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# SOME REMARKS ABOUT THE CONVOLUTION OF UNIMODAL FUNCTIONS

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In the note a new class of functions called $\alpha$-quasi-concave is introduced and it is proved that the convolution of two unimodal functions each having some additional concavity properties belongs to this class. Other results concerning the convolution of unimodal functions are also studied and the extensions of some of them are also given in the paper.

**1.** Anderson [1], generalizing the concept of 1-dimensional unimodality due to Khintchine [14], called the function $f\colon R^n \to R^1_+$, $(n \geq 1)$, unimodal if its upper level sets

$$A(f, u) := \{x \in R^n\colon f(x) \geq u\} \tag{1.1}$$

are convex for all $u \geq 0$, or, equivalently, if

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\} \tag{1.2}$$

for all $x, y \in R^n$ and $0 \leq \lambda \leq 1$.

In more recent literature these functions are called quasiconcave. In what follows we shall use the latter name for these functions.

The function $f\colon R^n \to R^1_+$ is called symmetric if $f(x) = f(-x)$ for all $x \in R^n$. The set $A \subset R^n$ is called symmetric if $A = -A$. The characteristic (indicator) function of a set $A$ is denoted $\chi_A$. The convolution

$$f * g(y) := \int_{R^n} f(x)g(y - x)\,dx, \quad y \in R^n, \tag{1.3}$$

of two quasiconcave (unimodal) functions $f$ and $g$ has been for many decades a subject of intensive research.

In the 1-dimensional case $(n = 1)$ the following three "classical" results had been proved:

1. If both $f$ and $g$ are symmetric and quasiconcave, then $f * g$ is quasiconcave (Wintner [22]).
2. There is a nonsymmetric quasiconcave function $f$ such that $f * f$ is not quasiconcave (Chung [3], see Feller [9] page 164).
3. Let $f$ be quasiconcave. Then $f * g$ is quasiconcave for all quasiconcave $g$ if and only if $f$ is logconcave (Ibragimov [12]).

For higher dimensions the convolution of two symmetric quasiconcave func-

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tions need not be quasiconcave:

EXAMPLE. (Sherman [19]). Let $f = 2\chi_A + \chi_B$, where $A := \{x \in R^2: |x_1| \leq 1, |x_2| \leq 1\}$ and $B := \{x \in R^2: |x_1| \leq 1, |x_2| \leq 5\}$. Then $f * \chi_A$ is not quasiconcave. □

First results for the general case $n \geq 1$ are due to Anderson [1] and Sherman [19]. Namely, using Brunn-Minkowski inequality (see e.g. [10]) we can easily prove:

LEMMA. (Sherman [19]). *If $A, B \subset R^n$ are convex sets, then $\chi_A * \chi_B$ is quasiconcave.* □

Using the obvious identity

$$f * g(y) = \int_{R^2_+} \chi_{A(f,u)} * \chi_{A(g,v)}(y)\, du\, dv, \tag{1.4}$$

this lemma easily implies:

THEOREM 1.1. ([1], [19]). *If $f, g: R^n \rightarrow R^1_+$ are symmetric and quasiconcave, then*

$$f * g(\lambda y) \geq f * g(y) \tag{1.5}$$

*for all $y \in R^n$ and $0 \leq \lambda \leq 1$.* □

Using the identity (1.4) we can in fact easily prove the following extension of this theorem.

THEOREM 1.2. *Let $f, g: R^n \rightarrow R^1_+$ be two quasiconcave functions such that their translates $f(x + a)$ and $g(x + b)$ are symmetric functions of $x$ for some $a, b \in R^n$. Then the condition*

$$f * g(a + b) = f * g(\theta) \tag{1.6}$$

*is sufficient and in the case $a + b \neq \theta$ is also necessary for $f * g$ to have the property* (1.5) *for all $y \in R^n$ and $0 \leq \lambda \leq 1$. ($\theta$ is the zero vector of $R^n$.)* □

Theorem 1.1 had been both sharpened and generalized by many authors ([19], [8], [13]; see reviews in [6], [7] or [20]). In all results the symmetry about the origin of both convolved functions had been always assumed. Theorem 1.2 shows how far we can go with translations of the symmetric functions so that the property of their convolution remained the same.

