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ASYMPTOTICALLY BALANCED FUNCTIONS AND STOCHASTIC COMPACTNESS OF SAMPLE EXTREMES

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Necessary and sufficient conditions are given under which all partial limit distributions for properly normalized sample extremes of i.i.d. random variables are proper and nondegenerate. In the process we study a new type of extended regular variation called asymptotic balance that should be useful in other contexts as well.

1. Introduction; formulation in terms of inverse functions. Suppose $Y_1, Y_2, \cdots$ are independent identically distributed (i.i.d.) random variables with distribution F. Set $X_n = \bigvee_{i=1}^n Y_i$ $(n = 1, 2, \cdots)$.

DEFINITION 1. The sequence of sample maxima $\{X_n\}$ is *stochastically compact* if there exist $\{a_n > 0, b_n \in \mathbb{R}, n \geq 1\}$ such that every sequence $\{(X_{n(k)} - b_{n(k)})/a_{n(k)}, k \geq 1\}$ contains a subsequence whose distributions converge weakly to a nondegenerate probability distribution. Such a limit distribution is called a *partial limit distribution* for F. We also occasionally say that F is stochastically compact if the above holds. The constants $\{a_n, b_n\}$ are called normalizing constants.

EXAMPLE. The geometric distribution satisfies the definition with b_n = const. log n and $a_n = 1$ but is not in a domain of attraction.

Corresponding notions for partial sums are developed in Feller (1966), Simons and Stout (1978), Maller (1981), de Haan and Resnick (1984). For maxima the special case $b_n = 0$ (with no exclusion of degenerate distribution but excluding an atom at zero) was treated in de Haan and Ridder (1979).

Our aim is to give conditions for stochastic compactness of $\{X_n\}$ in terms of the distribution function F. We start by analytically expressing stochastic compactness in terms of the inverse function of the distribution function F. The next section gives conditions for stochastic compactness in terms of that inverse function. In Section 3 we then derive conditions in terms of F itself. The final section gives special cases and examples.

Stochastic compactness of F means vague subsequential limits of $\{F^n(a_n x + b_n)\}$ are proper and nondegenerate. Suppose for some sequence of integers $\{n(i)\}$

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satisfying $n(i) \to \infty$

$$F^{n(i)}(a_{n(i)}x + b_{n(i)}) \to G(x).$$

This is equivalent to

$$(n(i)(1 - F(a_{n(i)}x + b_{n(i)})))^{-1} \to (-\log G(x))^{-1} \tag{1}$$

and (1) is often the most convenient way of expressing the existence of subsequential limit distributions for maxima. In the sequel we shall use the symbol $f(t) \asymp g(t)$ to mean $f(t) > 0$, $g(t) > 0$ and $\log(f(t)/g(t))$ is bounded. From (1), if $1 - F_1(t) \asymp 1 - F_2(t)$ $(t \uparrow x_0)$ and F_1 is stochastically compact, then so is F_2. Also if F is stochastically compact then so is $1 - (1 - F)^\alpha$ for $\alpha > 0$. Further it is clear that $\{a_n\}$ and $\{b_n\}$ can be replaced by $\{a_n^*\}$ and $\{b_n^*\}$ in the definition of stochastic compactness if and only if $a_n^* \asymp a_n$ and $a_n^{-1}(b_n^* - b_n)$ is bounded. This gives the extent to which the normalizing constants are unique in the definition.

If U is a nondecreasing function define for $x \in (\inf U, \sup U)$

$$U^{\leftarrow}(x) = \inf\{s: U(s) \geq x\}$$

so that $U^{\leftarrow}$ is nondecreasing, left continuous and $t < U^{\leftarrow}(x)$ iff $U(t) < x$. Throughout this paper, convergence of a family of nondecreasing functions means weak convergence, i.e. $U_n \to U$ means $U_n(x) \to U(x)$ for all continuity points x of U. It follows that $U_n \to U$ iff $U_n^{\leftarrow} \to U^{\leftarrow}$.

For the distribution F define the end points

$$x_1 = \inf\{x: F(x) > 0\}, \quad x_0 = \sup\{x: F(x) < 1\}$$

and set $\Psi(x) = (1/(1 - F))^{\leftarrow}(x)$ so that $\Psi: (1, \infty) \to \mathbb{R}$. Note Ψ is bounded if $-\infty < x_1 < x_0 < \infty$.

We now express the property of stochastic compactness in terms of Ψ. Inverting (1) we obtain

$$(\Psi(n(i)x) - b_{n(i)})/a_{n(i)} \to R_*(x) = (1/-\log G)^{\leftarrow}(x) \quad \text{(weakly on } (0, \infty)).$$

So partial limits for $F^n(a_n x + b_n)$ correspond to partial (or subsequential) limits for $a_n^{-1}\{\Psi(nx) - b_n\}$. The latter partial limits (generic notation P) then must be finite and nonconstant.

We claim that equivalently the partial limits (generic notation R) of $a_n^{-1}\{\Psi(nx) - \Psi(n)\}$ must be finite and not identically zero. This is obvious with regard to the finiteness of the limit functions. It is also obvious that if some P is constant, then the corresponding R exists and is identically zero. Conversely suppose R is identically zero. Take a further subsequence $n(i)$ such that $a_{n(i)}^{-1}\{\Psi(n(i)x) - b_{n(i)}\} \to P$. Combination with $a_{n(i)}^{-1}\{\Psi(n(i)x) - \Psi(n(i))\} \to 0$ (for all $x > 0$) gives $a_{n(i)}^{-1}\{\Psi(n(i)) - b_{n(i)}\} \to P(x)$ for all $x > 0$, a contradiction.

To summarize: If F is stochastically compact with normalizing constants $\{a_n, b_n\}$, then all partial limits of $a_n^{-1}\{\Psi(nx) - \Psi(n)\}$ are finite and not identically zero. Conversely if this condition on Ψ holds, F is stochastically compact with normalizing constants $\{a_n, \Psi(n)\}$. We now give a refinement of this characterization.

Proposition. *If F is stochastically compact with normalizing constants $\{a_n, b_n\}$ then for any sequence of reals $t_{n'} \to \infty$ there exists a subsequence $\{t_n\} \subset \{t_{n'}\}$ with*

$$\frac{\Psi(t_n x) - \Psi(t_n)}{a(t_n)} \to H(x) \quad \text{weakly on } (0, \infty) \tag{2}$$

where $a(t) = a_{[t]}$ and $H(x)$ is finite for all $x > 0$ and $H(x) \not\equiv 0$. Conversely, if (2) *holds, F is stochastically compact with normalizing constants $\{a_n, \Psi(n)\}$.*

Proof. From the remarks preceding the Proposition, it is clear that (2) implies stochastic compactness so let us suppose F is stochastically compact with normalizing constants $\{a_n, b_n\}$. Observe the inequalities

$$\frac{a_{2[t]}}{a_{[t]}} \frac{\Psi(2[t]\, x/3) - \Psi(2[t])}{a_{2[t]}} \le \frac{\Psi(tx) - \Psi(t)}{a(t)} \le \frac{\Psi([t]2x) - \Psi([t])}{a_{[t]}}. \tag{3}$$

If for $x > 0$ as $t_n \to \infty$

$$\{\Psi(t_n x) - \Psi(t_n)\}/a(t_n) \to H(x)$$

weakly on $(0, \infty)$, then taking further subsequences if necessary and using (3) gives

$$c\, H_d(x/3) \le H(x) \le H_d(2x) \tag{4}$$

where H_d is a partial limit of $\{(\Psi(nx) - \Psi(n))/a_n\}$. Since F is stochastically compact, the remark preceding the proposition gives that H_d is finite and not identically zero on $(0, \infty)$ and these same properties must hold for H by (4) if the constant c is finite and positive.

