The Annals of Probability
1984, Vol. 12, No. 2, 635–639

A NOTE ON THE RATE OF CONVERGENCE IN THE MARTINGALE CENTRAL LIMIT THEOREM

BY ERICH HAEUSLER

University of Munich

It is shown that a method recently developed by Bolthausen permits an extension (up to a logarithmic factor) of an estimate of the rate of convergence in the martingale central limit theorem due to Heyde and Brown.

1. Results. Let the real random variables $X_1, \cdots, X_n$ form a martingale difference sequence w.r.t. the σ-fields $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$, i.e. suppose that $E(|X_i|) < \infty$, $X_i \in \mathscr{F}_i$ and $E(X_i \mid \mathscr{F}_{i-1}) = 0$ a.s. for $1 \le i \le n$. For convenience, write $\sigma_i^2 = E(X_i^2 \mid \mathscr{F}_{i-1})$ and $a_n = \sup_{t \in \mathbb{R}} |P(\sum_{i=1}^n X_i \le t) - \Phi(t)|$, where Φ denotes the standard normal distribution function.

In martingale central limit theory, various sets of conditions for $a_n \to 0$ are known, and many authors derived explicit bounds on a_n. For $0 < \delta \le 1$, Heyde and Brown (1970) showed

$$a_n \le c_\delta (L_{n,2\delta} + N_{n,2\delta})^{1/(3+2\delta)} \tag{1.1}$$

where $L_{n,2\delta} = \sum_{i=1}^n E(|X_i|^{2+2\delta})$ and $N_{n,2\delta} = E(|\sum_{i=1}^n \sigma_i^2 - 1|^{1+\delta})$ and where c_δ is a finite constant depending only on δ. The virtue of an estimate like (1.1) is that it provides a rate of convergence under basic conditions of the martingale central limit theorem, demanding that these conditions hold in an L_p-norm, not only in probability.

The proof of Heyde and Brown is based on the martingale version of the Skorokhod embedding scheme. This method seems to be unsuited to obtain (1.1) for $\delta > 1$. It is the aim of the present note to show that the method developed in Bolthausen (1982) yields (1.1) for all $\delta > ½$ up to a logarithmic factor. The following theorem is a preliminary result.

THEOREM. *Suppose $\sum_{i=1}^n \sigma_i^2 = 1$ a.s. For any $\delta > ½$ there exists a finite constant c_δ depending only on δ such that $a_n \le c_\delta L_{n,2\delta}^{1/(3+2\delta)} |\log L_{n,2\delta}|$ whenever $L_{n,2\delta} \le ½$.*

If the assumption of $\sum_{i=1}^n \sigma_i^2$ being a constant a.s. is deleted, we obtain as a corollary the following:

MAIN RESULT. For any $\delta > ½$ there exists a finite constant c_δ depending only on δ such that $a_n \le c_\delta (L_{n,2\delta} + N_{n,2\delta})^{1/(3+2\delta)} |\log(L_{n,2\delta} + N_{n,2\delta})|$ whenever $L_{n,2\delta} + N_{n,2\delta} \le ½$.

Received March, 1983.
AMS 1980 *subject classifications.* Primary 60F05, 60G42.
Key words and phrases. Martingales, central limit theorem, rates of convergence.

For a stationary martingale difference sequence $(X_i)_{i\in\mathbb{Z}_+}$ such that $E(X_1^2 \mid X_j, j \le 0) = 1$ a.s. and $E(|X_1|^{2+2\delta}) < \infty$ for some $\delta > 1$ we infer

$$\sup_{t\in\mathbb{R}} |P(\sum_{i=1}^n X_i \le tn^{1/2}) - \Phi(t)| = O(n^{-\delta/(3+2\delta)}\log n).$$

For $\delta > 3/2$ this rate is better than $n^{-1/4}$, and apparently it does not follow from the results of Kato (1979) and Bolthausen (1982) which seem to be the only known results establishing rates faster than $n^{-1/4}$.

