Let $X(t)$, $t \in Z^2$, be a stationary random field. Define $\mathcal{M}_X(S) = \vee_{t \in S} X(t)$, the span closure of $X(t)$, $t \in S$, in $L^2(\Omega, \mathcal{B}, P)$. $\mathcal{M}_X^{t_1\infty}$, $\mathcal{M}_X^{\infty t_2}$, $\mathcal{M}_X^{t_1 t_2}$, $\mathcal{M}_X^{t_1 \vee t_2}$, $\mathcal{M}_X$ stand for $\mathcal{M}_X(S)$ whenever S is $\{s \in Z^2 : s_1 \leq t_1, s_2 \in Z\}$, $\{s \in Z^2 : s_1 \in Z, s_2 \leq t_2\}$, $\{s \in Z^2 : s_1 \leq t_1 \text{ and } s_2 \leq t_2\}$, $\{s \in Z^2 : s_1 \leq t_1 \text{ or } s_2 \leq t_2\}$ or Z^2 respectively.

It is well known [6] that corresponding to a stationary random field $X(t)$, $t \in$

Z^2, there exists a random measure Φ on T^2 and with values in $\mathscr{M}_X$ such that

$$E\Phi(S_1)\overline{\Phi(S_2)} = \int_{S_1S_2} dF(w) \tag{2.1}$$

for $S_1, S_2 \subset T^2$, and

$$X(t) = \int_{T^2} w^t \, d\Phi(w), \tag{2.2}$$

where F is a positive finite measure on T^2. F is called the spectral measure of the process $X(t)$. Conversely corresponding to each positive finite measure F on T^2 there is a stationary random field $X(t)$, $t \in Z^2$, and a random measure Φ satisfying (2.1) and (2.2) [6]. Whenever the spectral measure F is absolutely continuous with respect to the Lebesgue measure, the corresponding process is said to have a spectral density $f(w) = (dF/dm_2)(w)$. Clearly

$$EX(t)\overline{X(s)} = \int_{T^2} w^{t-s} \, dF(w). \tag{2.3}$$

(2.3) defines the so called Kolmogorov isomorphism between time domain $\mathscr{M}_X$ and spectral domain $\mathscr{A}_X = L^2(T^2, F)$. Define $\mathscr{A}_X(S) = \vee_{t \in S} w^t$, the span closure of w^t, $t \in S$, in $L^2(T^2, F)$. $\mathscr{A}_X(S)$ is isomorphic to $\mathscr{M}_X(S)$ for any $S \subset Z^2$. $\mathscr{A}_X^{t_1\infty}$, $\mathscr{A}_X^{\infty t_2}$, $\mathscr{A}_X^{t_1t_2}$, $\mathscr{A}_X^{t_1 \vee t_2}$ are defined in a similar way.

Let $h \in H^2(U^2)$. Then the function $f(w) = |h^*(w)|^2$ is a positive function of class $L^1(T^2)$. Let $X(t)$, $t \in Z^2$, be a stationary random field with density f, and let Φ be the corresponding random measure. Define

$$\xi^*(S) = \int_S \bar{h}^{*-1}(w) \, d\Phi(w), \quad S \subset T^2. \tag{2.4}$$

Then it is easy to verify that

(a) $\xi^*(S)$ is a random variable for each set $S \subset T^2$,
(b) $\xi^*(S_1 + S_2) = \xi^*(S_1) + \xi^*(S_2)$, where $S_1, S_2 \subset T^2$ with $S_1S_2 = \varnothing$,
(c) $E\xi^*(S_1)\overline{\xi^*(S_2)} = \int_{S_1S_2} dm_2$ (which, in particular, implies that ξ^* has independent increments), and
(d)

$$X(t) = \int_{T^2} w^t \overline{h^*}(w) \, d\xi^*(w). \tag{2.5}$$

NOTE. Since the set of finite linear combination of indicator functions is dense in $L^2(T^2)$, the identity (c) given above establishes an isomorphism between $\mathscr{A}_{\xi^*} = \vee\{\xi^*(S), S \subset T^2\}$ and $L^2(T^2)$ which corresponds to each $\not{f} \in \mathscr{A}_{\xi^*}$ a unique $f \in L^2(T^2)$ with $\not{f} = \int f \, d\xi^*$. Also for $\not{f}_1, \not{f}_2 \in \mathscr{A}_{\xi^*}$, $\langle \not{f}_1, \not{f}_2 \rangle_{\mathscr{A}_{\xi^*}} = \langle f_1, f_2 \rangle_{L^2(T^2)}$ where $f_i \in L^2(T^2)$ corresponds to $\not{f}_i \in \mathscr{A}_{\xi^*}$, $i = 1, 2$. Now (2.5) implies that $X(t) \in \mathscr{A}_{\xi^*}$ and $X(t)$ corresponds to $w^t\overline{h^*}$ in $L^2(T^2)$.