It remains to prove that if

$$a_{n_k}^{-1}(\Psi(n_k x) - \Psi(n_k)) \to H(x) \quad \text{and} \quad a_{n_k}^{-1} a_{2n_k} \to c,$$

then c is finite and positive. Now

$$\frac{\Psi(2nx) - \Psi(2n)}{a_{2n}} = \left(\frac{\Psi(2nx) - \Psi(n)}{a_n} - \frac{\Psi(2n) - \Psi(n)}{a_n}\right) \frac{a_n}{a_{2n}}.$$

Taking partial limits we get for partial limit functions H_0 and H_1

$$H_1(x) = \{H_0(2x) - H_0(2)\}\, c^{-1}.$$

Now $c = \infty$ is impossible since H_1 is not identically zero and H_0 is finite. Also $c = 0$ is impossible since H_1 is finite and $H_0(2x) - H_0(2)$ is not identically zero. The proof is complete.

Functions Ψ with property (2) are studied in the next section.

Let $U: \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing. Then U is of *bounded increase* (BI) or of *dominated variation* if

$$\lim_{x \to \infty} \frac{\log \limsup_{t \to \infty}(U(tx)/U(t))}{\log x} < \infty.$$

U is of *positive increase* (PI) if

$$\lim_{x\to\infty} \frac{\log \liminf_{t\to\infty}(U(tx)/U(t))}{\log x} > 0.$$

See Feller (1966), Goldie (1977), de Haan and Ridder (1979), Matuszewska (1962), Simons and Stout (1978), Seneta (1976).

We will need the following properties of BI and PI. If $U \in$ BI and PI then

a. for some $\alpha > 0$,

$$\int_1^\infty \frac{ds}{s^\alpha U(s)} < \infty \quad \text{and} \quad x^{1-\alpha}U(x) \int_x^\infty \frac{ds}{s^\alpha U(s)} \asymp 1.$$

b. for some $\alpha, m > 0$,

$$\int_1^\infty \frac{ds}{s^\alpha U(s)} < \infty \quad \text{and} \quad x^m \int_x^\infty \frac{ds}{s^\alpha U(s)} \quad \text{is increasing.}$$

c. $U^{\leftarrow} \in$ BI and PI.

2. Properties of Ψ. We now begin our study of functions satisfying (2). For any family of nondecreasing real functions $\{f_t(x)\}_{t\in\mathbb{R}_+}$ and any sequence $t_n' \to \infty$ there exists a subsequence $t_n \to \infty$ such that $f_{t_n}(x)$ converges weakly to some nondecreasing function $g(x)$ (possible $\pm\infty$). Such a function g is called a *partial limit function* for $\{f_t(x)\}$.

Suppose now $\Psi:(q, \infty) \to \mathbb{R}$ is nondecreasing (possibly bounded) for some $q \in \mathbb{R}$. (For convenience we suppose $q \geq 1$.)

DEFINITION 2. Ψ is *asymptotically balanced* if there exists a positive function $a(\cdot)$ such that all partial limits of

$$\frac{\Psi(tx) - \Psi(t)}{a(t)}$$

for $t \to \infty$ are finite and not identically zero. The function $a(\cdot)$ is called the auxiliary function. It is clear that if $a(\cdot)$ is an auxiliary function then $a_1(\cdot)$ also serves as an auxiliary function if and only if $a_1(t) \asymp a(t)$ $(t \to \infty)$.

From Section 1, F is stochastically compact iff the Ψ defined there is asymptotically balanced.

Functions satisfying a relation similar to the one described by Definition 2 have been studied by Bingham and Goldie (1979). They do not assume Ψ is monotone but require $a(\cdot)$ to be regularly varying.

We will give necessary and sufficient conditions for a function Ψ to be asymptotically balanced. We start with some lemmas.

LEMMA 1. *If Ψ is asymptotically balanced then for all $x > 0$*

$$\limsup_{t\to\infty} \frac{a(tx)}{a(t)} < \infty.$$

PROOF. Suppose not, then there exists $x_0 > 0$ and $t_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{a(t_n x_0)}{a(t_n)} = \infty.$$

Take $\{t_{n'}\} \subset \{t_n\}$ such that

$$\lim_{n'\to\infty} \frac{\Psi(t_{n'}x) - \Psi(t_{n'})}{a(t_{n'})} = H_1(x)$$

weakly and convergence holds for $x = x_0$.

Take now $\{t_{n''}x_0\} \subset \{t_{n'}x_0\}$ such that

$$\lim_{n''\to\infty} \frac{\Psi(t_{n''}x_0x) - \Psi(t_{n''}x_0)}{a(t_{n''}x_0)} = H_2(x) \quad \text{weakly.}$$

Pick $x > 0$ such that (according to Definition 2) $H_2(x) \neq 0$ and x is a continuity point of H_2. Then

$$\lim_{n''\to\infty} \frac{a(t_{n''}x_0)}{a(t_{n''})}$$

$$= \lim_{n''\to\infty} \left\{\frac{\Psi(t_{n''}x_0x) - \Psi(t_{n''})}{a(t_{n''})} - \frac{\Psi(t_{n''}x_0) - \Psi(t_{n''})}{a(t_{n''})}\right\} \Big/ \left\{\frac{\Psi(t_{n''}x_0x) - \Psi(t_{n''}x_0)}{a(t_{n''}x_0)}\right\}$$

$$= \frac{H_1(xx_0) - H_1(x_0)}{H_2(x)} < \infty,$$

which is a contradiction.

COROLLARY 1. *Suppose Ψ is asymptotically balanced with auxiliary function a. There exist positive constants t_0, c_i, $1 < i < 8$ and constants ρ_0, $\rho(\rho_0 \le \rho)$, x_0 such that for $x \ge x_0$*

(i) $x^{\rho_0} \le \liminf_{t\to\infty} a(tx)/a(t) \le \limsup_{t\to\infty} a(tx)/a(t) \le x^{\rho}$
(ij) $(\Psi(tx) - \Psi(t))/a(t) \le c_1 x^{\rho}$ *for* $t \ge t_0$
(iij) $\Psi(t) \le c_2 t^{\rho}$ *for* $t \ge t_0$.

REMARKS. (a) If $\rho_0 > 0$ then lower bounds of the order of x^{ρ_0} and t^{ρ_0} are valid in (ij) and (iij) respectively. If $\rho_0 < 0$, the lower bounds are noninformative.

(b) With regard to stochastic compactness of maxima, (ij) says partial limit distributions have a right tail bounded above by const. $x^{-\rho^{-1}}$ provided the right end point of the limit distribution is infinite.

PROOF. In what follows, c is a positive constant, perhaps different with each use. Let $\ell(x) := \limsup_{t\to\infty} a(tx)/a(t)$ for $x > 0$. Then $\ell(xy) \le \ell(x)\ell(y)$ so that by the theory of subadditive functions (Matuszewska, 1962, Hille, 1948), $\lim_{x\to\infty}(\log \ell(x)/\log x)$ exists and is finite. From this, the right-most inequality in (i) follows and the other inequality in (i) is obtained in a similar way.