**2.** No symmetry assumptions are needed if we restrict ourselves to some subclasses of quasiconcave functions. The subclasses come from the observation that $\min\{a, b\}$ appearing on the right hand side of (1.2) is "at one end" of the following class of means. Let $a, b \geq 0$, $-\infty < \alpha < +\infty$, $\alpha \neq 0$ and denote

$$(2.1) \qquad M_\alpha^\lambda(a, b) := \begin{cases} 0 & \text{if } a \cdot b = 0 \\ (\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha} & \text{if } a \cdot b > 0. \end{cases}$$

For $\alpha = -\infty, 0, +\infty$ we take limits to get: $M_0^\lambda(a, b) := a^\lambda \cdot b^{1-\lambda}$, $M_{-\infty}^\lambda(a, b) := \min\{a, b\}$, $M_{+\infty}^\lambda(a, b) := \{0$ if $a \cdot b = 0$ and $\max\{a, b\}$ if $a \cdot b > 0\}$. $M_\alpha^\lambda(a, b)$ is for $\lambda$, $a$, $b$ fixed a nondecreasing function of $\alpha$ on $-\infty \leq \alpha \leq +\infty$ (see, e.g. [11]).

We call the function $f\colon R^n \to R_+^1$ *$\alpha$-concave*, $-\infty \leq \alpha \leq +\infty$, if

$$(2.2) \qquad f(\lambda x + (1-\lambda)y) \geq M_\alpha^\lambda(f(x), f(y))$$

for all $x, y \in R^n$ and $0 \leq \lambda \leq 1$. (Compare with $\alpha$-unimodal functions of Das Gupta [6]; see remarks in the next section.)

We call the function $f\colon R^n \to R_+^1$ *$\alpha$-quasiconcave*, $-\infty < \alpha < +\infty$, if

$$(2.3) \qquad f(\lambda x + (1-\lambda)y) \geq \min\{\lambda^\alpha f(x), (1-\lambda)^\alpha f(y)\}$$

for all $x, y \in R^n$ and $0 \leq \lambda \leq 1$.

THEOREM 2.1. *Let $\alpha + \beta \geq 0$. The convolution of an $\alpha$-concave function with a $\beta$-concave one is*

$$\text{(A)} \qquad (1/\alpha + 1/\beta + n)^{-1}\text{-concave} \qquad \text{if } -1/n \leq \frac{\alpha\beta}{\alpha + \beta} \leq +\infty,$$

$$\text{(B)} \qquad (1/\alpha + 1/\beta + n)\text{-quasiconcave} \quad \text{if } -\infty \leq \frac{\alpha\beta}{\alpha + \beta} \leq -1/n. \quad \square$$

REMARK 2.1. Of course, we always assume that the functions in question are such that their convolutions exist (say, both belong to $L^2(R^n) \cap L^1(R^n)$). □

REMARK 2.2. Davidovich, Korenblum and Hacet [5] proved (A) for $\alpha = \beta = 0$ and Borell [2] for $\alpha, \beta \geq 0$. Das Gupta [6] announced (A) for the convolution of an $\alpha$-unimodal function with a $\beta$-unimodal one. Our class of $\alpha$-concave functions is more general than that of $\alpha$-unimodal ones (see remarks in the next section). We have to note that there is a misprint in Das Gupta's paper: in [6] he announced (A) under the condition $\alpha \cdot \beta < 0$ instead of $\alpha + \beta \geq 0$. This misprint can easily be corrected using his paper (use Theorem 4.1, [6] page 307 and property (3), [6] page 306). □

REMARK 2.3. The condition $\alpha + \beta \geq 0$ is in a weaker sense also necessary for (A) to hold. Namely, if (A) were true for $\alpha = 0$ and a sequence of $\beta_i$, $i = 1, 2, \cdots$, such that $\beta_i \to_{i \to +\infty} -\infty$, then taking limit, (A) would imply: (a) "The convolution of a 0-concave function with a $-\infty$-concave (quasiconcave) one is quasiconcave." Similarly, assume that (A) holds for an infinite sequence of pairs $(\alpha_i, \beta_i)$ such that $\alpha_i < -\beta_i$ and $\alpha_i \to -\infty$ and $\beta_i \to +\infty$ as $i \to +\infty$; taking the limit case, we would get: (b) "If $f$ is quasiconcave and $K$ a convex set, then $f * \chi_K$ is

$1/n$-concave". Using the example of Sherman (Section 1) Das Gupta showed that (a) cannot be true (see [6], page 308, 8th row from above). The same example shows directly that neither (b) can be true. Instead of (b) a much weaker statement is true, see Corollary 2.1 below. □

The proof of Theorem 2.1 is based on the following two lemmas.