Based on ξ^*, define three random measures ξ_1 on $T \times Z$, ξ_2 on $Z \times T$ and ξ on Z^2 by

$$\xi_1(A) = \int_{T^2} \left\{ \frac{1}{\sqrt{2\pi}} \sum_{t \in A_{w_1}} w_2^t \right\} d\xi^*(w), \tag{2.6}$$

$$\xi_2(B) = \int_{T^2} \left\{ \frac{1}{\sqrt{2\pi}} \sum_{t \in B_{w_2}} w_1^t \right\} d\xi^*(w)$$

and

$$\xi(D) = \int_{T^2} 1_D^\wedge(w)\, d\xi^*(w), \tag{2.7}$$

where $A \subset T \times Z$ with $A_{w_1} = \{t \in Z: (w_1, t) \in A\}$. $B \subset Z \times T$ with $B_{w_2} = \{t \in Z: (t, w_2) \in B\}$ and $D \subset Z^2$ with $1_D^\wedge(w) = \sum_{t \in D} w^t$, $w = (w_1, w_2)$. It is easy to see that the random measures ξ_1, ξ_2, and ξ have independent increments and for any $f \in L^2(T^2)$ the following holds:

$$\int_{T^2} f(w)\, d\xi^*(w) = \int_T \sum_{t \in Z} f(w_1, \overset{\vee}{\cdot})(t)\, \xi_1(dw_1, t)$$

$$= \int_T \sum_{t \in Z} f(\overset{\vee}{\cdot}, w_2)(t) \xi_2(t, dw_2) = \sum_{t \in Z^2} f^\vee(t) \xi(t).$$

This in particular implies that

$$X(t) = \int_T \sum_{s=-\infty}^{t_2} w_1^{t_1} \overline{h^*}(w_1, \overset{\vee}{\cdot})(s - t_2) \xi_1(dw_1, s) \tag{2.8}$$

$$X(t) = \int_T \sum_{s=-\infty}^{t_1} w_2^{t_2} \overline{h^*}(\overset{\vee}{\cdot}, w_2)(s - t_1) \xi_2(s, dw_2),$$

and

$$X(t) = \sum_{s_1=-\infty}^{t_1} \sum_{s_2=-\infty}^{t_2} \overset{\vee}{\overline{h^*}}\,(s_1 = t_1, s_2 = t_2) \xi(s_1, s_2). \tag{2.9}$$

Similarly to the Note given above $\mathscr{H}_{\xi_1}$, $\mathscr{H}_{\xi_2}$ and $\mathscr{H}_\xi$ are defined and are isomorphic to $L^2(T \times Z, m_1 \times c)$, $L^2(Z \times T, c \times m_1)$ and $L^2(Z^2, c \times c)$, where c is counting measure. Because of these isomorphisms we make no distinction between $\mathscr{H}_{\xi^*}$, $\mathscr{H}_{\xi_1}$, $\mathscr{H}_{\xi_2}$, $\mathscr{H}_\xi$ and the corresponding L^2 spaces.

Let

$$\mathscr{H}_{\xi_1}^{\infty t} = \{f(w_1, s) \in L^2(T \times Z): f(w_1, s) = 0 \text{ for } s > t \text{ and any } w_1 \in T\},$$

$$\mathscr{H}_{\xi_2}^{t\infty} = \{f(s, w_2) \in L^2(Z \times T): f(s, w_2) = 0 \text{ for } s > t,\ w_2 \in T\}$$

and

$$\mathscr{H}_\xi^{st} = \{f(s', t') \in L^2(Z^2): f(s', t') = 0 \text{ for } s' > s \text{ or } t' > t\}.$$