By the definition of asymptotic balance and Lemma 1, we have for some c, ρ,

t_0 that for $t \geq t_0$

$$\frac{\Psi(2t) - \Psi(t)}{a(t)} \leq c \quad \text{and} \quad \frac{a(2t)}{a(t)} \leq 2^{\rho}.$$

For $t \geq t_0$ it follows that

$$\frac{a(2^n t)}{a(t)} = \frac{a(2^n t)}{a(2^{n-1}t)} \cdots \frac{a(2t)}{a(t)} \leq 2^{n\rho}$$

and therefore

$$\frac{\Psi(2^n t) - \Psi(t)}{a(t)} = \frac{a(2^{n-1}t)}{a(t)} \frac{\Psi(2^n t) - \Psi(2^{n-1}t)}{a(2^{n-1}t)} + \cdots + \frac{\Psi(2t) - \Psi(t)}{a(t)}$$

$$\leq c\{2^{(n-1)\rho} + \cdots + 1\} \leq c \cdot 2^{n\rho}$$

and (ij) follows easily. The bound in (iij) follows from (ij) by setting $t = t_0$.

The inequality in (iij) and the relation $1 - F^n(y) \sim n(1 - F(y))$, $y \to \infty$ imply the next result.

COROLLARY 2. *If $\{X_n\}$ is stochastically compact, then for any fixed integer m, $E(\log X_m)_+ < \infty$.*

LEMMA 2. *If Ψ is asymptotically balanced with auxiliary function a, then any partial limit H of $\{a(t)\}^{-1}\{\Psi(tx) - \Psi(t)\}$ $(t - \infty)$ satisfies $H(x) > 0$ for some $x > 1$.*

PROOF. Suppose not; i.e. for some $t_n \to \infty$

$$\lim_{n\to\infty} \frac{\Psi(t_n x) - \Psi(t_n)}{a(t_n)} = 0 \quad \text{for all} \quad x > 1. \tag{5}$$

The definition of asymptotic balance will be contradicted if we find a sequence $r_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\Psi(r_n x) - \Psi(r_n)}{a(r_n)} = 0 \quad \text{for all} \quad x > 0.$$

From Corollary 1 and (5) we note that there is an n_2 such that for $n > n_2$

$$\frac{\Psi(2t_n) - \Psi(t_n)}{a(t_n\sqrt{2})} < \frac{1}{2}$$

and, in general, there exists for any k an n_k such that for $n \geq n_k$

$$\frac{\Psi(kt_n) - \Psi(t_n)}{a(t_n\sqrt{k})} < \frac{1}{k}.$$

Without loss, suppose $n_2 < n_3 < \cdots \to \infty$. Take $s_n := \max\{k : n_k \leq n\}$. Then

$s_n \to \infty$ $(n \to \infty)$ and

$$\lim_{n\to\infty} \frac{\Psi(t_n s_n) - \Psi(t_n)}{a(t_n\sqrt{s_n})} = 0.$$

Now for $x < 1$

$$0 \geq \frac{\Psi(t_n x\sqrt{s_n}) - \Psi(t_n\sqrt{s_n})}{a(t_n\sqrt{s_n})} \geq \frac{\Psi(t_n) - \Psi(t_n s_n)}{a(t_n\sqrt{s_n})} \to 0 \quad (n \to \infty)$$

and for $x > 1$

$$0 \leq \frac{\Psi(t_n x\sqrt{s_n}) - \Psi(t_n\sqrt{s_n})}{a(t_n\sqrt{s_n})} \leq \frac{\Psi(t_n s_n) - \Psi(t_n)}{a(t_n\sqrt{s_n})} \to 0 \quad (n \to \infty)$$

and the desired contradiction is obtained by setting $r_n = t_n\sqrt{s_n}$. From Corollary 1 we see that

$$(\Psi(tx) - \Psi(t))/(a(t)x^{\beta}) \leq c_1 x^{\rho-\beta}$$

(for $t \geq t_0$, $x \geq x_0$) and if we pick β sufficiently large the right side is Lebesgue integrable on $(1, \infty)$. It is convenient to choose β so that

$$\beta > 3\rho + 1. \tag{6}$$

THEOREM 1. *If Ψ is asymptotically balanced and β satisfies* (6)

$$a(t) \asymp t^{\beta-1} \int_t^\infty \Psi(s)\frac{ds}{s^\beta} - \frac{\Psi(t)}{\beta-1} = \frac{t^{\beta-1}}{\beta-1}\int_t^\infty \frac{d\Psi(u)}{u^{\beta-1}} \quad \textit{for} \quad t \to \infty.$$

PROOF. For any partial limit H we get from (6) and Corollary 1 that $\int_1^\infty H(x)x^{-\beta}\,dx < \infty$ and from Lemma 2 we get $\int_1^\infty H(x)x^{-\beta}\,dx > 0$. If

$$(\Psi(t_n x) - \Psi(t_n))/a(t_n) \to H(x)$$

then by Lebesgue's theorem on bounded convergence and Corollary 1 we get

$$\begin{aligned}\int_1^\infty H(x)x^{-\beta}\,dx &= \lim_{n\to\infty}\int_1^\infty \frac{\Psi(t_n x) - \Psi(t_n)}{a(t_n)}\frac{dx}{x^\beta}\\ &= \lim_{n\to\infty}\int_1^\infty \left(\int_{t_n}^{t_n x}\Psi(du)\right)x^{-\beta}\frac{dx}{a(t_n)} \qquad (7)\\ &= \lim_{n\to\infty}\frac{t_n^{\beta-1}}{\beta-1}\int_{t_n}^\infty \frac{u^{-(\beta-1)}\Psi(du)}{a(t_n)},\end{aligned}$$

the last step following by Fubini. So any sequence $t_{n'}$ has a subsequence $t_n \to \infty$ with

$$\lim_{n\to\infty}\frac{t_n^{\beta-1}}{(\beta-1)a(t_n)}\int_{t_n}^\infty \frac{d\Psi(u)}{u^{\beta-1}} \quad \text{finite and positive.}$$

The result follows by contradiction.

THEOREM 2. *If for some $\beta > 1$ the function*

$$K(x) := \int_x^\infty \frac{d\Psi(u)}{u^{\beta-1}} \tag{8}$$

is finite and $1/K(x)$ is of bounded and positive increase (cf. Section 1), then Ψ is asymptotically balanced. Conversely, if Ψ is asymptotically balanced, then for all β large enough, K is finite and $1/K(x)$ is of bounded and positive increase. Moreover for all such β we have the representation (for $x > p > q$)

$$\Psi(x) - \Psi(p) = (\beta - 1)\int_p^x K(s)s^{\beta-2}\,ds - K(x)x^{\beta-1} + p^{\beta-1}K(p). \tag{9}$$

PROOF. Suppose Ψ is asymptotically balanced and β satisfies (6). From Theorem 1

$$K(t) = r(t)a(t)/t^{\beta-1}$$

where

$$0 < c_1 \le \liminf_{t\to\infty} r(t) \le \limsup_{t\to\infty} r(t) \le c_2 < \infty.$$

On the one hand, for large x

$$\limsup_{t\to\infty}(K(tx)/K(t)) \le c_2 \limsup(a(tx)/a(t))x^{-(\beta-1)} \le c_2 x^{\rho-\beta+1}$$

and $\rho - \beta - 1 < 0$ ensuring $1/K \in \mathrm{PI}$, and on the other

$$\liminf_{t\to\infty} \frac{K(tx)}{K(t)} \ge c_1 x^{\rho_0-\beta+1} > 0$$

ensuring $1/K \in \mathrm{BI}$.

