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LIMIT LAWS FOR THE MAXIMUM OF WEIGHTED AND SHIFTED I.I.D. RANDOM VARIABLES

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Gnedenko's (1943) study of the class $\mathscr{G}$ of limit laws for the sequence of maxima $M_n \equiv \max\{X_0, \cdots, X_{n-1}\}$ of independent identically distributed random variables $X_0, X_1, \cdots$ is extended to limit laws for weighted sequences $\{w_n(\gamma)X_n\}$ (the simplest case $\{\gamma^n X_n\}$ has geometric weights $(0 \le \gamma < 1)$) and translated sequences $\{X_n - v_n(\delta)\}$ (the simplest case is $\{X_n - n\delta\}$ $(\delta > 0)$). Limit laws for these simplest cases belong to the family $\mathscr{G}$ characterized by Gnedenko; with more general weights or translates, limit laws outside $\mathscr{G}$ may arise.

1. Introduction. Let $\{X_n\}$ $(n = 0, 1, \cdots)$ be a sequence of independent, identically distributed (i.i.d.) random variables (r.v.'s) with distribution function (d.f.) F, and define

$$M_n = \max(X_1, \cdots, X_n) =_d \max(X_0, X_1, \cdots, X_{n-1}). \tag{1.1}$$

In a classic paper Gnedenko (1943) exhibited the class $\mathscr{G}$ of all possible non-degenerate limit laws that can arise from such sequences $\{M_n\}$, and discussed domains of attraction for the elements of $\mathscr{G}$ (that is, given the common d.f. F of the X_n, what properties of F determine whether $G \in \mathscr{G}$ will be its limit law?). This work has been expounded and extended in de Haan's (1970) tract, see also Balkema (1973) and Galambos (1978).

The present paper has its origins in the asymptotic behaviour of an extreme case of a storage model (Daley and Haslett, 1982, and Daley, 1984), namely, in studying possible limit laws as $\gamma \uparrow 1$ of the r.v.

$$Y(\gamma) \equiv \sup_{n\ge0}\{\gamma^n X_n\}, \quad 0 \le \gamma < 1, \tag{1.2}$$

in the case that $X_n \ge 0$ a.s. This observation led us to investigate limit laws using sequences $\{w_n(\gamma)\}$ of weight functions more general than the geometric, and also to consider the possibility of r.v.'s like

$$Z(\delta) \equiv \sup_{n\ge0}\{X_n - n\delta\} \quad (\gamma > 0) \tag{1.3}$$

with the prototype sequence of translates $\{n\delta\}$ replaced by the indexed sequence $\{v_n(\delta)\}$.

In what follows, we always take $0 \le \gamma < 1$, where

$$w_0(\gamma) \equiv 1 \ge w_n(\gamma) \ge w_{n+1}(\gamma) \to 0, \quad (n \to \infty), \quad w_n(\gamma) \to 1, \quad (\gamma \to 1),$$

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and $0 < \delta < \infty$, where

$$v_0(\delta) \equiv 0 \leq v_n(\delta) \leq v_{n+1}(\delta) \to \infty, \quad (n \to \infty) \quad \text{and} \quad v_n(\delta) \to 0, \quad (\delta \to 0).$$

In Section 2 we discuss the existence (i.e., a.s. finiteness) of r.v.'s like $Y(\gamma)$ and $Z(\delta)$, and investigate the limit law behaviour in Sections 3, 4 and 5. When the limit law of $\{M_n\}$ is of the double exponential type, limit laws exist for both $Y(\gamma)$ and $Z(\delta)$ and are also of double exponential type; when $\{M_n\}$ has any other limit law, a limit law for $Y(\gamma)$ will exist and for $Z(\delta)$ may exist, but not all three limit laws will be the same. The results are summarized in Section 6 where also a duality between $Y(\gamma)$ and $Z(\delta)$ is exploited. Connections between the functional equation of Gnedenko's general theory and the present limit laws are exhibited in Section 3, and there and in the subsequent section we exhibit examples of non-degenerate limit laws lying outside $\mathscr{G}$.

For the sequences $\{w_n\}$ involved, the supremum is evidently zero unless $\Pr\{X_0 > 0\} > 0$, while for $\{v_n\}$, if either of the r.v.'s $Z(\delta)$ and $Z(\delta) + A$ has a limit law (where A is any constant), then so does the other and the laws are of the same type. Accordingly we shall assume throughout that

(1.4) $$\Pr\{X < 0\} = 0, \quad \Pr\{X > 0\} > 0,$$

and define the positive (and possibly infinite) quantity

(1.5) $$\ell \equiv \sup\{x\colon F(x) < 1\}.$$

More generally than (1.2) and (1.3) we define

(1.6) $$M(\gamma) \equiv M(\gamma;\, w) \equiv \sup\nolimits_{n\geq 0}\{w_n(\gamma)X_n\},$$

(1.7) $$Z(\delta) \equiv Z(\delta;\, v) \equiv \sup\nolimits_{n\geq 0}\{X_n - v_n(\delta)\},$$

and

(1.8) $$M(\gamma, \delta) \equiv M(\gamma, \delta;\, w, v) \equiv \sup\nolimits_{n\geq 0}\{w_n(\gamma)X_n - v_n(\delta)\}.$$

Observe that if $w_n(\gamma) = 1$ or 0 as $n(1-\gamma) \leq$ or > 1, then

(1.9) $$M(1 - 1/n;\, w) = M_{n+1},$$

so any results that we prove for $M(\gamma; w)$ must hold true for $\{M_n\}$.

2. Existence. In this section we discuss the a.s. finiteness of the r.v.

(2.1) $$M \equiv M(w, v) \equiv \sup\nolimits_{n\geq 0}\{w_n X_n - v_n\}$$

for given sequences $\{w_n\}$ and $\{v_n\}$ for which $v_n \geq 0$, $0 \leq w_n \leq 1$.

THEOREM 1. (a) *Either* $M < \infty$. a.s. *or* $M = \infty$ a.s.
(b) $M < \infty$ a.s. *if and only if either*
(i) $\ell < \infty$, *or*
(ii) $\ell = \infty$, *and* $\{v_n\}$ *and* $\{w_n\}$ *are such that*

(2.2) $$\sum\nolimits_{n=0}^{\infty} [1 - F((x + v_n)/w_n)] < \infty$$

for some finite x.

REMARKS. If $w_n = 0$ for any n then the corresponding term in the sum at (2.2) is taken equal to zero.

The condition at (2.2) can be expressed in terms of a functional inverse as follows, assuming that $(x + v_n)/w_n$ ultimately increases monotonically. This is true in particular when $\{v_n\}$ and $\{w_n\}$ are sequences as in Section 1. And, note that (2.2) cannot hold when $\ell = \infty$ unless either $v_n \to \infty$ or $w_n \to 0$ (or both) as $n \to \infty$; then it is no loss of generality to take $x = 1$ as in the following.