$\mathscr{A}_{\xi_1}^{\infty t}$, $\mathscr{A}_{\xi_2}^{t\infty}$, and $\mathscr{A}_{\xi}^{st}$ can be identified as subspaces of $L^2(T^2)$ as

$$\mathscr{A}_{\xi_1}^{\infty t} = \{f \in L^2(T^2): f(w_1, \overset{\vee}{\cdot})(s) = 0 \text{ for } s > t \text{ and a.e. } w_1\},$$

$$\mathscr{A}_{\xi_2}^{t\infty} = \{f \in L^2(T^2): f(\overset{\vee}{\cdot}, w_2)(s) = 0 \text{ for } s > t \text{ and a.e. } w_2\}$$

and

$$\mathscr{A}_{\xi}^{st} = \{f \in L^2(T^2): f^{\vee}(s', t') = 0 \text{ for } s' > s \text{ or } t' > t\}$$

where $f(w_1, \overset{\vee}{\cdot})$, $f(\overset{\vee}{\cdot}, w_2)$, $f^{\vee}$ are the inverse Fourier transform of $f_{w_1}(w_2) = f(w_1, w_2)$, $f_{w_2}(w_1) = f(w_1, w_2)$ and $f(w_1, w_2)$ respectively. By using a Fourier transform argument it is easy to check that

$$\mathscr{A}_{\xi_1}^{\infty t} = \mathscr{A}_{\xi}^{\infty t} \quad \text{and} \quad \mathscr{A}_{\xi_2}^{t\infty} = \mathscr{A}_{\xi}^{t\infty}, \quad \forall t \in Z \tag{2.10}$$

as a subspace of $L^2(T^2)$.

2.11 DEFINITION. The representations (2.8) and (2.9) are said to be canonical if $\mathscr{A}_X^{\infty t} = \mathscr{A}_{\xi_1}^{\infty t}$, $\mathscr{A}_X^{t\infty} = \mathscr{A}_{\xi_2}^{t\infty}$ for all $t \in Z$ and $\mathscr{A}_X^{t_1t_2} = \mathscr{A}_{\xi}^{t_1t_2}$ for all $t \in Z^2$ respectively.

2.12 THEOREM. *The representation* (2.9) *is canonical if and only if the representations* (2.8) *are canonical and*

$$\mathscr{A}_X^{t_1t_2} = \mathscr{A}_X^{t_1\infty} \cap \mathscr{A}_X^{\infty t_2} \quad \textit{for all} \quad t = (t_1, t_2) \in Z^2. \tag{2.13}$$

PROOF. Suppose the representations of X in (2.8) are canonical and (2.13) is satisfied. Then $\mathscr{A}_X^{\infty t} = \mathscr{A}_{\xi_1}^{\infty t}$ and $\mathscr{A}_X^{s\infty} = \mathscr{A}_{\xi_2}^{s\infty}$. But

$$\mathscr{A}_X^{t_1t_2} = \mathscr{A}_X^{\infty t_2} \cap \mathscr{A}_X^{t_1\infty} = \mathscr{A}_{\xi}^{\infty t_2} \cap \mathscr{A}_{\xi}^{t_1\infty} = \mathscr{A}_{\xi}^{t_1t_2},$$

where the first equality is by the assumption, the second equality comes from (2.10) and the fact that $\mathscr{A}_X \leftrightarrow \vee_t w^t \overline{h^*}$, and the fact that ξ has independent increments gives the third equality. Therefore the representation (2.9) is canonical. Now suppose that representation (2.9) is canonical, i.e. $\mathscr{A}_X^{t_1t_2} = \mathscr{A}_{\xi}^{t_1t_2}$ for all $t \in Z^2$. But

$$\mathscr{A}_X^{\infty t_2} = \cup_{t_1} \mathscr{A}_X^{t_1t_2} = \lim_{t_1 \to \infty} \mathscr{A}_X^{t_1t_2} = \lim_{t_1 \to \infty} \mathscr{A}_{\xi}^{t_1t_2} = \mathscr{A}_{\xi}^{\infty t_2}.$$

Also by the same argument we obtain that $\mathscr{A}_X^{t_1\infty} = \mathscr{A}_{\xi}^{t_1\infty}$. But

$$\mathscr{A}_X^{t_1t_2} = \mathscr{A}_{\xi}^{t_1t_2} = \mathscr{A}_{\xi}^{t_1\infty} \cap \mathscr{A}_{\xi}^{\infty t_2} = \mathscr{A}_X^{t_1\infty} \cap \mathscr{A}_X^{\infty t_2}$$

giving the result.

The following lemmas give the necessary and sufficient conditions in terms of h in order that the representations (2.8) be canonical. The proofs can be carried out similarly to the proofs of corresponding propositions given by Karhunen [4] page 155.