Conversely suppose K given by (8) satisfies $1/K \in \mathrm{BI} \cap \mathrm{PI}$. Inverting (8) we have

$$\frac{\Psi(tx) - \Psi(t)}{t^{\beta-1}K(t)} = (\beta - 1)\int_1^x \frac{K(ts)}{K(t)} s^{\beta-2}\,ds - x^{\beta-1}\frac{K(tx)}{K(t)} + 1.$$

For any sequence $t_{n'} \to \infty$ there is a subsequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{K(t_n x)}{K(t_n)} = S(x)$$

weakly with $S(x)$ finite and positive for all x. It follows that

$$\lim_{n\to\infty} \frac{\Psi(t_n x) - \Psi(t)}{t_n^{\beta-1}K(t_n)} = (\beta - 1)\int_1^x S(u)u^{\beta-2}\,du - x^{\beta-1}S(x) + 1 =: H(x).$$

Obviously $H(x)$ is finite for all $x > 0$. We show $H(x) > 0$ for some $x > 1$. If not and $H(x) \equiv 0$, $x \geq 1$ we have

(10) $$(\beta - 1) \int_1^x S(u)u^{\beta-2}\, du = x^{\beta-1}S(x) - 1 \quad \text{for} \quad x > 1.$$

Differentiation gives $S'(x) \equiv 0$. Substitution of $S(x) = c(x > 1)$ in (10) then gives $c = 1$, which means that $1/K$ is not of positive increase.

REMARK. The following parallel statement can be proved: U is asymptotically balanced if and only if for some $\beta > 0$ the function P defined by $P(t) = \int_0^t v^\beta U(dv)$ is of bounded and positive increase. Also then $a(t) \asymp t^{-\beta}P(t)$ $(t \to \infty)$. This will be used and proved in a forthcoming paper by de Haan and Stadtmüller.

We can now construct smoother versions of Ψ.

COROLLARY 3. *If Ψ is asymptotically balanced, there is a continuous and strictly increasing Ψ_1 such that $\Psi_1(t) > \Psi(t)$ and $\Psi_1(t) - \Psi(t) \asymp a(t)$ as $t \to \infty$. Even more: There exists a twice differentiable Ψ_2 with $\Psi_2(t) > \Psi(t)$ and $\Psi_2(t) - \Psi(t) \asymp a(t)$. Both Ψ_1 and Ψ_2 are asymptotically balanced with $a(\cdot)$ as auxiliary function. If we set $\Psi_2^*(t) = \Psi_2(t^{1/(\beta-1)})$ then $-1/x(\Psi_2^*(x))'' \in \mathrm{BI} \cap \mathrm{PI}$ and $-x(\Psi_2^*(x))'' \asymp (\Psi_2^*(x))'$.*

PROOF. Let $\Psi_1(x) = (\beta - 1)x^{\beta-1} \int_x^\infty \Psi(s)(ds/s^\beta)$ and from Theorem 1 we obtain $(\Psi_1(t) - \Psi(t)) \asymp a(t)$. To check if Ψ_1 is asymptotically balanced use (8): Partial integration gives

$$\Psi_1(x) = \Psi(x) + x^{\beta-1}K(x)$$

and from (9)

(11) $$\Psi_1(x) = \Psi(p) + (\beta - 1) \int_p^x K(s)s^{\beta-2}\, ds + p^{\beta-1}K(p)$$

so that

$$\frac{\Psi_1(tx) - \Psi_1(t)}{t^{\beta-1}K(t)} = (\beta - 1) \int_1^x \frac{K(ts)}{K(t)} s^{\beta-2}\, ds.$$

As in Theorem 2, we obtain that Ψ_1 is asymptotically balanced and from Theorem 1 we see that an auxiliary function is $t^{\beta-1}K(t)\, a(t)$. Similarly, define $\Psi_2(x) = (\beta - 1)x^{\beta-1} \int_x^\infty \Psi_1(s)(ds/s^\beta)$. By analogy with the above paragraph, Ψ_2 is asymptotically balanced with auxiliary function $a(\cdot)$ and $\Psi_2(t) - \Psi_1(t) \asymp a(t)$ and so $\Psi_2(t) - \Psi(t) \asymp a(t)$.

Set $\Psi_0 = \Psi$ and $\Psi_i^*(t) = \Psi_i(t^{1/(\beta-1)})$, $i = 0, 1, 2$, and we get $\Psi_i^*(x) = x \int_x^\infty (\Psi_{i-1}^*(u)/u^2)\, du$, $i = 1, 2$. Then

(12) $$(\Psi_i^*(x))' = x^{-1}(\Psi_i^*(x) - \Psi_{i-1}^*(x))$$

so that

$$(\Psi_2^*(x))'' = \frac{x\{(\Psi_2^*(x))' - (\Psi_1^*(x))'\} - (\Psi_2^*(x) - \Psi_1^*(x))}{x^2}$$

$$= \frac{x\{(\Psi_2^*(x))' - (\Psi_1^*(x))'\} - x(\Psi_2^*(x))'}{x^2} \quad \text{(from (12))}$$

$$= - \frac{(\Psi_1^*(x))'}{x}.$$

It is easy to check that Ψ_i is asymptotically balanced with auxiliary function $a(t)$ iff Ψ_i^* is asymptotically balanced with auxiliary function $a(t^{1/(\beta-1)})$. Therefore setting $K^*(t) = \int_t^\infty (d\Psi_0^*(u)/u)$ we get from Theorem 2 (with $\beta = 2$) that $1/K^* \in \mathrm{BI} \cap \mathrm{PI}$. From the analogue of (11) for Ψ_1^* (set $\beta = 2$ and replace K by K^*) we get

$$(\Psi_1^*(x))' = K^*(x)$$

and so

$$(\Psi_2^*(x))'' = \frac{-(\Psi_1^*(x))'}{x} = \frac{-K^*(x)}{x}.$$

It follows that $-1/x(\Psi_2^*(x))'' \in \mathrm{BI} \cap \mathrm{PI}$.

Lastly, from the expression for $(\Psi_2^*(x))''$ we have

$$(\Psi_2^*(x))' = \int_x^\infty s^{-1}K^*(s)\, ds$$

and by property a. of Section 1

$$\frac{(\Psi_2^*(x))'}{-x(\Psi_2^*(x))''} = \frac{\int_x^\infty s^{-1}K^*(s)\, ds}{K^*(x)} \asymp 1. \quad \square$$

EXAMPLE. The function

$$\Psi(x) = \int_1^x t^{-\alpha(\gamma+\sin\log\log t)}\, dt$$

($\alpha > 0$) is asymptotically balanced for $\gamma > \sqrt{2}$ and is not for $\gamma = \sqrt{2}$ (cf. de Haan and Ridder, 1979, example 7.2).

We conclude this section by giving a different formulation of the property of asymptotic balance.

PROPOSITION. *A nondecreasing function Ψ is asymptotically balanced if and only if there is a positive function $a(t)$ such that*

$$\limsup_{t\to\infty} \left| \frac{\Psi(tx) - \Psi(t)}{a(t)} \right| < \infty \quad \textit{for all} \quad x > 0$$

and

$$\liminf_{t\to\infty} \frac{\Psi(tx) - \Psi(t)}{a(t)} > 0 \quad \textit{for some} \quad x > 1.$$

PROOF. It is clear that Definition 2 follows from the properties given by the proposition. Conversely the two properties follow from the representation (6): For $x > 1$ we have

$$c\, x^{\rho} \geq (\beta - 1) \int_1^x \frac{K(ts)}{K(t)}\, s^{\beta-2}\, ds + 1 \geq \frac{\Psi(tx) - \Psi(t)}{t^{\beta-1}K(t)}$$

$$\geq (x^{\beta-1} - 1)\frac{K(tx)}{K(t)} - x^{\beta-1}\frac{K(tx)}{K(t)} + 1 = 1 - \frac{K(tx)}{K(t)}$$

and the lim inf ($t \to \infty$) of the last expression is positive for large x. The necessary result for $x < 1$ is similarly checked.