LEMMA 2.1. *Let $a, b, c, d \geq 0$ and $\alpha + \beta \geq 0$. Then*

$$M_\alpha^\lambda(a, b) \cdot M_\beta^\lambda(c, d) \geq M_{\alpha\beta/(\alpha+\beta)}^\lambda(ac, bd). \quad \square \tag{2.4}$$

PROOF. Assume that $a, b, c, d > 0$, $0 < \lambda < 1$, $-\infty < \alpha, \beta < +\infty$, $\alpha \cdot \beta \neq 0$ and $\alpha + \beta > 0$. Denote $p = (\alpha + \beta)/\beta$, $q = (\alpha + \beta)/\alpha$; $r = \alpha\beta/(\alpha + \beta)$, $x = \lambda^{1/p}a^r$, $y = \lambda^{1/q}c^r$, $u = (1 - \lambda)^{1/p}b^r$, $v = (1 - \lambda)^{1/q}d^r$. Write the Hölder inequality:

$$\begin{aligned} &x \cdot y + u \cdot v \leq (x^p + u^p)1/p \cdot (y^q + v^q)1/q \\ &\qquad\qquad\qquad\qquad \text{if} \quad p, q > 1, \quad 1/p + 1/q = 1, \end{aligned} \tag{2.5}$$

and the reverse inequality if $\{0 < p < 1, q < 0\}$ or $\{0 < q < 1, p < 0\}$, $1/p + 1/q = 1$. The inequality (2.5) implies

$$(x \cdot y + u \cdot v)^{1/r} \leq (x^p + u^p)^{1/pr} \cdot (y^q + v^q)^{1/qr}, \tag{2.6}$$

where $r$ should be positive in the first case and negative in the reverse case. The condition $\{p, q > 1, r > 0\}$ is equivalent to $\{\alpha, \beta > 0\}$ and conditions $\{0 < p < 1, q < 0, r < 0\}$ or $\{0 < q < 1, p < 0, r < 0\}$ are equivalent to $\{\alpha + \beta > 0, \alpha \cdot \beta < 0\}$. Taking limits we can prove (2.4) for the remaining cases. We note that (2.4) is in general not true if $\alpha + \beta < 0$. □

LEMMA 2.2. ([4]). *Let $f, g\colon R^n \to R^1_+$ be Borel-measurable functions. Let $0 \leq \lambda \leq 1$, $-\infty \leq \gamma \leq +\infty$ and denote $h(t) := \text{ess-sup}_{\lambda x+(1-\lambda)y=t} M_\gamma^\lambda(f(x), g(y))$. Then*

$$\int_{R^n} h(t)\, dt \geq \begin{cases} M^\lambda_{\gamma/(1+n\gamma)}\left(\int_{R^n} f(x)\, dx, \int_{R^n} g(x)\, dx\right), & \textit{if} \quad -1/n \leq \gamma \leq +\infty, \\ \min\left\{\lambda^{n+1/\gamma} \cdot \int_{R^n} f(x)\, dx, (1-\lambda)^{n+1/\gamma} \cdot \int_{R^n} g(x)\, dx\right\} & \\ & \textit{if} \quad -\infty \leq \gamma \leq -1/n. \quad \square \end{cases} \tag{2.7}$$

PROOF. See [4] Theorem 3.1, Theorem 3.3 and Remark on page 398. □

PROOF OF THEOREM 2.1. Let $f$ be $\alpha$-concave, $g$ be $\beta$-concave. Then

$$f(t) \geq \sup_{\lambda x+(1-\lambda)y=t} M_\alpha^\lambda(f(x), f(y)), \tag{2.8}$$

$$g(\lambda u + (1 - \lambda)v - t) \geq \sup_{\lambda x+(1-\lambda)y=t} M_\beta^\lambda(g(u - x), g(v - y)). \tag{2.9}$$

The product of suprema is not smaller than the supremum of product of means, so assuming $\alpha + \beta \geq 0$ and using Lemma 2.1 we get

$$f * g(\lambda u + (1 - \lambda)v) \geq \int_{R^n} \sup_{\lambda x+(1-\lambda)y=t} M_\gamma^\lambda(f(x)g(u - x), f(y)g(v - y))\, dt, \tag{2.10}$$

where $\gamma = \alpha\beta/(\alpha + \beta)$.