2. Proofs.

PROOF OF THE THEOREM. Since the proof is similar to the proof of Theorem 2 in Bolthausen (1982) we only sketch the main steps for which we use a notation which is as close to Bolthausen's as possible. As a notational convention, let the symbol c always denote a generic absolute constant. Let $Z_1, \cdots, Z_n$ be standard normal variables and ξ be a centered normal variable with variance $\kappa^2 > 0$ such that $\mathscr{F}_n, Z_1, \cdots, Z_n$ and ξ are independent. Putting $s = 1$ we obtain as in (4.2)–(4.4) of Bolthausen (1982)

$$\begin{aligned} a_n \le c[&\sup_{t\in\mathbb{R}} \textstyle\sum_{m=1}^n E(|X_m|^3\lambda_m^{-3}|\varphi''(T_m - \Theta_m X_m\lambda_m^{-1})|) \\ &+ \sup_{t\in\mathbb{R}} \textstyle\sum_{m=1}^n E(|\sigma_m Z_m|^3\lambda_m^{-3}|\varphi''(T_m - \Theta'_m\sigma_m Z_m\lambda_m^{-1})|) + \kappa] \end{aligned} \tag{2.1}$$

where $U_m = \sum_{j=1}^{m-1} X_j$, $T_m = \lambda_m^{-1}(t - U_m)$, $\lambda_m^2 = \sum_{j=m+1}^n \sigma_j^2 + \kappa^2$ and $0 \le \Theta_m, \Theta'_m \le 1$ and where φ denotes the standard normal density. Here the assumption $\sum_{i=1}^n \sigma_i^2 = 1$ a.s. is essential.

Fix $\beta \in (0, \min(\kappa^2/3, 1/2))$ and define stopping times $\tau_0 \le \tau_1 \le \cdots \le \tau_{[\beta^{-1}]+1}$ by $\tau_0 = 0$, $\tau_j = \inf\{k: \sum_{i=1}^k \sigma_i^2 \ge j\beta\}$ for $1 \le j \le [\beta^{-1}]$ and $\tau_{[\beta^{-1}]+1} = n$, where $[\beta^{-1}]$ denotes the integer part of β^{-1}. Fix $t \in \mathbb{R}$ and write

$$\begin{aligned} &\textstyle\sum_{m=1}^n E(|X_m|^3\lambda_m^{-3}|\varphi''(T_m - \Theta_m X_m\lambda_m^{-1})|) \\ &\quad\le c\textstyle\sum_{j=1}^{[\beta^{-1}]+1} [E(\sum_{m=\tau_{j-1}+1}^{\tau_j} |X_m|^3\lambda_m^{-3} I(|X_m| \le \beta^{1/2}) \\ &\qquad\qquad I(\sigma_m^2 \le \beta)|\varphi''(T_m - \Theta_m X_m\lambda_m^{-1})|) \\ &\qquad\qquad + E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} |X_m|^3\lambda_m^{-3} I(\sigma_m^2 > \beta)) \\ &\qquad\qquad + E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} |X_m|^3\lambda_m^{-3} I(|X_m| > \beta^{1/2}))] \\ &\quad =: c\textstyle\sum_{j=1}^{[\beta^{-1}]+1} [I_j + II_j + III_j]. \end{aligned}$$

Let us consider I_j for a fixed $j \in \{1, \cdots, [\beta^{-1}] + 1\}$. On $\{\tau_{j-1} < m \le \tau_j\} \cap \{\sigma_m^2 \le \beta\} \cap \{|X_m| \le \beta^{1/2}\}$ we have

$$\begin{aligned} \underline{\lambda}_j^2 &:= 1 - j\beta - \beta + \kappa^2 \le \lambda_m^2 \le \bar{\lambda}_j^2 \\ &:= 1 - (j-1)\beta + \kappa^2 \quad \text{and} \quad |\Theta_m X_m \lambda_m^{-1}| \le 1. \end{aligned}$$

Furthermore, $\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2\, I(\sigma_m^2 \le \beta) \le 2\beta$ a.s. and $\kappa^2 \ge 3\beta$. Therefore, the arguments leading from (4.6) to (4.9) of Bolthausen (1982) may be copied to obtain

$$I_j \leq c\underline{\lambda}_j^{-3}\beta^{3/2}[a_n + (1-(j-1)\beta)^{1/2} + \bar{\lambda}_j].$$