REMARK. This definition should make it possible to study the property of asymptotic balance in a nonmonotone context. It also proves that one always can take $a(t) = \Psi(tx_0) - \Psi(t)$ for some $x_0 > 1$.

3. Conditions on F for stochastic compactness. From the previous work, we know F is stochastically compact if and only if $\Psi(t) = (1/(1 - F))^{\leftarrow}(t)$ is asymptotically balanced. Corollary 1 (iij) informs us that for large t we have $\Psi(t) \leq ct^{\rho}$. Upon inverting we find that if $x_0 = \infty$, $1 - F(t) \leq c't^{-1/\rho}$. So some β satisfies (6), i.e. $\beta - 1 > 3\rho$, iff ultimately $(1 - F(t))^{\beta-1} < c't^{-3}$ iff

$$\Psi^*(t) = \Psi(t^{1/(\beta-1)}) \leq ct^{1/3}.$$

This choice of β guarantees that $\int_t^\infty \int_y^\infty (1 - F(s))^{\beta-1}\, ds\, dy < \infty$.

We begin with a lemma which shows a stochastically compact distribution can be replaced by a smooth distribution.

LEMMA 3. *Suppose F is stochastically compact and $\Psi = (1/(1 - F))^{\leftarrow}$. Define Ψ_1, Ψ_2 as in Corollary 3 and define distributions $F_i (i = 1, 2)$ by*

$$\frac{1}{1 - F_i} = \Psi_i^{\leftarrow}.$$

Then $1 - F \asymp 1 - F_i$ as $t \uparrow x_0$ $(i = 1, 2)$.

PROOF. Without loss of generality we may suppose $\beta = 2$ which amounts to replacing F by the stochastically compact F_* with tail $1 - F_*(x) := (1 - F(x))^{\beta-1}$. Also Ψ is replaced by $\Psi_*(x) := \Psi(x^{1/(\beta-1)})$. Using (8) and the fact that $a(t) \asymp tK(t) = \Psi_1(t) - \Psi(t)$ we have for a typical subsequence $\{t_n\}$, $t_n \to \infty$ for which $t_nK(t_n)/a(t_n) \to c$ and $(K(t_n))^{-1}K(t_nx) \to S(x)$, that

$$\lim_{n\to\infty} \frac{\Psi(t_nx) - \Psi_1(t_n)}{a(t_n)} = \lim_{n\to\infty} c \int_1^x \frac{K(t_ns)}{K(t_n)}\, ds - cx\, \frac{K(t_nx)}{K(t_n)} = H(x)$$

weakly with $H(x) = c \int_1^x S(v)\,dv - cxS(x)$. Inverting we obtain

$$\lim_{n\to\infty} \frac{1}{t_n(1 - F(\Psi_1(t_n) + xa(t_n)))} = H^{\leftarrow}(x) \quad \text{weakly.}$$

By passing to a further subsequence if necessary, we may suppose convergence holds at $x = 0$. Since $H(1) < 0$ and $H(x) > 0$ for some $x > 1$, we get for this inverse: $H^{\leftarrow}(0) > 0$ and (draw a picture!) $H^{\leftarrow}(\xi) < \infty$ for some $\xi > 0$. Since $1 - F_1(\Psi_1(t_n)) = t_n^{-1}$ we obtain on setting $x = 0$ that

$$\lim_{n\to\infty} \frac{1}{t_n(1 - F(\Psi_1(t_n)))} = \lim_{n\to\infty} \frac{1 - F_1(\Psi_1(t_n))}{1 - F(\Psi_1(t_n))}$$

exists finite and strictly positive.

Because Ψ_1 is continuous and strictly increasing, we conclude any sequence $s_n \to \infty$ has a further subsequence $\Psi_1(t_n)$ such that $1 - F_1(\Psi_1(t_n)) \asymp 1 - F(\Psi_1(t_n))$. The result follows for $1 - F_1$ and a similar proof works for $1 - F_2$.

We provide a representation theorem.

THEOREM 3. *F is stochastically compact if and only if there exists a distribution $F_\#$ satisfying $1 - F(x) \asymp 1 - F_\#(x)$ as $x \uparrow x_0$ and for some $z_0 < x_0$*

$$1 - F_\#(x) = \exp\left\{-\int_{z_0}^{x} \frac{dt}{f(t)}\right\}, \quad z_0 < x < x_0 \tag{13}$$

with $f(x) > 0$ and $f'(x)$ bounded on $(z_0, x_0]$.

PROOF. Suppose F is stochastically compact. As in the previous lemma, we may without loss of generality suppose $\beta = 2$. Then Ψ_2 from Corollary 3 satisfies

$$\frac{-x\Psi_2''(x)}{\Psi_2'(x)} \asymp 1$$

and $\Psi_2(x) - \Psi(x) \asymp a(x)$, $x \to \infty$. Set $\phi := \Psi_2^{\leftarrow}$ and $1 - F_\# = 1/\phi$. Lemma 3 assures us that $1 - F \asymp 1 - F_\#$. Now $f = \phi/\phi' = \phi\Psi_2'(\phi)$ and hence $f' = (\phi/\phi')' = 1 + (\phi\Psi_2''(\phi)/\Psi_2'(\phi))$. The assertion follows.

Conversely suppose $F_\#$ has a representation as in (10). It is sufficient to prove that a distribution function with tail $(1 - F_\#)^\alpha$ for some $\alpha > 0$ is stochastically compact. Take α such that $\alpha^{-1}f'(x) \le c < 1$. Set $\Psi = (1/(1 - F_\#)^\alpha)^{\leftarrow}$. For some $\varepsilon > 0$, $M < \infty$ we have

$$\varepsilon \le \frac{-y\Psi''(y)}{\Psi'(y)} \le M \quad \text{for all} \quad y \in (z_0, x_0).$$

Now $1/\Psi'$ is of positive and bounded increase since $\varepsilon y^{-1} \le (d/dy)\log(1/\Psi'(y)) \le My^{-1}$. It follows, since

$$\frac{\Psi(tx) - \Psi(t)}{t\Psi'(t)} = \int_1^x \frac{\Psi'(ts)}{\Psi'(t)}\,ds,$$

that Ψ is asymptotically balanced, hence $F_\#$ is stochastically compact.

COROLLARY 4. *If F is stochastically compact*

$$1 - F(x-) \asymp 1 - F(x) \quad \text{for} \quad x \uparrow x_0.$$

EXAMPLE. Von Mises' well known example of a distribution function for which the normalized sample maxima do not converge

$$F(x) = 1 - e^{-x-\sin x} \quad (x \geq 0)$$

clearly satisfies the requirement of the theorem with $f(t) \equiv 1$.

COROLLARY 5. *F is stochastically compact if and only if for some $z_0 < x_0$*

$$1 - F(x) = c(x)\exp\left\{-\int_{z_0}^{x} \frac{g(t)}{f(t)}\, dt\right\}$$

with $g(x) \asymp c(x) \asymp 1$, $f > 0$ and $f'(x)$ bounded on (z_0, x_0).