2.14 LEMMA. *The first representation in* (2.8) *is canonical if and only if there does not exist a function* $k \in L^2(T \times Z^+)$ *satisfying*

$$\sum_{t=0}^{\infty} h(w_1, t^{\wedge})k(w_1, t + s) = 0 \tag{2.15}$$

for almost every $w_1 \in T$ *and all* $s \geq 0$.

2.16 LEMMA. (2.15) *is satisfied if and only if*

$$h^*(w_1, w_2) = \lim_{0 \leq r < 1, r \to 1} h^*(w_1, rw_2),$$

where

$$h^*(w_1, rw_2) = a(w_1)\exp\left\{\frac{1}{2\pi}\int_T \frac{w + z}{w - z} \log |h(w_1, w)| \, dm_1\right\},$$

with $z = rw_2$ *and* $a(w_1)$ *is some complex measurable function with* $|a(w_1)| = 1$.

From Lemmas (2.14) and (2.16) it follows that the representation (2.8) is canonical if and only if $h^*_{w_1}(\cdot) = h^*(w_1, \cdot)$ and $h^*_{w_2}(\cdot) = h^*(\cdot, w_2)$ are the outer functions on T for almost all w_1 and w_2 respectively. Now by using Theorem 2.12 we arrive at the following theorem.

2.17 THEOREM. *The representation* (2.9) *is canonical if and only if* $h^*(w_1, \cdot)$, $h^*(\cdot, w_2)$ *are outer functions for almost all* w_1, w_2 *respectively and* $\mathscr{M}_X^{t_1t_2} = \mathscr{M}_X^{t_1\infty} \cap \mathscr{M}_X^{\infty t_2}$ *for all* $t = (t_1, t_2) \in Z^2$.

As we observed earlier $\mathscr{M}_X^{t_1t_2}$ is isomorphic to $\vee\{w^s\overline{h^*}: s_1 \leq t_1, s_2 \leq t_2\}$ in $L^2(T^2, dm_2)$. Since $\mathscr{M}_X^{t_1t_2} = U_t\mathscr{M}_X^{00}$, where U_t is the shift operator corresponding to w^t, $t = (t_1, t_2)$. The representation (2.8) being canonical is equivalent to saying that $\vee\{w^th^*: t \in Z^{2+}\}$ coincides with $\{f \in L^2(T^2): f^{\vee}(t) = 0 \text{ for } t \notin Z^{2+}\}$. The second class defines the space of Hardy functions of class H^2. Thus, combining this with Theorem 2.17 and Lemma 1.3(e) and Theorem 1.5 we obtain the following theorem which is an extension of Beurling's Theorem to the functions of several variables.

2.18 THEOREM. *Let* $h \in H^2$. *Then* $\vee\{z^th: t \in Z^{2+}\} = H^2$ *if and only if h is outer and* $\mathscr{M}_X^{t_1t_2} = \mathscr{M}_X^{t_1\infty} \cap \mathscr{M}_X^{\infty t_2}$, *where* $X(t)$ *is a stationary random field corresponding to* h^* *through* (2.5).

3. Regular stationary random fields. With the same notations as in Section 2, a stationary random field $X(t)$, $t \in Z^2$, is called

(a) strongly regular (*s*-regular) if $\bigcap_{t \in (-Z^{2+})} \mathscr{M}_X^{t_1 \vee t_2} = \{0\}$, where $t = (t_1, t_2)$
(b) horizontally regular (*h*-regular) if $\bigcap_{t_2 < 0} \mathscr{M}_X^{\infty t_2} = \{0\}$
(c) vertically regular (*v*-regular) if $\bigcap_{t_1 < 0} \mathscr{M}_X^{t_1\infty} = \{0\}$
(d) *hv*-regular if it is *h*-regular and *v*-regular
(e) weakly regular (*w*-regular) if $\bigcap_{t \in (-Z^{2+})} \mathscr{M}_X^{t_1t_2} = \{0\}$, $t = (t_1, t_2)$.