Let $h(y)$ be any nondecreasing function satisfying for $y \geq 1$

$$h(y) \leq \inf\{n: (1 + v_n)/w_n > y\} \leq h(y) + 1. \tag{2.3}$$

Then (2.2) is equivalent to the condition

$$Eh(X) < \infty. \tag{2.4}$$

For example, when $v_n \equiv 0$ and $w_n = \gamma^n$, we can take $h(y) = (\log y)/\log(1/\gamma)$, and the finiteness of

$$E \log(\max(1, X)) = E \log(1 + (X - 1)_+)$$

ensures that $Y(\gamma)$ at (1.2) is well-defined, as asserted in the introduction.

Similarly, when $w_n \equiv 1$ and $v_n = n\delta$, we can take $h(y) = (y - 1)/\delta$, and so the a.s. finiteness of $Z(\delta)$ at (1.3) is equivalent to requiring $EX_+ < \infty$.

PROOF OF THEOREM 1. (a) Since M is a function of the independent r.v.'s $\{w_n X_n - v_n\}$, the zero-one law implies that $\Pr\{M < \infty\} = 0$ or 1.

(b) If (i) holds then $M < \infty$ a.s. by inspection of the defining relation (2.1). In proving (ii) we may assume that $w_n > 0$ for all n. Now $M = \infty$ a.s. if and only if for all x,

$$\Pr\{w_n X_n - v_n > x \text{ infinitely often}\} = 1,$$

and by the Borel-Cantelli lemma for independent events, this condition is equivalent to

$$\infty = \textstyle\sum_{n=0}^{\infty} \Pr\{w_n X_n - v_n > x\} = \sum_{n=0}^{\infty} [1 - F((x + v_n)/w_n)].$$

3. Gnedenko's class $\mathscr{G}$ and limit laws for $M(\gamma, \delta)$.

It is appropriate at this stage to recall certain facts about the class $\mathscr{G}$. First, by a *limit law* G (understood to be nondegenerate) we mean that for some sequence of constants $\{a_n\}$, $\{b_n\}$ the sequence of d.f.'s $\{F_n\}$ has

$$F_n(a_n x + b_n) \to G(cx + d), \quad (n \to \infty)$$

for all points of continuity $cx + d$ of G. Here, $c > 0$ and d is any constant. Thus, we do not distinguish between $G(x)$ and $G(cx + d)$ in identifying this d.f. as a limit law: we say that $G(x)$ and $G(cx + d)$ are of the same type.

Next, recall that Gnedenko (1943) identified the class $\mathscr{G}$ of limit laws for $\{M_n\}$ as comprising all d.f.'s G such that, for every integer $k = 2, 3, \cdots$ there exist constants a_k, b_k for which

$$(G(x))^k = G(a_k x + b_k) \quad \text{(all real } x). \tag{3.1}$$

Moreover, $\mathscr{G} = \mathscr{G}_\Phi \cup \mathscr{G}_\Psi \cup \mathscr{G}_\Lambda$ where the limit laws in these three classes are as follows (the parameter α is any positive constant):

$$G(x) = \Phi_\alpha(x) = \begin{cases} 0 & (x \le 0), \\ \exp(-x^{-\alpha}) & (x > 0); \end{cases} \tag{3.2}$$

$$G(x) = \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & (x < 0), \\ 1 & (x \ge 0); \end{cases} \tag{3.3}$$

$$G(x) = \Lambda(x) = \exp(-e^{-x}) \quad (-\infty < x < \infty). \tag{3.4}$$

Suppose that for families of sequences $\{w_n(\gamma)\}$ and $\{v_n(\delta)\}$ as in Section 1 we can write $\delta = \delta(\gamma)$ such that $\delta(\gamma) \downarrow 0$ as $\gamma \uparrow 1$, and that there exist functions $a(\gamma)$ and $b(\gamma)$ with $a(\gamma) > 0$ such that

$$aM + b \equiv a(\gamma)M(\gamma, \delta(\gamma); w, v) + b(\gamma) \tag{3.5}$$

converges to the limit law H as $\gamma \uparrow 1$.

THEOREM 2. *If $aM + b$ at* (3.5) *converges weakly to the limit law H as $\gamma \uparrow 1$, then for every integer $k = 1, 2, \cdots$*

$$M_k \equiv M_k(\gamma, \delta(\gamma); w, v) \equiv \sup_{n>0}\{w_{nk}(\gamma)X_{nk} - v_{nk}(\delta(\gamma))\} \tag{3.6}$$

has $aM_k + b$ converging weakly to $H_k \equiv (H)^{1/k}$. Conversely, the convergence of $aM_k + b$ for any integer k implies the convergence of $aM + b$ to a limit law H and hence of $aM_k + b$ to $(H)^{1/k}$ for every k.

PROOF. Since $w_n(\gamma) \uparrow 1$ and $v_n(\delta(\gamma)) \downarrow 0$ as $\gamma \uparrow 1$,

$$\sup_{0\le n\le r}\{w_n(\gamma)X_n - v_n(\delta(\gamma))\} \to \sup_{0\le n\le r} X_n, \quad (\gamma \uparrow 1)$$

for each positive integer r. Consequently, defining

$$M(r) = \sup_{n\ge r}\{w_n(\gamma)X_n - v_n(\delta(\gamma))\},$$

it follows from (1.4) that for each such r, $\Pr\{M = M(r)\} \to 1$ as $\gamma \uparrow 1$. Then, for any $a(\gamma)$ and $b(\gamma)$, we have as $\gamma \uparrow 1$

$$\begin{aligned} &\sup_{-\infty<y<\infty} | \Pr\{aM + b \le y\} - \Pr\{aM(r) + b \le y\} | \\ &\qquad \le \Pr\{M \ne M(r)\} \to 0. \end{aligned} \tag{3.7}$$

Fix the integer $k \ge 1$, and let $H^{(\gamma,r)}$ denote the d.f. of $aM(r) + b$ so that

$$\begin{aligned} H^{(\gamma,rk)}(y) &= \textstyle\prod_{n=rk}^{\infty} F[(y - b(\gamma) + a(\gamma)v_n(\gamma))/a(\gamma)w_n(\gamma)] \\ &= \textstyle\prod_{j=0}^{k-1} \prod_{n=r}^{\infty} F[(y - b(\gamma) + a(\gamma) + a(\gamma)v_{nk+j}(\gamma))/a(\gamma)w_{nk+j}(\gamma)]. \end{aligned}$$

By the monotonicity properties of $\{w_n(\gamma)\}$ and $\{v_n(\gamma)\}$ in n, each infinite product on the right-hand side is bounded above and below by $H_k^{(\gamma,r+1)}$ and $H_k^{(\gamma,r)}$, respectively, where

$$H_k^{(\gamma,r)} \equiv \textstyle\prod_{n=r}^{\infty} F[(y - b(\gamma) + a(\gamma)v_{nk}(\gamma))/a(\gamma)w_{nk}(\gamma)].$$

(Since $F(0+) < 1$, we need only consider $y - b(\gamma) + a(\gamma)v_{nk}(\gamma) \geq 0$.) Therefore

$$H^{(\gamma,rk)}(y) \leq (H_k^{(\gamma,r+1)}(y))^k \leq H^{(\gamma,(r+1)k)}(y).$$

But it follows from (3.7) that for each r and k,

$$\sup_{-\infty<y<\infty} | H^{(\gamma,rk)}(y) - H^{(\gamma,(r+1)k)}(y) | \to 0$$

as $\gamma \uparrow 1$, and by taking $r = 0$ and using (3.7) a second time it also follows that

$$\sup_{-\infty<y<\infty} | \Pr\{aM + b \leq y\} - (H_k^{(\gamma,1)}(y))^k | \to 0. \tag{3.8}$$

The argument establishing (3.7) can be used to show that

$$\sup_{-\infty<y<\infty} | H_k^{(\gamma,1)}(y) - H_k^{(\gamma,0)}(y) | \to 0 \tag{3.9}$$

for $\gamma \uparrow 1$. Combining (3.9) and (3.8) proves the results as claimed.