PROOF. It is easy to check, for example by looking at the inverse functions, that if $U \in \mathrm{BI} \cap \mathrm{PI}$, then the two probability distributions $F_{\#}$ and G related by

$$\frac{1}{1-G} = U \circ \frac{1}{1-F_{\#}}$$

are either both stochastically compact or neither is.

Set

$$F_{\#}(x) = 1 - \exp\left\{-\int_{z_0}^{x} \frac{1}{f(t)}\, dt\right\}, \quad z_0 < x < x_0$$

and

$$U(x) = \exp\left\{\int_{z_0}^{x} \frac{g \circ (1/(1-F_{\#}))^{\leftarrow}(s)}{s}\, ds\right\}.$$

One readily checks $U \in \mathrm{BI} \cap \mathrm{PI}$ and

$$\left(\log U\left(\frac{1}{1-F_{\#}(x)}\right)\right)' = \frac{g(x)}{f(x)}.$$

THEOREM 4. *F is stochastically compact if and only if $\int_x^{x_0} (1-F(s))^{\beta-1}\, ds$ is finite for some $\beta > 1$ and*

$$\begin{aligned}
&0 < \liminf_{x\uparrow x_0} \frac{\int_x^{x_0} (1-F(s))^{\beta}\, ds}{(1-F(x)) \int_x^{x_0} (1-F(s))^{\beta-1}\, ds} \\
(14)\qquad &\leq \limsup_{x\uparrow x_0} \frac{\int_x^{x_0} (1-F(s))^{\beta}\, ds}{(1-F(x)) \int_x^{x_0} (1-F(s))^{\beta-1}\, ds} < 1.
\end{aligned}$$

PROOF. Suppose F is stochastically compact and, as in Section 2, set $K(x) = \int_x^{\infty} u^{-(\beta-1)}\, d\Psi(u)$ so that $1/K(x) \in \mathrm{BI} \cap \mathrm{PI}$ by Theorem 2. Return to (7) and note that this relation holds for β replaced by $\beta + 1$; i.e.

$$\lim_{n\to\infty} \frac{t_n^\beta}{\beta a(t_n)} \int_{t_n}^\infty \frac{d\Psi(u)}{u^\beta} = \int_1^\infty H(x) \frac{dx}{x^{\beta+1}}. \tag{15}$$

Dividing (15) by (7) gives

$$\begin{aligned}\lim_{n\to\infty} \frac{t_n \int_{t_n}^\infty u^{-\beta}\, d\Psi(u)}{\int_{t_n}^\infty u^{-(\beta-1)}\, d\Psi(u)} &= \frac{\beta \int_1^\infty H(x)x^{-(\beta+1)}\, dx}{(\beta - 1) \int_1^\infty H(x)x^{-\beta}\, dx} \\ &= \frac{\int_1^\infty H(x)\, d(1 - (1/x^\beta))}{\int_1^\infty H(x)\, d(1 - (1/x^{\beta-1}))}.\end{aligned} \tag{16}$$

The extreme right hand side of (16) is clearly greater than zero and also less than one since by Lemma 2, $H(x) \not\equiv 0$ for $x > 1$. Using the transformation theorem for integrals to change variables in (16), we thus obtain

$$\begin{aligned} 0 &< \lim\inf_{t\to\infty} \frac{t \int_{\Psi(t)}^{x_0} (1 - F(u))^\beta\, du}{\int_{\Psi(t)}^{x_0} (1 - F(u))^{\beta-1}\, du} \\ &\le \lim\sup_{t\to\infty} \frac{t \int_{\Psi(t)}^{x_0} (1 - F(u))^\beta\, du}{\int_{\Psi(t)}^{x_0} (1 - F(u))^{\beta-1}\, du} < 1. \end{aligned} \tag{17}$$

Replacing t by $(1 + \varepsilon)/(1 - F(t))$ and using $\Psi((1 - \varepsilon)/(1 - F(t))) \le t \le \Psi((1 + \varepsilon)/(1 - F(t)))$ gives

$$\begin{aligned} 0 &< \lim\inf_{t\uparrow x_0} \frac{(1 + \varepsilon)}{1 - F(t)} \frac{\int_{\Psi((1+\varepsilon)/(1-F(t)))}^{x_0} (1 - F(u))^\beta\, du}{\int_{\Psi((1+\varepsilon)/(1-F(t)))}^{x_0} (1 - F(u))^{\beta-1}\, du} \\ &\le \lim\inf_{t\uparrow x_0} \frac{(1 + \varepsilon)}{1 - F(t)} \frac{\int_t^{x_0} (1 - F(u))^\beta\, du}{\int_{\Psi((1+\varepsilon)/(1-F(t)))}^{x_0} (1 - F(u))^{\beta-1}\, du}. \end{aligned}$$

The integral in the denominator is $K((1 + \varepsilon)/(1 - F(t)))$ and because

$$c_1 < \lim\inf_{t\to\infty} K((1 + \varepsilon)t)/K((1 - \varepsilon)t)$$

($1/K \in \mathrm{BI}$) we get

$$\begin{aligned} 0 &< \lim\inf_{t\uparrow x_0} \frac{1}{1 - F(t)} \frac{\int_t^{x_0} (1 - F(u))^\beta\, du}{\int_{\Psi((1+\varepsilon)/(1-F(t)))}^{x_0} (1 - F(u))^{\beta-1}\, du} \\ &< \lim\inf_{t\uparrow x_0} \frac{\int_t^{x_0} (1 - F(u))^\beta\, du}{(1 - F(t)) \int_t^{x_0} (1 - F(u))^{\beta-1}\, du}, \end{aligned}$$

giving the left inequality of (14). The right inequality is more delicate. Set $J(x) := \int_x^\infty s^{-2}K(s)\, ds = x^{-1}K(x) - \int_x^\infty u^{-\beta}\, d\Psi(u)$ so that from (16)

$$\lim\inf_{x\to\infty} \frac{J(x)}{x^{-1}K(x)} > 0. \tag{18}$$

By property b of Section 1 there exists $m > 0$ such that $x^m J(x)$ is increasing.

Hence for any $\varepsilon > 0$

$$J((1-\varepsilon)t)/J((1+\varepsilon)t) \le ((1-\varepsilon)^{-1}(1+\varepsilon))^m. \tag{19}$$

From the definition of J and the transformation theorem for integrals

$$J(x) = \int_x^\infty u^{-(\beta-1)}(x^{-1} - u^{-1})\, d\Psi(u) = \int_{\Psi(x)}^{x_0} (1 - F(s))^{\beta-1}(x^{-1} - (1 - F(s)))\, ds.$$

It follows that

$$J\left(\frac{1-\varepsilon}{1-F(t)}\right) \le \int_{\Psi((1+\varepsilon)/(1-F(t)))}^{x_0} (1 - F(s))^{\beta-1}((1 - F(t)) - (1 - F(s)))\, ds$$

and holding the nonnegative integrand fixed and using $\Psi((1+\varepsilon)/(1 - F(t))) \ge t$ we get this is at most

$$\begin{aligned} &\int_t^{x_0} (1 - F(s))^{\beta-1}((1-F(t)) - (1 - F(s)))\, ds \\ &\qquad = (1 - F(t)) \int_t^{x_0} (1 - F(s))^{\beta-1}\, ds - \int_t^{x_0} (1 - F(s))^{\beta}\, ds. \end{aligned} \tag{20}$$

Therefore

$$\begin{aligned} 0 &< \liminf_{x\to\infty} x\, J(x)/K(x) && \text{(from (18))} \\ &< \liminf_{t\uparrow x_0} \frac{(1-\varepsilon)J((1-\varepsilon)/(1-F(t)))}{(1-F(t))K((1-\varepsilon)/(1-F(t)))} \\ &\le \liminf_{t\uparrow x_0} \frac{(1+\varepsilon)^m J((1+\varepsilon)/(1-F(t)))}{(1-\varepsilon)^{m-1}(1-F(t))K((1-\varepsilon)(1-F(t)))} && \text{(from (19))} \\ &\le \liminf_{t\uparrow x_0} \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m-1}} \\ &\quad \cdot \left\{\frac{(1-F(t)) \int_t^{x_0} (1-F(s))^{\beta-1}\, ds - \int_t^{x_0} (1-F(s))^{\beta}\, ds}{(1-F(t)) \int_t^{x_0} (1-F(s))^{\beta-1}\, ds}\right\} \end{aligned}$$

(from (20) and the form of K).