Obviously s-regular fields are hv-regular and hv-regular fields are w-regular. h-regular fields (so v-regular fields) are discussed in [1] where necessary and sufficient conditions for h-regularity in terms of the spectral measure are given, namely a stationary random field with spectral measure F is h-regular if and only if F is absolutely continuous with respect to $F(dw_1, \pi)\, dm_1$ and for almost all w_1 (relative to the measure $F(dw_1, \pi)$)

$$\int \left| \log \frac{dF(w_1, w_2)}{F(dw_1, \pi)\, dm_1} \right| dm_1(w_2) < \infty, \quad \text{where} \quad F(A, \pi) = F(A \times T), \quad A \subset T.$$

Our aim in this section is to obtain necessary conditions and sufficient conditions for s-regularity in terms of the spectral measure F. We start with the following lemma.

3.1 Lemma. *$\mathscr{A}_X^{-1\vee -1^{\perp}}$ is isomorphic to $\{g \in H^1 \colon \int (|g|^2/F_a')\, dm_2 < \infty\}$, where $\perp$ stands for the orthogonal complement in $L^2(F)$, F_a' for the derivative of the absolutely continuous part of F and $H^1 = H^1(U^2)$.*

Proof. Let $k \in \mathscr{A}_X^{-1\vee -1^{\perp}}$. Then $k \perp w^t$ in $L^2(F)$ for all $t = (t_1, t_2) \in Z^2$ with $t_1 \leq -1$ or $t_2 \leq -1$, i.e. $\int_{T^2} \bar{w}^t k\, dF = 0$ for $t \notin Z^{2+}$. This by the extension of Bochner's theorem [2] page 184 implies that $k(w)\, dF(w)$ is absolutely continuous with respect to Lebesgue measure m_2, giving $k\, dF = kF_a'\, dm_2$ which implies that k is zero on the singular part of F. Now $\int \bar{w}^t kF_a'\, dm_2 = 0$ for $t \notin Z^{2+}$ which by Theorem 1.1(b) implies that $kF_a' \in H^1$. Let $g = kF_a'$. Then $g \in H^1$ and $\int (|g|^2/F_a')\, dm_2 = \int |k|^2 F_a'\, dm_2 = \int |k|^2\, dF < \infty$. The rest is self-evident.

3.2 Lemma. *Let $\mathscr{A}_X^{-1\vee -1^{\perp}} \neq \{0\}$. Then $\mathscr{A}_{-\infty} = \bigcap_{t \in (-Z^{2+})} \mathscr{A}_X^{t_1 \vee t_2} = L^2(F_s)$ and $\log F_a' \in L^1(T^2)$, where F_s is the singular part of F.*

Proof. Note that $\mathscr{A}_X = (\bigcap_{t \in (-Z^{2+})} \mathscr{A}_X^{t_1 \vee t_2}) \oplus (\bigcap_{t \in (-Z^{2+})} \mathscr{A}_X^{t_1 \vee t_2})^{\perp}$ and also $\mathscr{A}_X = L^2(F) = L^2(F_a) \oplus L^2(F_s)$. So the result follows by showing that $\bigvee_t \mathscr{A}_X^{t_1 \vee t^{\perp}} = L^2(F_a)$. Since the elements of $\mathscr{A}_X^{t_1 \vee t_2^{\perp}}$ are zero on F_s we obtain that $\bigvee_t \mathscr{A}_X^{t_1 \vee t_2^{\perp}} \subseteq L^2(F_a)$. Now let $k \in L^2(F_a)$ with $k \perp \bigvee \mathscr{A}_X^{t_1 \vee t_2^{\perp}}$. Since $\mathscr{A}_X^{t_1 \vee t_2^{\perp}} = w^t \mathscr{A}_X^{0 \vee 0^{\perp}}$ we have $\int \bar{w}^t k \bar{g} F_a' = 0$ for all $t \in Z^2$ and $g \in \mathscr{A}_X^{0 \vee 0^{\perp}}$ which gives that $k\bar{g}F_a' = 0$ a.e. m_2. But $\bar{w}gF_a' \in H^1$ giving $k = 0$ a.e. m_2. To show $\log F_a' \in L^1(T^2)$ note that $\bar{w}gF_a' \in H^1$. Therefore, by Theorem 1.1(a), $\log |\bar{g}F_a'| \in L^1(T^2)$ and $\int_{T^2} \log[|g|^2 F_a']\, dm_2 \leq \|g\|_{L^2(F)}$. Now the result follows from the identity that $\log F_a' = \log[|g| F_a']^2 - \log[|g|^2 F_a']$.

3.3 Lemma. *If* (i) $\log F_a' \in L^1(T^2)$ *and* (ii) $(\log F_a')^{\vee}(t) = 0$ *for* $t \notin Z^{2+} \cup (-Z^{2+})$ *then* $\mathscr{A}_X^{-1\vee -1^{\perp}} \neq \{0\}$.