COROLLARY 2.1. *$M(\gamma, \delta(\gamma))$ has a limit law in $\mathscr{G}$ if and only if for each k, $M_k(\gamma, \delta(\gamma))$ has a limit law in $\mathscr{G}$. In this case the limit laws of M and M_k are of the same type.*

PROOF. Recall (cf. (3.1)) that the class $\mathscr{G}$ has the property that $G \in \mathscr{G}$ if and only if $(G)^q \in \mathscr{G}$ for every positive rational q, and that the laws G and $(G)^q$ are then of the same type. The assertion now follows from the theorem.

Another corollary also follows immediately:

COROLLARY 2.2 *$H \in \mathscr{G}$ if and only if each H_k is of the same type as H.*

THEOREM 3. *If for each k there are constants a_k, b_k and a function $f_k(\gamma)$ such that $M_k(\gamma) =_d a_k M(f_k(\gamma)) + b_k$ then $H \in \mathscr{G}$.*

PROOF. Let $a(\gamma)M(\gamma) + b(\gamma)$ converge weakly as $\gamma \uparrow 1$ to the limit law H. Then by Theorem 2, $a(\gamma)M_k(\gamma) + b(\gamma)$ converges weakly with limit law $H_k = (H)^{1/k}$. But we can also express the convergence of M as $a(f_k(\gamma))M(f_k(\gamma)) + b(f_k(\gamma))$ converging weakly with limit law H. Consequently, by Theorem 2.1.1 of de Haan (1970), H and H_k are of the same type, and the theorem follows from Corollary 2.2.

EXAMPLE 1. *Limit law for $Y(\gamma)$.* Referring to (1.2), it is clear that

$$Y_k(\gamma) \equiv \sup_{n\leq 0}\{\gamma^{nk} X_{nk}\} =_d Y(\gamma^k),$$

so limit laws for Y are in $\mathscr{G}$.

EXAMPLE 2. *Limit law for $Z(\delta)$.* Referring to (1.3),

$$Z_k(\delta) \equiv \sup_{n\geq 0}\{X_{nk} - nk\delta\} =_d Z(k\delta),$$

so limit laws for Z are in $\mathscr{G}$.

It is important to note here that no claim is made about the limit laws of $Y(\gamma)$ and $Z(\delta)$ being the same, or being the same as for M_n.

EXAMPLE 3. *Limit laws for polynomial weights.* Suppose

$$w_n(\gamma) = (1 + n(1 - \gamma))^{-r}, \quad v_n(\delta) \equiv 0.$$

Then taking $h(y) = (y^{1/r} - 1)/(1 - \gamma)$(cf. (2.3)), a.s. finiteness of $M(\gamma; w)$ is ensured by the finiteness of $EX^{1/r}$ (cf. Theorem 1). Further,

$$M_k(\gamma; w) =_d M(1 - k(1 - \gamma); w),$$

so limit laws for $\sup_{n\geq 0}\{X_n/(1 + n(1 - \gamma))^r\}$ are in $\mathscr{G}$.

EXAMPLE 4. *Limit law not in* $\mathscr{G}$. Given i.i.d. $\{X_n\}$, suppose that both $\{M_n\}$ and $Z(\delta)$ have limit laws but that the limit laws differ. Introduce $w_n(\delta) \equiv 1$ and $v_n(\delta) = (r(\delta) - n)_+\delta$ for some integer-valued function $r(.)$ to be specified. Then

$$M(\delta) = \sup_{n\geq 0}\{X_n - (r(\delta) - n)_+\delta\} =_d \max(M_{r(\delta)}, Z(\delta)),$$

with $M_{r(\delta)}$ and $Z(\delta)$ independent. By choosing $r(\cdot)$ so that $r(\delta) \to \infty$ $(\delta \to 0)$ and $\Pr\{M(\delta) = Z(\delta)\} = 1 - \Pr\{M(\delta) = M_{r(\delta)}\} \not\to 0$ or 1 as $\delta \to 0$, then any limit law for $M(\delta) \notin \mathscr{G}$ because the d.f. is the product of two different types of d.f. in $\mathscr{G}$.

In a little more detail, for example, suppose the d.f. of X_i is Φ_α for some $\alpha > 1$, and define $r(\delta)$ as the integer part of $1/\delta^{\alpha/(\alpha-1)}$. Since the law of $\varepsilon M_{[1/\varepsilon^\alpha]}$ equals Φ_α whenever $1/\varepsilon^\alpha$ is an integer, then the limit law of $\delta^{1/(\alpha-1)}M_{r(\delta)}$ equals Φ_α, while the limit law of $\delta^{1/(\alpha-1)}Z(\delta)$ (cf. Theorem 7 below) is

$$\lim_{\delta\to 0}\Pr\{\delta^{1/(\alpha-1)}Z(\delta) \leq y\} = \exp(-y^{-(\alpha-1)}/(\alpha - 1)).$$

4. Domains of attractions for weight functions. In detailing the precise analytical form of a (nontrivial) limit law as $\gamma \uparrow 1$ for the r.v.'s $M(\gamma, \delta(\gamma); w, v)$, equivalently of a limit for

$$\begin{aligned} H^{(\gamma)}(y) &\equiv \Pr\{a(\gamma)(M(\gamma, \delta(\gamma); w, v) + b(\gamma)) \leq y\} \\ &= \textstyle\prod_{n=0}^{\infty} F[(b(\gamma) + v_n(\gamma) + y/a(\gamma))/w_n(\gamma)], \end{aligned}$$

it is evident that some assumptions are needed concerning the way that $F(x) \to 1$ as $x \to \ell$, and that for each fixed γ we must have

$$\liminf_{n\to\infty}\{(b(\gamma) + v_n(\gamma) + y/a(\gamma))/w_n(\gamma)\} \geq \ell. \tag{4.1}$$

The condition (4.1) implies that

$$\begin{aligned} &-\log H^{(\gamma)}(y) \\ &= -\textstyle\sum_{n=0}^{\infty} \log F[(b(\gamma) + v_n(\gamma) + y/a(\gamma))/w_n(\gamma)] \qquad (4.2)\\ &= (1 + o(1)) \textstyle\sum_{n=0}^{\infty} [1 - F((b(\gamma) + v_n(\gamma) + y/a(\gamma))/w_n(\gamma))] \end{aligned}$$

where, because by assumption $H^{(\gamma)}(y)$ has a nontrivial limit as $\gamma \uparrow 1$, the term $o(1)$ converges to zero uniformly on bounded intervals for y. All the results in this section essentially start from this representation (4.2).