Applying Lemma 2.2 to (2.10) we get the theorem. □

It is clear that the class of $+\infty$-concave functions coincides with the class $\{\chi_K, K \text{ convex sets}\}$. So we have

COROLLARY 2.1. *If $K \subset R^n$ is convex and $f\colon R^n \to R^1_+$ is quasiconcave, then $f * \chi_K$ is n-quasiconcave.* □

PROOF. $\chi_K$ is $+\infty$-concave, $f$ is $-\infty$-concave. Taking the limit case of (B) of the theorem, i.e. letting tend $\alpha \to -\infty$, $\beta \to +\infty$ so that $\alpha + \beta \to 0+$ and $\alpha\beta/(\alpha + \beta) \to -\infty$, we get the result. □

**3.** Das Gupta [6] called the nonnegative function $f$ defined on the open convex set $W \subset R^n$ $\alpha$-unimodal (see [6], page 304) if: for $-\infty \leq \alpha \leq 0$, $f$ fulfills the condition (2.2) for all $x, y \in W$ and $0 \leq \lambda \leq 1$; for $0 < \alpha < +\infty$, $f^\alpha$ is concave; and for $\alpha = +\infty$, $f(\lambda x + (1 - \lambda)y) \geq \max\{f(x), f(y)\}$ for all $x, y \in W$ and $0 \leq \lambda \leq 1$. It is clear that for arbitrary $W$ any $\alpha$-unimodal function is $\alpha$-concave (after extending its domain to $R^n$ by $f \equiv 0$ on $R^n \backslash W$). On the other hand, the support $\operatorname{supp} f := \{x \in R^n\colon f(x) > 0\}$ of any $\alpha$-concave function is clearly convex, but not necessarily open (take e.g. $\chi_K$, $K$ convex nonopen). Borel [2] defined his functions similarly. We see that our Theorem 2.1 is an extension of their results also in this sense.

Olshen and Savage [18] called the function $f\colon R^n \to R^1_+$ *$\alpha$-unimodal*, $-\infty < \alpha < +\infty$, if

$$f(\lambda x) \geq \lambda^{\alpha-n} f(x) \tag{3.1}$$

for all $x \in R^n$ and $0 \leq \lambda \leq 1$. For these functions we have ([18]):

> The convolution of an $\alpha$-unimodal function with a $\beta$-unimodal one is $\alpha + \beta$-unimodal. □

An easy corollary of this result is

COROLLARY 3.1. *If $f$ and $g$ are quasiconcave functions such that $f(\theta) \geq f(x)$, $g(\theta) \geq g(x)$ for all $x \in R^n$, then $f * g$ is $2n$-unimodal.* □

The condition of $n$-quasiconcavity principally differs from the conditions (3.1) for $\alpha = n$ and $\alpha = 2n$ ($n$-unimodality and $2n$-unimodality). So Corollary 2.1 conveys new information also in the cases treated by Theorem 1.1 ($K$, $f$ are symmetric) and by Corollary 3.1 ($\theta \in K$, $f(\theta) \geq f(x)$ $\forall\, x \in R^n$). The corollary throws new light on the example in Section 1 as well. It seems to be interesting also in the 1-dimensional case. The comparison of $\alpha + n$-unimodality ((3.1) for $\alpha := \alpha + n$) with $\alpha$-quasiconcavity is also illustrative using $A(f, u)$: $f$ is $\alpha + n$-unimodal iff $\lambda A(f, u) \subseteq A(f, \lambda^{\alpha} u)$ for all $u \geq 0$ and $0 \leq \lambda \leq 1$; $f$ is $\alpha$-quasiconcave if and only if $\lambda A(f, u) + (1 - \lambda)A(f, u) \subseteq A(f, \min\{\lambda^{\alpha} u, (1 - \lambda)^{\alpha} u\})$ for all $u \geq 0$ and $0 \leq \lambda \leq 1$ (algebraic sum of sets).

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