Proof. Let ϕ be $\log F_a'$ in (1.4). Then the corresponding function h belongs to H^1 with $|h^*| = F_a'$ a.e. m_2. Now $\int (|h^*|^2/F_a')\, dm_2 = \int F_a'\, dm_2 < \infty$. Apply Lemma 3.1.

Let us summarize the lemmas given above in the following theorem.

3.4 THEOREM. *Let $X(t)$, $t \in Z^2$, be a stationary random field with spectral measure F. Then:*

(a) *If $X(t)$, $t \in Z^2$, is s-regular, then F is absolutely continuous with respect to Lebesgue measure and* $\log F' \in L^1(T^2)$.

(b) *If F is absolutely continuous with respect to Lebesgue measure and* $\log F' \in L^1(T^2)$ *with* $(\log F')^\vee(t) = 0$ *for $t \notin Z^{2+} \cup (-Z^{2+})$, then $X(t)$ is s-regular.*

(c) *If* 3.3(i), (ii) *are satisfied, then* $\mathscr{A}_{-\infty} = L^2(F_s)$.

3.5 DEFINITION. A random field $X(t)$, $t \in Z^2$, is called singular if $\mathscr{A}_X^{t_1 \vee t_2} = \mathscr{A}_X$ for all $t = (t_1, t_2) \in Z^2$.

From Part (c) of Theorem 3.4 and the fact that $\mathscr{A}_X^{t_1 \vee t_2^\perp} = \mathscr{A}_{X_1}^{t_1 \vee t_2^\perp}$ for all $t = (t_1, t_2) \in Z^2$, where $X_1(t)$, $t \in Z^2$, is a stationary random field with spectral measure F_a, we arrive at the following theorem which gives the so-called Wold-Cramér concordance theorem for the stationary random fields in the theory of extrapolation.

3.6 THEOREM (Wold-Cramér concordance). *Let $X(t)$, $t \in Z^2$, be a stationary random field with spectral measure F. Let F_a and F_s denote the absolutely continuous part and the singular part of F respectively with respect to Lebesgue measure. Suppose the density of F_a satisfies the conditions* (i) *and* (ii) *of Lemma* 3.3. *Then $X(t)$, $t \in Z^2$, can be uniquely decomposed in the form $X(t) = X_1(t) \oplus X_2(t)$, $t \in Z^2$, where the field $X_1(t)$ is s-regular, $X_2(t)$ is singular and $\mathscr{A}_{X_1} \perp \mathscr{A}_{X_2}$. Moreover $\mathscr{A}_X^{t_1 \vee t_2} = \mathscr{A}_{X_1}^{t_1 \vee t_2} \oplus \mathscr{A}_{X_2}^{t_1 \vee t_2}$ and $F_{X_1} = F_a$ and $F_{X_2} = F_s$.*

PROOF. $X(t) = \int w^t \, d\Phi$, where Φ is the random spectral measure of $X(t)$. Let $X_1(t) = \int w^t 1_{A^c} \, d\Phi$, $X_2(t) = \int w^t 1_A \, d\Phi$, where A is the support of F_s. Clearly $X(t) = X_1(t) \oplus X_2(t)$. The rest is plain.

REMARK. The following problem is left open by this section: whether the condition that $\log F' \in L^1(T^2)$ is sufficient for s-regularity, (which amounts to proving or disproving that if f is positive function satisfying the conditions that $f \in L^1(T^2)$ and $\log f \in L^1(T^2)$, then there exists a function $h \in H^1(T^2)$ with $\int_{T^2} (|h|^2/f) \, dm_2 < \infty$)).

4. Moving average representations. The problem of obtaining a moving average representation for a process $X(t)$, $t \in Z^2$, is to present $X(t)$ as a filtered white noise. Such a representation is extremely important and useful in dealing with the prediction and filtering problems. In this section, by combining the results of the previous sections, we give necessary and sufficient conditions for a random field $X(t)$, $t \in Z^2$, to admit such a representation which leads to a recipe formula for the best linear extrapolator. For the best linear interpolator see [8]. We start with the definition of a moving average representation.