Recall that a function U mapping $\mathbb{R}^+ \equiv (0, \infty)$ into itself is said to *vary regularly* (at infinity) with *exponent* ρ, $-\infty < \rho < \infty$, when

$$\lim_{x\to\infty} U(tx)/U(x) = t^{\rho} \quad (\text{all } x \in \mathbb{R}^+). \tag{4.3}$$

Gnedenko showed that M_n has a limit law in $\mathscr{G}_\Phi$ if and only if $\ell = \infty$ and $1 - F(x)$ varies regularly with exponent $-\alpha$, and that Φ_α is then its limit law, while its limit is in $\mathscr{G}_\Psi$ if and only if $\ell < \infty$ and $1 - F(\ell - x^{-1})$ varies regularly with exponent $-\alpha$, and that Ψ_α is then its limit law.

THEOREM 4. *Suppose that* $1 - F(x)$ *is regularly varying with exponent* $-\alpha < 0$, *and that*

$$\infty > \textstyle\sum_{n=0}^{\infty} [1 - F(1/w_n(\gamma))] \to \infty, \quad (\gamma \uparrow 1). \tag{4.4}$$

Then the limit law of $M(\gamma; w)$ *exists and equals* $\Phi_\alpha \in \mathscr{G}$.

PROOF. The finiteness of the sum at (4.4) allows us to conclude from Theorem 1 that $M(\gamma; w) < \infty$ a.s.

From the monotonicity of $\{w_n(\gamma)\}$ it follows that the function $a(\gamma)$ defined by

$$a(\gamma) = \sup\{a\colon \textstyle\sum_{n=0}^{\infty} [1 - F(1/a\, w_n(\gamma))] \le 1\} \tag{4.5}$$

decreases monotonically in γ and $\to 0$ as $\gamma \to 1$. By the right-continuity of F and the strict monotonicity in n of each sequence $\{w_n(\gamma)\}$,

$$\begin{aligned} 1 \ge \textstyle\sum_{n=0}^{\infty} [1 - F(1/a(\gamma)w_n(\gamma))] &\ge 1 - \textstyle\sum_{n=0}^{\infty} \Pr\{X = 1/a(\gamma)w_n(\gamma)\} \\ &\ge 1 - \Pr\{X \ge 1/a(\gamma)\} \to 1, \quad (\gamma \to 1). \end{aligned}$$

Consequently,

$$\textstyle\sum_{n=0}^{\infty} [1 - F(1/a(\gamma)w_n(\gamma))] \to 1, \quad (\gamma \to 1). \tag{4.6}$$

For each $y > 0$ we have from (4.2) with $v_n(\gamma) \equiv 0$ and $b(\gamma) = 0$ that

$$-\log H^{(\gamma)}(y) = (1 + o(1)) \textstyle\sum_{n=0}^{\infty} [1 - F(y/a(\gamma)w_n(\gamma))],$$

where the term of $o(1)$ is bounded by $[1 - F(y/a(\gamma))]/2F(y/a(\gamma))$. By the regular variation assumption concerning F,

$$[1 - F(y/a(\gamma)w_n(\gamma))]/[1 - F(1/a(\gamma)w_n(\gamma))] \to y^{-\alpha}$$

as $\gamma \to 1$, and this convergence is uniform in n because $1 = w_0(\gamma) > w_n(\gamma)$ (all n and γ). Thus

$$\begin{aligned} -\log H^{(\gamma)}(y) &= (1 + o(1))y^{-\alpha} \textstyle\sum_{n=0}^{\infty} [1 - F(1/a(\gamma)w_n(\gamma))] \\ &\to y^{-\alpha} = -\log \Phi_\alpha(y), \quad (\gamma \to 1). \end{aligned}$$

EXAMPLE 5. Suppose $F(x) = \Phi_\alpha(x)$. Then, much as in example 6.1 of Daley and Haslett (1982), for $0 < \gamma < 1$ and $y > 0$ we have

$$\Pr\{(1 - \gamma^{\alpha})^{1/\alpha} \sup_{n\ge0}\{\gamma^n X_n\} \le y\} = \exp(-y^{-\alpha}) = \Phi_\alpha(y).$$

The limit behaviour is trivial!

THEOREM 5. *Suppose that $\ell < \infty$ and $1 - F(\ell - x^{-1})$ is regularly varying with exponent $-\alpha$. Then the limt law of $Y(\gamma)$ exists and equals $\Psi_{\alpha+1} \in \mathscr{G}$.*

PROOF. By rescaling we can and shall assume that $\ell = 1$. Define $a(\gamma)$ by

$$a(\gamma) = \inf\{a > 0: a^{-1}(1 - F(1 - a^{-1})) \le -(\alpha + 1) \log \gamma\},$$

so that as $\gamma \uparrow 1$, $a(\gamma) \to \infty$.

Write

$$H^{(\gamma)}(y) = \Pr\{a(\gamma)(Y(\gamma) - 1) \le -y\}, \quad (0 < y < \infty)$$
$$= \Pr\{\sup_{n\ge 0}\gamma^n X_n \le 1 - y/a(\gamma)\}.$$

Since $a(\gamma) \to \infty$ $(\gamma \uparrow 1)$, we may assume that $y < a(\gamma)$, and then there is a least integer $N(\gamma)$ such that $\gamma^{N(\gamma)} < 1 - y/a(\gamma)$. For $n \ge N(\gamma)$, $\gamma^n X_n \le 1 - y/a(\gamma)$ a.s., and therefore, as at (4.2),

$$-\log H^{(\gamma)}(y) = -\log \Pr(\cap_{n=0}^{N(\gamma)-1} \{\gamma^n X_n \le 1 - y/a(\gamma)\})$$
$$= (1 + o(1)) \textstyle\sum_{n=0}^{N(\gamma)-1} [1 - F(\gamma^{-n}(1 - y/a(\gamma)))]$$

where, since $1 - y/a(\gamma) \to 1$, $(\gamma \uparrow 1)$, the term $o(1)$ is bounded for given y by $[1 - F(1 - y/a(\gamma))]2F(1 - y/a(\gamma))$.