Pick $\varepsilon > 0$ sufficiently small and we obtain

$$\limsup_{t\to x_0} \frac{\int_t^{x_0} (1-F(s))^{\beta}\, ds}{(1-F(t)) \int_t^{x_0} (1-F(s))^{\beta-1}\, ds} < 1$$

as required.

Conversely suppose (14) holds and set

$$r(x) = \frac{\int_x^{x_0} (1-F(s))^{\beta}\, ds}{(1-F(x)) \int_x^{x_0} (1-F(s))^{\beta-1}\, ds}.$$

Observe that

$$\frac{d}{dx}\left\{\frac{\int_x^{x_0}(1-F(u))^\beta\,du}{\int_x^{x_0}(1-F(u))^{\beta-1}\,du}\right\}=\left(\frac{(1-F(x))^\beta}{\int_x^{x_0}(1-F(u))^{\beta-1}\,du}\right)(r(x)-1)<0$$

for sufficiently large x. So there is a continuous strictly increasing function F_1 such that for sufficiently large x

$$1-F_1(x)=\int_x^{x_0}(1-F(u))^\beta\,du\Big/\int_x^{x_0}(1-F(u))^{\beta-1}\,du$$

and

$$0<\liminf_{x\uparrow x_0}\frac{1-F_1(x)}{1-F(x)}<\limsup_{x\uparrow x_0}\frac{1-F_1(x)}{1-F(x)}<1. \tag{21}$$

It suffices to verify that F_1 is stochastically compact and this we do by means of Corollary 5.

Observe that

$$\frac{d}{dx}(-\log(1-F_1(x)))=\frac{(1-r(x))}{\int_x^{x_0}(1-F(u))^\beta\,du/(1-F(x))^\beta}$$

$$=\frac{(1-r(x))(1-F(x))^\beta/(1-F_1(x))^\beta}{\int_x^{x_0}(1-F(u))^\beta\,du/(1-F_1(x))^\beta}$$

and setting

$$f(x)=\frac{\int_x^{x_0}(1-F(u))^\beta\,du}{(1-F_1(x))^\beta}$$

and

$$g(x)=\frac{(1-r(x))(1-F(x))^\beta}{(1-F_1(x))^\beta},$$

we obtain the representation of Corollary 5.

REMARK. This criterion corresponds to Theorem 2.8.1 of de Haan (1970) for weak convergence of the sequence $\{X_n\}$.

An alternative set of conditions is contained in the next theorem.

THEOREM 5. *F is stochastically compact if and only if for some $\beta>0$*

$$\int_x^{x_0}\int_y^{x_0}(1-F(s))^\beta\,ds\,dy<\infty$$

and

$$\frac{1}{2} < \liminf_{x\uparrow x_0} \frac{(1-F(x))^\beta \int_x^{x_0}\int_y^{x_0}(1-F(s))^\beta\,ds\,dy}{(\int_x^{x_0}(1-F(y))^\beta\,dy)^2} \tag{22}$$

$$< \limsup_{x\uparrow x_0} \frac{(1-F(x))^\beta \int_x^{x_0}\int_y^{x_0}(1-F(s))^\beta\,ds\,dy}{(\int_x^{x_0}(1-F(y))^\beta\,dy)^2} < \infty.$$

PROOF. Suppose F is stochastically compact. Without loss of generality we may suppose $\beta = 1$ in Theorem 4. With this convention in mind, we proceed by establishing a sequence of identities.

First we observe by the transformation theorem for integrals

$$\int_{\Psi(x)}^{x_0} (1-F(s))\,ds = \int_x^\infty u^{-1}\,d\Psi(u) = K(x). \tag{23}$$

Next observe that by Fubini and (23)

$$\int_{\Psi(x)}^{x_0}\int_y^{x_0}(1-F(s))\,ds\,dy$$

$$= \int_{\Psi(x)}^{x_0} s(1-F(s))\,ds - \Psi(x)K(x) = \int_x^\infty \Psi(u)\,\frac{d\Psi(u)}{u} - \Psi(x)K(x)$$

$$= \int_x^\infty\int_x^u d\Psi(s)\,\frac{d\Psi(u)}{u} = \int_x^\infty\left(\int_s^\infty \frac{d\Psi(u)}{u}\right)d\Psi(s)$$

$$= \int_x^\infty s\left(\int_s^\infty \frac{d\Psi(u)}{u}\right)\frac{d\Psi(s)}{s} \tag{24}$$

$$= \int_x^\infty\int_x^s dv \int_s^\infty \frac{d\Psi(u)}{u}\,\frac{d\Psi(s)}{s} + x\int_x^\infty\int_s^\infty \frac{d\Psi(u)}{u}\,\frac{d\Psi(s)}{s}$$

$$= \int_x^\infty\left(\int_v^\infty\int_s^\infty \frac{d\Psi(u)}{u}\,\frac{d\Psi(s)}{s}\right)dv + \frac{xK^2(x)}{2}$$

$$= \left\{\int_x^\infty K^2(v)\,dv + xK^2(x)\right\}\Big/2.$$

Finally we have (recalling 23)

$$\frac{\int_{\Psi(x)}^{x_0}\int_y^{x_0}(1-F(s))\,ds\,dy}{x(\int_{\Psi(x)}^{x_0}(1-F(s))\,ds)^2} = \frac{1}{2}\left\{\frac{\int_x^\infty K^2(u)\,du}{xK^2(x)} + 1\right\}.$$

Because $1/K \in \mathrm{BI} \cap \mathrm{PI}$, we get from property a of Section 1 that

$$\frac{1}{2} < \liminf_{x\to\infty} \frac{\int_{\Psi(x)}^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy}{x(\int_{\Psi(x)}^{x_0} (1 - F(s))\, ds)^2}$$

$$< \limsup_{x\to\infty} \frac{\int_{\Psi(x)}^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy}{x(\int_{\Psi(x)}^{x_0} (1 - F(s))\, ds)^2} < \infty.$$

We then get (22) by replacing x by $1/(1 - F(t))$; this step is made rigorous in exactly the same manner as the analogous problem was handled in Theorem 3.