Note that Bolthausen always employs the inequality $a_n \leq \delta(n, s, \gamma) \leq \delta(n-1, s, 2\gamma)$ in his proof; this step is deleted here. Obviously

$$II_j + III_j \leq c\kappa^{-3}\beta^{1/2-\delta}E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} |X_m|^{2+2\delta}),$$

hence

$$\begin{aligned}\sup_{t\in\mathbb{R}} \textstyle\sum_{m=1}^n E(|X_m|^3\lambda_m^{-3}|\varphi''(T_m - \Theta_m X_m \lambda_m^{-1})|)\\ \leq c[a_n\beta^{3/2} \textstyle\sum_{j=1}^{[\beta^{-1}]+1} \underline{\lambda}_j^{-3} + \beta^{3/2} \sum_{j=1}^{[\beta^{-1}]+1} \underline{\lambda}_j^{-3}(1-(j-1)\beta)^{1/2}\\ + \textstyle\sum_{j=1}^{[\beta^{-1}]+1} \underline{\lambda}_j^{-3}\bar{\lambda}_j + \kappa^{-3}\beta^{1/2-\delta}L_{n,2\delta}]\\ \leq c[a_n\beta^{1/2}|\log\beta|(\kappa^3 - 2\beta)^{-1/2} + \beta^{1/2}|\log\beta| + \kappa^{-3}\beta^{1/2-\delta}L_{n,2\delta} + \kappa]\end{aligned}$$

where the last inequality is the result of simple calculations. By the same modifications of Bolthausen's arguments on the second half of page 679 of his paper, we obtain exactly the same estimate for the second supremum in (2.1). Combining, we arrive at

$$a_n \leq c^* a_n\beta^{1/2}|\log\beta|(\kappa^2 - 2\beta)^{-1/2} + c^*\beta^{1/2}|\log\beta| + c^*\kappa^{-3}\beta^{1/2-\delta}L_{n,2\delta} + c^*\kappa.$$

This inequality is true for all $\kappa^2 > 0$ and all $\beta \in (0, \min(\kappa^2/3, \tfrac{1}{2}))$, and c^* is an absolute constant. We take now $\kappa^2 = 3\beta + 4c^{*2}\beta|\log\beta|^2$ for $\beta \in (0, \tfrac{1}{2})$. Then

$$a_n \leq a_n/2 + c^*\beta^{1/2}|\log\beta| + 3^{-3/2}\beta^{-1-\delta}L_{n,2\delta} + 3^{1/2}\beta^{1/2} + 2c^*\beta^{1/2}|\log\beta|,$$

hence for all $\beta \in (0, \tfrac{1}{2})$ and an absolute constant $c < \infty$

$$a_n \leq c(\beta^{1/2}|\log\beta| + \beta^{-1-\delta}L_{n,2\delta}).$$

Putting $\beta = L_{n,2\delta}^{2/(3+2\delta)}$, we have $\beta < \tfrac{1}{2}$ for $L_{n,2\delta} < 2^{-(3+2\delta)/2}$ and $a_n \leq c_\delta L_{n,2\delta}^{1/(3+2\delta)} |\log L_{n,2\delta}|$, where c_δ depends only on δ. This estimate remains true for $L_{n,2\delta} \leq \tfrac{1}{2}$ if c_δ is suitably enlarged. □

Proof of the Main Result. The proof is based on an idea of Dvoretzky (1972). Let $L(F, G)$ denote the Lévy-distance of the two distribution functions F and G, i.e.

$$L(F, G) = \inf\{\varepsilon > 0: F(x-\varepsilon) - \varepsilon \leq G(x) \leq F(x+\varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\}.$$

The following inequalities are well known and easily proved:

$$L(F, G) \leq \sup_{x\in\mathbb{R}}|F(x) - G(x)|, \tag{2.2}$$

$$\sup_{x\in\mathbb{R}}|F(x) - G(x)| \leq (1 + \|g\|_\infty)L(F, G), \tag{2.3}$$

if G has a bounded density g, and

$$L(F, G) \leq 2E(|X - Y|^s)^{1/(1+s)} \quad \text{for all} \quad s > 0 \tag{2.4}$$

if the random variables X and Y are distributed according to F and G, respectively.