Since the terms in the last summation decrease monotonically in n, and each term $\to 0$ as $\gamma \uparrow 1$, the sum can be approximated by the integral

$$\int_0^{\log(1-y/a(\gamma))/\log\gamma} [1 - F(e^{-u\log\gamma}(1 - y/a(\gamma)))]\, du \tag{4.7}$$

$$= (-\log\gamma)^{-1} \int_{a(\gamma)/y}^{\infty} [1 - F(1 - v^{-1})]v^{-1}(v - 1)^{-1}\, dv$$

on substituting $1 - v^{-1} = e^{-u\log\gamma}(1 - y/a(\gamma))$,

$$= (-\log\gamma)^{-1}(1 + o(1))(y/(\alpha + 1)a(\gamma))[1 - F(1 - y/a(\gamma))]$$

by the integral theorem for the tails of regularly varying functions (e.g., Theorem 1 of VIII.9 of Feller, 1966). But by definition of $a(\gamma)$, for any $\varepsilon > 0$,

$$1 \le [1 - F(1 - 1/(a(\gamma) - \varepsilon))]/[(a(\gamma) - \varepsilon)(\alpha + 1)(-\log\gamma)]$$
$$= \frac{1 - F(1 - 1/(a(\gamma) - \varepsilon))}{1 - F(1 - 1/a(\gamma))} \cdot \frac{a(\gamma)}{a(\gamma) - \varepsilon} \cdot \frac{1 - F(1 - 1/a(\gamma))}{a(\gamma)(\alpha + 1)(-\log\gamma)}$$
$$= (1 + o(1))[1 - F(1 - 1/a(\gamma))]/[a(\gamma)(\alpha + 1)(-\log\gamma)] \le 1 + o(1)$$

as $\gamma \to 1$, and thus $[1 - F(1 - 1/a(\gamma))]/a(\gamma)(\alpha + 1)(-\log\gamma) \to 1$ as $\gamma \to 1$. Consequently,

$$-\log H^{(\gamma)}(y) = (1 + o(1))y^{\alpha+1}, \quad (\gamma \to 1),$$

proving the theorem.

Comparison of Theorems 4 and 5 prompts the question as to whether Theorem 5 may hold with a general class of weight functions as in Theorem 4. The following example shows that any such result would require such a class to be more restricted than the general class of Theorem 4.

EXAMPLE 6. Suppose we are given i.i.d $\{X_n\}$ with $1 - F(1 - x^{-1})$ regularly varying with exponent $-\alpha$. Then the limit law for $\{M_n\}$ is Ψ_α, while the limit law for $Y(\gamma)$ is $\Psi_{\alpha+1}$. Much as in Example 4, consider the weights $\gamma^{(n-r(\gamma))_+}$ where $r(\gamma)$ is an integer. Then

$$M^{(\gamma)} \equiv \sup_{n\geq 0}\{\gamma^{(n-r(\gamma))} + X_n\} =_d \max(M_{r(\gamma)}, Y(\gamma))$$

with $M_{r(\gamma)}$ and $Y(\gamma)$ independent. By choosing $r(\gamma)$ appropriately, a limit law may be exhibited for $M^{(\gamma)}$ as the product of the limit laws Ψ_α and $\Psi_{\alpha+1}$ of $M_{r(\gamma)}$ and $Y(\gamma)$. Hence, it is not in $\mathscr{G}$.

For example, if $\Pr\{X_n \leq 1 - x\} = \min(1, e^{-x})$,

$$\Pr\{r(\gamma)(M_{r(\gamma)} - 1) \leq -x\} = e^{-x}.$$

Let $a(\gamma)$ be determined for $Y(\gamma)$ as in Theorem 5. If we now set $r(\gamma)$ equal to the integer part of $a(\gamma)$, then a nontrivial limit law for $a(\gamma)(M^{(\gamma)} - 1)$ exists and is a product as asserted. If either $r(\gamma) = o(a(\gamma))$ or $a(\gamma) = o(r(\gamma))$, then any limit law for $M^{(\gamma)}$ is trivial (i.e., equals 0 or 1).

THEOREM 6. *Suppose that $\ell \leq \infty$ and is such that $\{M_n\}$ has Λ as its limit law. Then a limit law for $Y(\gamma)$ exists and it too equals Λ.*

PROOF. Appealing to (4.2), we seek functions $a(\gamma)$ and $b(\gamma)$ such that

$$\sum_{n=0}^{\infty} [1 - F(\gamma^{-n}(b(\gamma) + y/a(\gamma)))] \to e^{-y} \quad (\gamma \uparrow 1). \tag{4.8}$$

Observe that, because $\{M_n\}$ has limit law Λ,

$$[1 - F(x + yR(x))]/[1 - F(x)] \to e^{-y} \tag{4.9}$$

as $x \to \ell$ where

$$R(x) = [1 - F(x)]^{-1} \int_x^{\ell} [1 - F(u)]\, du \tag{4.10}$$

(see e.g. Theorem 2.5.1 of De Haan, 1970), for which as $x \uparrow \ell$,

$$\frac{R(x)}{x} \to 0 \quad \text{if} \quad \ell = \infty \quad \text{or} \quad \frac{R(x)}{\ell - x} \to 0 \quad \text{if} \quad \ell < \infty. \tag{4.11}$$

Further, from (4.9) it can be checked that the convergence there is uniform on compact sets, and therefore (as will be needed below)

$$1 \geq \int_x^\infty u^{-1}[1 - F(u)]\, du/x^{-1} \int_x^\infty [1 - F(u)]\, du$$

$$(4.12) \qquad = \int_0^\infty \frac{1 - F(x + yR(x))}{1 - F(x)} \cdot \frac{dy}{1 + yR(x)/x}$$

$$\geq 1 - e^{-y'} \quad (x \to \infty),$$

by restricting the range of integration to the closed interval $[0, y']$ for some finite y'.

Supposing $a(\cdot)$ and $b(\cdot)$ are given, and that $y > b(\gamma)$ (for otherwise the sum at (4.8) is not convergent), the sum at (4.8) is approximated as at (4.7) by

$$\int_0^{\log[(b(\gamma)+y/a(\gamma))/\ell]/\log\gamma} [1 - F(e^{-u\log\gamma}(b(\gamma) + y/a(\gamma)))]\, du$$

where the upper limit $= \infty$ if $\ell = \infty$, and the approximation is asymptotically exact provided $F(b(\gamma) + y/a(\gamma)) \to 1$. Assume this last holds. Substituting v for the argument of F, the integral equals

$$(-\log \gamma)^{-1} \int_{b(\gamma)+y/a(\gamma)}^{\ell} v^{-1}(1 - F(v))\, dv.$$

We now treat the cases $\ell = \infty$ and $\ell < \infty$ separately.