4.1 DEFINITION. A random field $X(t)$, $t \in Z^2$, is said to have a moving average representation if it has a canonical representation of the form (2.9). Namely, there is a sequence of scalars a_t, $t = (t_1, t_2) \in Z^{2+}$, with $\sum_{t\in Z^{2+}} |a_t|^2 < \infty$ and

$$(4.2)\qquad \begin{aligned} X(t) &= \textstyle\sum_{s_1=-\infty}^{t_1} \sum_{s_2=-\infty}^{t_2} a_{(t_1-s_1,\, t_2-s_2)}\xi(s_1, s_2) \\ \mathscr{A}_X^{t_1t_2} &= \mathscr{A}_\xi^{t_1t_2} \quad \text{for all} \quad t = (t_1, t_2) \in Z^2 \end{aligned}$$

where ξ is a random measure on Z^2 with $E\xi(A)\overline{\xi(B)} = \langle 1_A, 1_B\rangle_{L^2(Z^2, c\times c)}$.

4.3 THEOREM. *Let $X(t)$, $t \in Z^2$, be a stationary random field with spectral measure F. Then $X(t)$ has a moving average representation if and only if $X(t)$ has a density $f = (dF/dm_2)$ and*

(i) $\log f \in L^1$,
(ii) $(\log f)^\vee(t) = 0$ *for* $t \notin Z^{2+} \cup (-Z^{2+})$, *and*
(iii) $\mathscr{A}_X^{t_1t_2} = \mathscr{A}_X^{t_1\infty} \cap \mathscr{A}_X^{t_2\infty}$, *for* $t = (t_1, t_2) \in Z^2$.

PROOF. *Sufficiency*: Let $h(z) = \exp\{\int_{T^2} C(z, w)\log f(w)\, dm_2\}$. Then by Theorem 1.2 and Lemma 1.3(d), h is an outer function of class H^2. Because of condition 4.4(iii), Theorem 2.18 or 2.17 can be applied to h, giving that the representation (2.9) is canonical. This proves that $X(t)$ has a moving average representation. Note that in this case, in (4.2), $a_t = h^{*\vee}(t)$, $t \in Z^{2+}$.

Necessity: Suppose $X(t)$, $t \in Z^2$, is given by (4.2). Let $h^*(w) = \sum_{t\in Z^2} a_t w^t$. Then h^* defines an H^2 function and $X(t) = \int_{T^2} w^t \overline{h^*}\, d\,\xi^*(w)$, where ξ^* is a random measure on T^2 given by $\xi^*(A) = \sum_{t\in Z^2} 1_A^\vee(t)\xi(t)$, $A \subset T^2$, $1_A^\vee = (1/\sqrt{2\pi}) \int w^t 1_A \cdot dm_2$. Clearly $|h^*|^2 = f$ a.e. m_2. Now by Theorem 2.18 $\mathscr{A}_X^{t_1t_2} = \mathscr{A}_\xi^{t_1t_2}$ implies that h^* is outer and 4.3(iii) is satisfied. (i) is trivial and (ii) follows from Lemma 1.3(c). Proof of the theorem is complete.

4.4 COROLLARY. *Every random field which admits a moving average representation is s-regular.*

PROOF. Use Theorems 4.3 and 3.4(b).

The condition that $\mathscr{A}_X^{t_1t_2} = \mathscr{A}_X^{t_1\infty} \cap \mathscr{A}_X^{\infty t_2}$ which appeared in Theorems 2.18 and 4.3 plays an important role in these theorems. In the following theorem we give a family of spectral densities for which the corresponding processes satisfy this condition.

4.5 THEOREM. *Let $X(t)$, $t \in Z^2$, be a stationary random field with density f. If $0 < c_1 \le f \le c_2$ a.e. m_2, where c_1 and c_2 are positive constants, then*

$$\mathscr{A}_X(S_1 \cap S_2) = \mathscr{A}_X(S_1) \cap \mathscr{A}_X(S_2) \quad \textit{for any} \quad S_1, S_2 \subseteq Z^2.$$