Conversely, suppose (22) holds and again without loss of generality let $\beta = 1$. As in the proof of Theorem 3 we find that for large x

$$1 - F_0(x) := \left(\int_x^{x_0} (1 - F(s))\, ds\right)^2 \Big/ \int_x^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy$$

is a distribution tail and $1 = F_0 \asymp 1 - F$. Furthermore let

$$h(x) = (1 - F(x)) \int_x^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy \Big/ \left(\int_x^{x_0} (1 - F(s))\, ds\right)^2$$

and we find

$$\frac{d}{dx} (-\log(1 - F_0(x))) = \frac{(2h(x) - 1)}{\int_x^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy / \int_x^{x_0} (1 - F(s))\, ds}$$

and setting $g(x) = 2h(x) - 1$,

$$f(x) = \int_x^{x_0} \int_y^{x_0} (1 - F(s))\, ds\, dy \Big/ \int_x^{x_0} (1 - F(s))\, ds$$

enables us to verify the representation in Corollary 5 is satisfied. Thus F_0 and hence F is stochastically compact.

REMARK. This criterion is comparable to that of Theorem 2.5.2 of de Haan (1970).

4. Particular cases and examples.

One can distinguish two particular cases: $a(x) \asymp 1$ and $a(x) \asymp \Psi(x)$ corresponding to the situations where either no scaling or no shift is necessary. We now show how the conditions particularize. First we have the following connection.

THEOREM 6. *If the sequence of maxima $X_1, X_2, \cdots$ is stochastically compact with norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ $(n = 1, 2, \cdots)$, then for some positive sequence $\{\beta_n\}$ all partial limit laws of $\{X_n/\beta_n\}$ are proper and have no atom at the origin (but may possibly be degenerate) and $\Psi \in \mathrm{BI}$.*

PROOF. From the representation (9)

$$\liminf_{x\to\infty} \frac{\Psi(x)}{x^{\beta-1}K(x)} \geq \liminf_{x\to\infty} \frac{\Psi(x) - \Psi(p)}{x^{\beta-1}K(x)}$$
$$\geq \int_\varepsilon^1 \liminf_{x\to\infty} \frac{K(sx)}{K(s)} (\beta - 1)s^{\beta-2}\, ds - 1.$$

Since this holds for every ε, we may let $\varepsilon \downarrow 0$ and obtain

$$\liminf_{x\to\infty} \frac{\Psi(x)}{x^{\beta-1}K(x)} \geq \int_0^1 \liminf_{x\to\infty} \frac{K(xs)}{K(x)}\, ds - 1 > 0.$$

Hence for $x > 1$

$$\limsup_{t\to\infty} \frac{\Psi(tx)}{\Psi(t)} - 1 \leq \limsup_{t\to\infty} \frac{\Psi(tx) - \Psi(t)}{t^{\beta-1}K(t)} \limsup_{t\to\infty} \frac{t^{\beta-1}K(t)}{\Psi(t)} < \infty.$$

Hence Ψ is of bounded increase and, by de Haan and Ridder (1979, Remark 4.1), the result follows.

The exponential distribution shows that indeed not all limit laws of $\{X_n/\beta_n\}$ are necessarily nondegenerate.

The particular case in which no shift is necessary, i.e. in which $b_n = 0$ $(n = 1, 2, \cdots)$ is a possible choice to obtain proper and nondegenerate limit laws, corresponds to the case

$$t^{\beta-1}K(t) \asymp a(t) \asymp \Psi(t) \quad (t \to \infty)$$

or

$$\Psi(t) \asymp t^{\beta-1} \int_t^\infty \frac{d\Psi(s)}{s^{\beta-1}} = (\beta - 1)t^{\beta-1} \int_t^\infty \frac{\Psi(s)}{s^\beta}\, ds - \Psi(t),$$

corresponding to Ψ of bounded and positive increase as it should. This can also be expressed as $1/(1 - F) \in \mathrm{BI} \cap \mathrm{PI}$ (property c of Section 1).

Similarly, the particular case when no scaling is necessary, i.e., $a_n = 1$ $(n = 1, 2, \cdots)$, to obtain proper and nondegenerate limit laws, corresponds to

$$t^{\beta-1}K(t) \asymp a(t) \asymp 1$$

i.e.

$$t^{\beta-1} \int_t^\infty \frac{d\Psi(s)}{s^{\beta-1}} = (\beta - 1)t^{\beta-1} \int_t^\infty \frac{\Psi(s)}{s^\beta}\, ds - \Psi(t) \asymp 1.$$

This again corresponds to

$$\limsup_{t\to\infty} \Psi(tx) - \Psi(t) < \infty \quad \text{for all} \quad x > 1$$

and

$$\liminf_{t\to\infty} \Psi(tx) - \Psi(t) > 0 \quad \text{for some} \quad x > 1.$$

This can also be expressed as $(1/(1 - F)) \circ \log \in \mathrm{BI} \cap \mathrm{PI}$. Cf. Remark (b) after Corollary 1 and also Anderson (1970).

EXAMPLE. The distribution function

$$F(x) = 1 - (\log x)^{-1} \quad \text{for} \quad x \geq e$$

is not stochastically compact since the tail is not bounded by a power function.

EXAMPLE. The Poisson distribution

$$F(x) = \textstyle\sum_{k \leq x} e^{-\lambda} (\lambda^k/k!)$$

satisfies

$$\lim_{n\to\infty} \frac{1 - F(n-1)}{1 - F(n)} = \infty$$

and hence is not stochastically compact according to Corollary 4.

EXAMPLE. The partial limit distributions for the geometric distribution

$$F(x) = 1 - e^{-[x]}, \quad x > 0$$

($[x]$ = integral part of x) are (with the choice $b_n = \log n$, $a_n = 1$)

$$G(x) = \exp\{-\exp\{-[x + \varepsilon]\}\}, \quad -\infty < x < \infty$$

with $0 < \varepsilon \leq 1$.

EXAMPLE. The distribution function

$$F(x) = 1 - \exp\left\{-\int_0^x \frac{dt}{2t + \cos t}\right\}$$

is stochastically compact by the representation of Theorem 3.

EXAMPLE. The distribution function

$$F(x) = 1 - \exp\left\{-\int_0^x \frac{dt}{t(2 + \cos t)}\right\}$$

is stochastically compact by the representation of Corollary 5. Set $g(t) = (2 + \cos t)^{-1}$, $f(t) = t$. It is not clear how to fit this distribution into the representation of Theorem 3.

EXAMPLE. For the distribution function

$$F(x) = 1 - \exp\{-\alpha(\sqrt{2} + \sin \log \log x) \log x\} \quad (x \geq e)$$

one has (de Haan and Ridder, 1979, example 7.2.) $1/(1 - F) \notin$ PI and so by Theorem 6 and property c of Section 1, F is not stochastically compact.

EXAMPLE. We apply the criterion of Theorem 5 to von Mises' example

$$F(x) = 1 - e^{-x-\sin x} \quad (x > 0).$$

Choose β such that $e^{4\beta} < 2$. We have for $x \leq 1$

$$e^{-\beta x-\beta} \leq \{1 - F(x)\}^{\beta} \leq e^{-\beta x+\beta}$$

hence

$$\beta^{-1}e^{-\beta x-\beta} \leq \int_x^\infty \{1 - F(t)\}^{\beta}\, dt \leq \beta^{-1}e^{-\beta x+\beta}$$

and

$$\beta^{-2}e^{-\beta x-\beta} \leq \int_x^\infty \int_y^\infty \{1 - F(t)\}^{\beta}\, dt\, dy \leq \beta^{-2}e^{-\beta x+\beta}.$$

It follows for $x \geq 1$

$$2^{-1} \leq e^{-4\beta} \leq \frac{\{1 - F(x)\}\{\int_x^\infty \int_y^\infty (1 - F(t))\, dt\, dy\}}{\{\int_x^\infty (1 - F(t))\, dt\}^2} \leq e^{4\beta}.$$

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