Supposing $\ell = \infty$, it follows from (4.12), written as

$$\int_x^\infty v^{-1}[1 - F(v)]dv/x^{-1} \int_x^\infty [1 - F(v)]\, dv \to 1, \quad (x \to \infty),$$

that the sum at (4.8) equals

$$(1 + o(1))(-\log \gamma)^{-1}[b(\gamma) + y/a(\gamma)]^{-1} \int_{b(\gamma)+y/a(\gamma)}^\infty (1 - F(u))\, du$$

$$= (1 + o(1))(-\log \gamma)^{-1}[b(\gamma) + y/a(\gamma)]^{-1}$$

$$\cdot [1 - F(b(\gamma) + y/a(\gamma))]R(b(\gamma) + y/a(\gamma)).$$

Define $b(\gamma)$ as the root in $(0, \infty)$ of

$$(4.13) \qquad \int_{b(\gamma)}^{\ell} (1 - F(u))\, du = b(\gamma)(-\log \gamma) = (1 - F(b(\gamma)))R(b(\gamma))$$

and set $a(\gamma) = 1/R(b(\gamma))$, so that $b(\gamma) \to \ell = \infty$ and $b(\gamma) + y/a(\gamma) \to \infty$ $(\gamma \uparrow 1)$. Then from (4.9),

$$1 - F(b(\gamma) + y/a(\gamma)) = 1 - F(b(\gamma) + y\, R(b(\gamma)))$$

$$= e^{-y}(1 + o(1))(1 - F(b(\gamma)));$$

from (4.11),

$$b(\gamma) + y/a(\gamma) = b(\gamma)[1 + y\,R(b(\gamma))/b(\gamma)] = (1 + o(1))b(\gamma);$$

and from (2.5.25) of de Haan (1970),

$$R(b(\gamma) + y\,R(b(\gamma))) = (1 + o(1))R(b(\gamma)).$$

The sum at (4.8) is now seen to be equal to $(1 + o(1))e^{-y}$, and the case $\ell = \infty$ is established.

In the case $\ell < \infty$, with $a(\cdot)$ and $b(\cdot)$ as at (4.13), it is trivially true that when $b(\gamma) + y/a(\gamma) \to \ell$,

$$(-\log\gamma)^{-1} \int_{b(\gamma)+y/a(\gamma)}^{\ell} v^{-1}(1 - F(v))\,dv$$
$$= (-\log\gamma)^{-1}(1 + o(1))\ell^{-1} \int_{b(\gamma)+y/a(\gamma)}^{\ell} (1 - F(v))\,dv$$

and the similar analysis as for $\ell = \infty$ follows to establish the result.

Theorems 5 and 6 with the case $w_n(\gamma) = \gamma^n$ of Theorem 4 can be summed up as follows:

Suppose M_n has a limit law $G \in \mathscr{G}$; then $Y(\gamma)$ has a limit law in $\mathscr{G}$, being equal to G unless $G = \Psi_\alpha$ in which case the limit law is $\Psi_{\alpha+1}$.

Equivalently, the d.f.'s F yielding limit laws for M_n in $\mathscr{G}$ yield limit laws for $Y(\gamma)$ in $\mathscr{G}\backslash\{\Psi_\alpha\colon 0 < \alpha \le 1\}$. From Example 1, all the limit laws for $Y(\gamma)$ belong to $\mathscr{G}$ so there remains open the question as to whether there exists any F yielding a limit law for $Y(\gamma)$ in $\{\Psi_\alpha\colon 0 < \alpha \le 1\}$.

5. Domains of attraction for location functions. The same prefatory remarks to Section 4 apply in considering possible limit laws for r.v.'s like $\{Z(\delta)\colon \delta > 0\}$. Our results are not quite as general as for $M(\gamma; w)$ in that it is only for certain d.f.'s for which Λ is the limit law of $\{M_n\}$ that we have obtained results with fairly general sequences $\{v_n(\delta)\}$ (see part (b) of Theorem 10 and Section 6, but note also Theorem 8 where $v_n(\delta) = n^{1/\beta}\delta$).

THEOREM 7. *Suppose that $1 - F(x)$ is regularly varying with exponent $-\alpha < -1$. Then the limit law of $Z(\delta)$ exists and equals $\Phi_{\alpha-1}$.*

PROOF. For all sufficiently small $\delta > 0$ define

$$a(\delta) = \sup\{a > 0\colon a^{-1}(1 - F(a^{-1})) \ge (\alpha - 1)\delta\},$$

so that $a(\delta) \downarrow 0$ as $\delta \downarrow 0$. Using (4.2), we study for $0 < y < \infty$,

$$H^{(\delta)}(y) \equiv \Pr\{a(\delta)\sup\{X_n - n\delta\} \le y\} = \textstyle\prod_{n=0}^{\infty} F(n\delta + y/a(\delta)),$$

so

$$(5.1)\qquad -\log H^{(\delta)}(y) = (1 + o(1)) \textstyle\sum_{n=0}^{\infty} [1 - F(n\delta + y/a(\delta))]$$

$$= (1 + o(1)) \int_0^{\infty} [1 - F(\delta u + y/a(\delta))]\, du$$

$$= (1 + o(1))\delta^{-1} \int_{y/a(\delta)}^{\infty} [1 - F(v)]\, dv$$

$$= (1 + o(1))\delta^{-1}(\alpha - 1)^{-1}(y/a(\delta))[1 - F(y/a(\delta))]$$

$$= (1 + o(1))y^{\alpha-1}(a(\delta))^{-1}[1 - F(1/a(\delta))]/(\alpha - 1)\delta$$

$$(5.2)\qquad = (1 + o(1))y^{\alpha-1}$$

provided

$$(5.3)\qquad (a(\delta))^{-1}[1 - F(1/a(\delta))]/(\alpha - 1)\delta \to 1 \quad \text{as } \delta \downarrow 0.$$

But, much as in the proof of Theorem 5,

$$1 \ge (1/a(\delta) - \varepsilon)[1 - F(1/a(\delta) - \varepsilon)]/(\alpha - 1)\delta$$

$$= \frac{1 - F(1/a(\delta) - \varepsilon)}{1 - F(1/a(\delta))} \cdot (1 - \varepsilon\, a(\delta)) \cdot \frac{1 - F(1/a(\delta))}{a(\delta)(\alpha - 1)\delta},$$

and since by right-continuity the last term ≥ 1 (all δ), and the other terms $\to 1$ as $a(\delta) \to 0$, (5.3) holds and the theorem is proved.

REMARK. A r.v. X with Φ_α as its d.f. has $EX < \infty$ if and only if $\alpha > 1$. Since

$$\sup_n \{X_n - n\delta\} < \infty \text{ a.s.}$$

if and only if

$$\infty > \textstyle\sum_{n=0}^{\infty} [1 - F(n\delta)] \approx \delta^{-1} \int_0^{\infty} [1 - F(v)]\, dv = \delta^{-1}EX,$$

the constraint on the exponent α in the theorem is seen to be necessary.

THEOREM 8. *Suppose that* $1 - F(x)$ *is regularly varying with exponent* $-\alpha$, *and for* $0 < \beta < \alpha$ *define*

$$(5.4)\qquad Z_\beta(\delta) \equiv \sup_{n\ge 0}\{X_n - n^{1/\beta}\delta\}.$$

Then a limit law for $Z_\beta(\delta)$ *exists and equals* $\Phi_{\alpha-\beta}$.

PROOF. Let $a(\delta)$ be a function to be defined later, with $a(\delta) \to 0$ as $\delta \to 0$, and such that $a(\delta)Z_\beta(\delta)$ has a nontrivial limit law. With $H^{(\delta)}(y) = \Pr\{a(\delta)Z_\beta(\delta) \le y\}$,

$$\begin{aligned} -\log H^{(\delta)}(y) &= (1 + o(1)) \textstyle\sum_{n=0}^{\infty} [1 - F(n^{1/\beta}\delta + y/a(\delta))] \\ &= (1 + o(1)) \int_0^\infty [1 - F(y/a(\delta) + \delta u^{1/\beta})]\, du \qquad (5.5) \\ &= (1 + o(1))\delta^{-\beta} \int_0^\infty \beta v^{\beta-1}[1 - F(y/a(\delta) + v)]\, dv. \end{aligned}$$

We now establish the following analogue of Theorem 2.6 of Seneta (1976).