PROOF. Note that for $S \subseteq Z^2$, $\mathscr{A}_X(S)^\perp \equiv \{g \in L^1(T^2); g^\vee(t) = 0 \text{ for } t \in S$ and $\int (|g|^2/f)\, dm_2 < \infty\}$. Let $\mathscr{A}_X^*(S) = \bigvee\{w^t, t \in S\}$ in $L^2(f^{-1})$. Clearly $\mathscr{A}_X^*(S^c) \subset \mathscr{A}_X(S)^\perp$, where S^c is the complement of S in Z^2. Now let $g \in \mathscr{A}_X(S)^\perp$. Then

$$\int |g|^2\, dm_2 = \int \frac{|g|^2}{f} f\, dm_2 \le c_2 \int \frac{|g|^2}{f}\, dm_2 < \infty,$$

giving that $g \in L^2(dm_2)$. Thus $g = \sum_{t \in S^c} g^\vee(t) w^t$ in $L^2(dm_2)$. Since $f^{-1} \le c_1^{-1}$, $g = \sum_{t \in S^c} g^\vee(t) w^t$ in $L^2(f^{-1})$, which implies that $g \in \mathscr{A}^*(S^c)$. Therefore $\mathscr{A}_X^*(S)^\perp = \mathscr{A}^*(S^c)$. Now

$$\begin{aligned}\mathscr{A}_X(S_1 \cap S_2)^\perp &= \mathscr{A}_X^*((S_1 \cap S_2)^c) = \mathscr{A}_X^*(S_1^c \cup S_2^c) = \mathscr{A}_X^*(S_1^c) \vee \mathscr{A}_X^*(S_2^c)\\ &= \mathscr{A}_X(S_1)^\perp \vee \mathscr{A}_X(S_2)^\perp = (\mathscr{A}_X(S_1) \cap \mathscr{A}_X(S_2))^\perp.\end{aligned}$$

Thus $\mathscr{A}_X(S_1 \cap S_2) = \mathscr{A}_X(S_1) \cap \mathscr{A}_X(S_2)$ finishing the proof.

As mentioned, the representation (4.2) is extremely important in prediction problems. Consider a Gaussian stationary random field $X(t)$, $t \in Z^2$, i.e. $X(t)$ and all finite linear combinations of $X(t)$, $t \in Z^2$, are Gaussian random variables. It is desirable to obtain the distribution of $X(T)$, conditional on the $X(t)$, $t \in \{t \in Z^2\colon t_1 \le s_1 \text{ and } t_2 \le s_2\}$, where $s = (s_1, s_2)$ is a fixed point in Z^2 and $T \notin \{t \in Z^2\colon t_1 \le s_1, t_2 \le s_2\}$. The conditional law is Gaussian with mean $m = E\{X(T) \mid X(t); t_1 \le s_1, t_2 \le s_2\}$ and variance $\sigma^2 = E[(X(T) - m)^2 \mid X(t); t_1 \le s_1, t_2 \le s_2]$. Now by using (4.2)

$$\begin{aligned} m &= E[\textstyle\sum_{r_1=-\infty}^{T_1} \sum_{r_2=-\infty}^{T_2} a_{(T_1-r_1, T_2-r_2)} \xi(r_1, r_2) \mid \mathscr{A}_X^{s_1 s_2}]\\ &= E[\textstyle\sum_{r_1=-\infty}^{T_1} \sum_{r_2=-\infty}^{T_2} a_{(T_1-r_1, T_2-r_2)} \xi(r_1, r_2) \mid \mathscr{A}_\xi^{s_1 s_2}]\\ &= \textstyle\sum_{r_1=-\infty}^{s_1} \sum_{r_2=-\infty}^{s_2} a_{(T_1-r_1, T_2-r_2)} \xi(r_1, r_2)\end{aligned}$$

and

$$\begin{aligned}\sigma^2 = {}& \textstyle\sum_{r_1=s_1+1}^{T_1} \sum_{r_2=-\infty}^{s_2} |a_{(T_1-r_1, T_2-r_2)}|^2\\ &+ \textstyle\sum_{r_1=-\infty}^{s_1} \sum_{r_2=s_2+1}^{T_2} |a_{(T_1-r_1, T_2-r_2)}|^2 + \sum_{r_1=s_1+1}^{T_1} \sum_{r_2=s_2+1}^{T_2} |a_{(T_1-r_1, T_2-r_2)}|^2.\end{aligned}$$

From this it follows that

$$P_{\mathscr{A}^{s_1 s_2}} X(T) = \int_{T^2} w^T \textstyle\sum_{t=T-s}^{\infty} a_t \bar{w}^t\, d\xi^*(w) = \displaystyle\int_{T^2} \frac{1}{\bar{h}^*} \{w^T \textstyle\sum_{t=T-s}^{\infty} a_t \bar{w}^t\}\, d\Phi,$$

where $\sum_{t=T-s}^{\infty}$ stands for $\sum_{t_1=T_1-s_1}^{\infty} \sum_{t_2=T_2-s_2}^{\infty}$.

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