Define the stopping time τ by $\tau = \max\{k \in \{0, 1, \cdots, n\}: \sum_{i=1}^{k} \sigma_i^2 \le 1\}$ (with $\sum_{i=1}^{0} \sigma_i^2 = 0$) and put $\tilde{X}_i = X_i I(i \le \tau)$ for $1 \le i \le n$ and $\tilde{X}_{n+1} = Y(1 - \sum_{i=1}^{\tau} \sigma_i^2)^{1/2}$ where Y is independent of $\mathscr{F}_n$ with $P(Y = 1) = ½ = P(Y = -1)$. $\tilde{X}_1, \cdots, \tilde{X}_{n+1}$ is a martingale difference sequence satisfying the assumptions of the Theorem so that

$$\sup_{t\in\mathbb{R}} | P(\sum_{i=1}^{n+1} \tilde{X}_i \le t) - \Phi(t) | \le c_\delta (\sum_{i=1}^{n+1} E(|\tilde{X}_i|^{2+2\delta})^{1/(3+2\delta)} |\log \sum_{i=1}^{n+1} E(|\tilde{X}_i|^{2+2\delta})| \tag{2.5}$$

whenever $\sum_{i=1}^{n+1} E(|\tilde{X}_i|^{2+2\delta}) \le ½$. But

$$\begin{aligned}\sum_{i=1}^{n+1} E(|\tilde{X}_i|^{2+2\delta}) &\le L_{n,2\delta} + E(| 1 - \sum_{i=1}^{\tau} \sigma_i^2 |^{1+\delta}) \\ &= L_{n,2\delta} + E(|1 - \sum_{i=1}^{n} \sigma_i^2|^{1+\delta} I(\tau = n)) + E(|1 - \sum_{i=1}^{\tau} \sigma_i^2|^{1+\delta} I(\tau < n)) \\ &\le L_{n,2\delta} + N_{n,2\delta} + E(\max_{1\le i\le n} \sigma_i^{2+2\delta}) \le 2(L_{n,2\delta} + N_{n,2\delta}) \le ½\end{aligned}$$

whenever $L_{n,2\delta} + N_{n,2\delta} \le ¼$. Let F and $\tilde{F}$ denote the distribution functions of $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n+1} \tilde{X}_i$, respectively. Then by (2.2)–(2.4)

$$\begin{aligned}\sup_{t\in\mathbb{R}} | P(\sum_{i=1}^{n} X_i \le t) - \Phi(t) | &\le c(L(F, \tilde{F}) + L(\tilde{F}, \Phi)) \\ &\le c[E(|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n+1} \tilde{X}_i|^{2+2\delta})^{1/(3+2\delta)} \\ &\quad + \sup_{t\in\mathbb{R}} | P(\sum_{i=1}^{n+1} \tilde{X}_i \le t) - \Phi(t) |],\end{aligned}$$

and because of (2.5) it is enough to show that the expectation is bounded by $c_\delta(L_{n,2\delta} + N_{n,2\delta})$. But

$$E(|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n+1} \tilde{X}_i|^{2+2\delta}) \le c_\delta[E(|\sum_{i=\tau+1}^{n} X_i|^{2+2\delta}) + E(|\tilde{X}_{n+1}|^{2+2\delta})],$$

$$E(|\tilde{X}_{n+1}|^{2+2\delta}) = E(|1 - \sum_{i=1}^{\tau} \sigma_i^2|^{1+\delta}) \le L_{n,2\delta} + N_{n,2\delta}$$

and

$$\begin{aligned}E(|\sum_{i=\tau+1}^{n} X_i|^{2+2\delta}) &\le c_\delta[E(|\sum_{i=\tau+1}^{n} \sigma_i^2|^{1+\delta}) + E(\max_{1\le i\le n} |X_i|^{2+2\delta})] \\ &\le c_\delta[E(|\sum_{i=1}^{n} \sigma_i^2 - 1|^{1+\delta}) + E(|1 - \sum_{i=1}^{\tau} \sigma_i^2|^{1+\delta}) + L_{n,2\delta}] \\ &\le c_\delta(L_{n,2\delta} + N_{n,2\delta})\end{aligned}$$

by a well known Burkholder inequality, cf. Burkholder (1973), Theorem 21.1. □

REFERENCES

[1] BOLTHAUSEN, E. (1982). Exact convergence rates in some martingale central limit theorems. *Ann. Probab.* **10** 672–688.

[2] BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19–42.

[3] DVORETZKY, A. (1972). Asymptotic normality for sums of dependent random variables. *Proc. 6th Berkeley Symp. Math. Statist. Probab.* **2** 513–535.

[4] HEYDE, C. C. and BROWN, B. M. (1970). On the departure from normality of a certain class of martingales. *Ann. Math. Statist.* **41** 2161–2165.
[5] KATO, Y. (1979). Rates of convergence in central limit theorems for martingale differences. *Bull. Math. Statist.* **18** 1–9.

MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D-8000 MUNICH 2
WEST GERMANY