LEMMA. *When $G(x)$ varies regularly with exponent $-\alpha$ and $\alpha > \beta > 0$, the function*

$$G_\beta(x) \equiv \int_0^\infty v^{\beta-1}G(x + v)\, dv \tag{5.6}$$

is regularly varying with exponent $\alpha - \beta$; specifically,

$$G_\beta(x)/x^\beta G(x) \to \int_0^\infty u^{\beta-1}(1 + u)^{-\alpha}\, du, \quad (x \to \infty). \tag{5.7}$$

Let $y > 0$ be fixed for the time being, and consider

$$\begin{aligned} &\frac{G_\beta(x)}{x^\beta G(x)} - \int_0^{yx} \left(\frac{v}{x}\right)^{\beta-1}\left(1 + \frac{v}{x}\right)^{-\alpha} \frac{dv}{x} \\ &= \int_0^{yx} \left(\frac{v}{x}\right)^{\beta-1}\left(1 + \frac{v}{x}\right)^{-\alpha}\left(\frac{(x + v)^\alpha G(x + v)}{x^\alpha G(x)} - 1\right) \frac{dv}{x} \qquad (5.8) \\ &\quad + \int_{yx}^\infty \left(\frac{v}{x}\right)^{\beta-1} \frac{G(x + v)}{G(x)} \cdot \frac{dv}{x}. \end{aligned}$$

For $0 < \beta < \alpha$, the integral

$$\int_0^\infty \left(\frac{v}{x}\right)^{\beta-1}\left(1 + \frac{v}{x}\right)^{-\alpha} \frac{dv}{x}$$

is a beta function, and on $0 \le v \le yx$, $(x + v)^\alpha G(x + v)/x^\alpha G(x) \to 1$ as $x \to \infty$. Consequently the first integral $\to 0$ as $x \to \infty$ by dominated convergence.

As in the proof of Theorem 2.6 in Seneta (1976), with $0 < \eta < \alpha - \beta$,

$$x^{-\beta} \int_{yx}^\infty v^{\beta-1}G(v + x)\, dv = x^{-\beta} \int_{yx}^\infty v^{\beta-1}(v + x)^{-\alpha}L(v + x)\, dv$$

where $L(v) = v^{\alpha}G(v)$ is a slowly varying function,

$$\leq x^{-\alpha+\eta} \sup_{v \geq x+yx}\{v^{-\eta}L(v)\} \cdot \int_{yx}^{\infty} \left(\frac{v}{x}\right)^{\beta-1}\left(1+\frac{v}{x}\right)^{-\alpha+\eta} \frac{dv}{x}$$

$$\sim (x+yx)^{-\eta}L(x+yx)x^{-\alpha+\eta} \int_{yx}^{\infty} \left(\frac{v}{x}\right)^{\beta-1}\left(1+\frac{v}{x}\right)^{-\alpha+\eta} \frac{dv}{x}$$

$$= (y+1)^{\alpha-\eta}G(x+yx) \int_{yx}^{\infty} \left(\frac{v}{x}\right)^{\beta-1}\left(1+\frac{v}{x}\right)^{-\alpha+\eta} \frac{dv}{x}.$$

It follows that the modulus of the second integral at (5.8) is at most

$$(y+1)^{\alpha-\eta}(G(x+yx)/G(x)) \int_{yx}^{\infty} \left(\frac{v}{x}\right)^{\beta-1}\left(1+\frac{v}{x}\right)^{-\alpha+\eta} \frac{dv}{x}$$

$$= (1+o(1))(y+1)^{-\eta} \int_{yx}^{\infty} \left(\frac{v}{x}\right)^{\beta-1}\left(1+\frac{v}{x}\right)^{-\alpha+\eta} \frac{dv}{x}$$

which may be made arbitrarily small by taking y sufficiently large. The lemma is now established.

Applying the lemma to the expression at (5.5), we have

$$\begin{aligned} -\log H^{(\delta)}(y) &= (1+o(1))\beta\delta^{-\beta}(y/a(\delta))^{\beta}[1 - F(y/a(\delta))] \\ &= (1+o(1))\beta\delta^{-\beta}y^{-(\alpha-\beta)}(a(\delta))^{-\beta}[1 - F(1/a(\delta))]. \end{aligned}$$

Thus, determining $a(\delta)$ by

(5.9) $$a(\delta) = \sup\{a > 0: [\beta(1 - F(1/a))]^{1/\beta}/\delta a \geq 1\},$$

and establishing the analogue of (5.3), $-\log H^{(\delta)}(y) \to y^{-(\alpha-\beta)}$ $(\delta \to 0)$ and Theorem 8 is proved.

THEOREM 9. *Suppose that* $\ell < \infty$ *and that* $1 - F(\ell - x^{-1})$ *is regularly varying with exponent* $-\alpha$. *Then the limit law of* $Z(\delta)$ *exists and equals* $\Psi_{\alpha+1}$.

PROOF. Much as in the proof of Theorem 5, we show that, taking $\ell = 1$ without loss of generality,

$$\Pr\{a(\delta)\sup\{X_n - n\delta\} \leq -y\} \to \exp(-y^{\alpha+1}), \quad (0 < y < \infty)$$

where for general ℓ

(5.10) $$a(\delta) = \inf\{a > 0: [1 - F(\ell - \ell\delta)]/\ell a \leq (\alpha+1)\delta\}.$$

The details are similar and are omitted.

THEOREM 10. (a) *Suppose that* F *is such that* $\{M_n\}$ *has* Λ *as its limit law. Then a limit law for* $Z(\delta)$ *exists and it too equals* Λ.

(b) *If additionally* $\ell = \infty$ *and*

(5.11)
$$R(x) = \int_x^\infty [1 - F(u)]\,du/[1 - F(x)] = E(X - x \mid X > x) \to 1/\alpha \quad (x \to \infty)$$

for some $0 < \alpha < \infty$, *then for sequences* $\{v_n(\delta)\}$ *satisfying*

(5.12)
$$\textstyle\sum_{n=0}^\infty \Pr\{X > v_n(\delta)\} < \infty,$$

(5.13)
$$\Pr\{\sup_{n\geq 0}\{x_n - v_n(\delta)\} - a(\delta) \leq y/b(\delta)\} \to \exp(-e^{-y}), \quad (\delta \downarrow 0)$$

where for sufficiently small $\delta > 0$,

(5.14)
$$a(\delta) = \sup\{a\colon \textstyle\sum_{n=0}^\infty [1 - F(a(\delta) + v_n(\delta))] \geq 1\}$$

and

(5.15)
$$b(\delta) = 1/R(a(\delta)).$$

PROOF. In the particular case $v_n(\delta) = n\delta$ of part (a), we can define $a(\delta)$ in place of (5.14) by the root of

$$1 = \delta^{-1}\int_{a(\delta)}^{\ell} [1 - F(v)]\,dv = \int_0^\infty [1 - F(a(\delta) + \delta u)]\,du = (1 + o(1))\textstyle\sum_{n=0}^\infty [1 - F(a(\delta) + n\delta)].$$

Now the convergence we seek to show is that

$$\begin{aligned}\exp(-e^{-y}) &= \lim_{\delta\downarrow 0}\Pr\{\sup\{X_n - v_n(\delta)\} - a(\delta) \leq y/b(\delta)\}\\ &= \lim_{\delta\downarrow 0} \textstyle\prod_{n=0}^\infty F(a(\delta) + y/b(\delta) + v_n(\delta))\\ &= \lim_{\delta\downarrow 0} H^{(\delta)}(y) \quad \text{say.}\end{aligned}$$

Much as before, and in the case $v_n(\delta) = n\delta$,

(5.16)
$$\begin{aligned}-\log H^{(\delta)}(y) &= (1 + o(1))\textstyle\sum_{n=0}^\infty [1 - F(a(\delta) + y/b(\delta) + n\delta)]\\ &= (1 + o(1))\int_0^\infty [1 - F(a(\delta) + y/b(\delta) + \delta u)]\,du\\ &= (1 + o(1))\delta^{-1}\int_{a(\delta)+y/b(\delta)}^\infty [1 - F(v)]\,dv\\ &= (1 + o(1))\delta^{-1}[1 - F(a(\delta) + yR(\delta))]R(a(\delta) + yR(\delta))\\ &= (1 + o(1))\delta^{-1}[1 - F(a(\delta))]e^{-y}R(a(\delta) + yR(\delta)),\end{aligned}$$

using Theorem 2.5.1 of de Haan (1970),

$$= (1 + o(1))e^{-y}R(a(\delta) + yR(\delta))/R(a(\delta))$$
$$\to e^{-y}$$

as at de Haan's equation (2.5.9). Part (a) is proved.

For part (b), we have in place of the relation above (5.16) that

$$-\log H^{(\delta)}(y) = (1 + o(1)) \textstyle\sum_{n=0}^{\infty} [1 - F(a(\delta) + y/b(\delta) + v_n(\delta))].$$

In view of (5.14), in which, much as in Section 4, the infinite sum is asymptotically equal to 1 for $\delta \downarrow 0$, it is therefore enough to show that we may write

$$\textstyle\sum_{n=0}^{\infty} [1 - F(a(\delta) + y/b(\delta) + v_n(\delta))]$$
$$= (1 + o(1))e^{-y} \textstyle\sum_{n=0}^{\infty} [1 - F(a(\delta) + y/b(\delta))].$$

While a direct proof can (presumably) be constructed, it is simpler to appeal to the duality argument of Section 6 and apply Theorem 4. Details are given in the next section.

6. Duality between scale and location functions. Since for positive $\{X_n\}$ and $0 < \gamma < 1$

$$\sup_{n\geq 0}\{\gamma^n X_n\} = \exp(\sup_{n\geq 0}\{\log X_n - n \log(\gamma^{-1})\})$$
$$= \exp(\sup_{n\geq 0}\{W_n - n\delta\})$$

where $W_n = \log X_n$ and $\delta = -\log \gamma$, it is proper to exhibit any relationship between the results of Sections 4 and 5. For brevity, write $DA_{\mathscr{M}}(\cdot)$, $DA_{\mathscr{Y}}(\cdot)$ and $DA_{\mathscr{Z}}(\cdot)$ to denote the domains of attraction of the limit law $(\cdot)$ for the respective sequences $\{M_n\}$, $\{Y(\gamma)\}$, and $\{Z(\delta)\}$. Then most of Theorems 4 to 10 can be phrased as follows:

$$F \in DA_{\mathscr{M}}(\Phi_\alpha) \Rightarrow F \in DA_{\mathscr{Y}}(\Phi_\alpha) \quad \text{and} \quad (\text{when } \alpha > 1)$$
$$F \in DA_{\mathscr{Z}}(\Phi_{\alpha-1});$$
$$F \in DA_{\mathscr{M}}(\Psi_\alpha) \Rightarrow F \in DA_{\mathscr{Y}}(\Psi_{\alpha+1}) \quad \text{and} \quad F \in DA_{\mathscr{Z}}(\psi_{\alpha+1});$$
$$F \in DA_{\mathscr{M}}(\Lambda) \Rightarrow F \in DA_{\mathscr{Y}}(\Lambda) \quad \text{and} \quad F \in DA_{\mathscr{Z}}(\Lambda).$$

When the tail of the d.f. $1 - F(x)$ of the r.v. X varies regularly with exponent $-\alpha$, $[1 - F(tx)]/[1 - F(x)] \to t^{-\alpha}$ $(x \to \infty)$ for $0 < t < \infty$. Consequently, since

$$\Pr\{\log W > y + \tau/\alpha\} = \Pr\{W > e^y e^{\tau/\alpha}\}$$
$$= (1 + o(1))e^{-\tau}\Pr\{W > e^y\}$$
$$= (1 + o(1))e^{-\tau}\Pr\{\log W > y\},$$

the r.v. $W = e^X$ has its d.f. $\in DA_{\mathscr{M}}(\Lambda)$ with $E(W - x \mid W > x) \to 1/\alpha$ as $x \to \infty$. Conversely, when $W = e^X$ satisfies these conditions, $1 - F(x)$ varies regularly with exponent $-\alpha$.

It is this converse statement which enables part (b) of Theorem 10 to be deduced from Theorem 4. To see this, write $w_n(\delta) = \exp(-v_n(\delta))$, $a_1(\delta) = \exp(-a(\delta))$, $W_n = \exp(X_n)$, so that

$$X_n - v_n(\delta) - a(\delta) \le y/b(\delta)$$

if and only if

$$a_1(\delta)w_n(\delta)W_n \le \exp(y/b(\delta)).$$

Now

$$\sum_{n=0}^{\infty} \Pr\{W_n > 1/w_n(\delta)\} = \sum_{n=0}^{\infty} \Pr\{X_n > v_n(\delta)\}$$

whose finiteness is assumed at (5.6), and

$$\begin{aligned} a_1(\delta) &= \exp(-\sup\{a\colon \textstyle\sum_{n=0}^{\infty} [1 - F(a(\delta) + v_n(\delta))] \ge 1\}) \\ &= \exp(-\sup\{a\colon \textstyle\sum_{n=0}^{\infty} \Pr\{W > 1/e^{-a(\delta)}w_n(\delta)\} \ge 1\}) \\ &= \sup\{a_1\colon \textstyle\sum_{n=0}^{\infty} \Pr\{W > 1/a_1 w_n(\delta)\} \le 1\}. \end{aligned}$$

The condition at (4.4) is satisfied and $a(\gamma)$ at (4.5) may be replaced by $a_1(\delta)$, and Theorem 4 therefore applies to $\sup\{w_n(\delta)W_n\}$.

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