The Annals of Probability
1984, Vol. 12, No. 2, 561–570

A RENEWAL THEOREM OF BLACKWELL TYPE

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Suppose $\{X_1, X_2, \cdots\}$ are i.i.d. random variables with finite mean $0 < E(X_1) < \infty$. If S_n stands for the nth partial sum, and $\{a(n)\}_n$ is a sequence of nonnegative numbers, then $G(x) = \sum_{n=0}^\infty a(n)P\{S_n \le x\}$ is a generalized renewal measure. We investigate the behaviour of $G(x+h) - G(x)$ as $x \to \infty$ for $\{a(n)\}_n$ regularly varying.

1. Introduction and results. Let $\{X_1, X_2, \cdots\}$ be a sequence of nonnegative independent identically distributed random variables with distribution function $F(x)$ and with $0 < EX_1 = \mu < \infty$, and write $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \ge 1$.

The object of this paper is to give renewal theorems of Blackwell type for generalized renewal measures, i.e. theorems on the asymptotic behaviour of

$$\sum_{n=1}^\infty a(n)P\{x < S_n \le x + h\}, \quad h > 0, \quad x \to \infty \tag{1.1}$$

for some sequence of nonnegative constants $\{a(n) \mid n \in \mathbb{N}\}$. When F is lattice, we suppose F is concentrated on the nonnegative integers and suppose its span equals 1. We then examine the asymptotic behaviour of

$$\sum_{n=1}^\infty a(n)P\{S_n = k\}$$

as $k \to \infty$.

Clearly connected with the problem above is the asymptotic behaviour of $\sum_{n=0}^\infty a(n)P\{S_n \le x\}$. When $a(n) \equiv 1$ this function is known as the renewal function $U(x) = \sum_{n=0}^\infty P\{S_n \le x\}$. Similarly in the lattice case, the renewal sequence $\{u_n\}_n$ is defined by $u_n = \sum_{k=0}^\infty P\{S_k = n\}$. Generalized renewal measures of the form $\sum_{n=0}^\infty a(n)P\{S_n \le x\}$ have been studied by many authors. See e.g. Embrechts and Omey (1983), Greenwood, Omey and Teugels (1982), Heyde (1966), Kalma (1972), Kawata (1961) and Smith (1964). As to the asymptotic behaviour of (1.1), Kawata (1961) gave a result for $\{a(n)\}_n$ such that $\sum_{k=1}^n a(k) = na + o(n^{1/2})$ for some $a \ge 0$ and various moment conditions, and Kalma (1972) studied the case where $a(n) = n^{-\alpha}$, $\alpha \in \mathbb{R}$.

The main result which we are going to prove in this paper is the following.

Theorem 1. *Let $a(x)$ be a positive function such that $a(x) \in RV_\alpha$, that is $a(x) = x^\alpha L(x)$, $L(x)$ being slowly varying. Let F be nonlattice.*

(a) *In case $\alpha > -1$, for all $h > 0$,*

$$\sum_{n=1}^\infty a(n)P\{x < S_n \le x + h\} \sim \frac{h}{\mu^{\alpha+1}} a(x), \quad \textit{as} \quad x \to \infty. \tag{1.2}$$

Received December, 1982; revised August, 1983

AMS 1980 *subject classifications.* Primary 60K05.

Key words and phrases. Generalized renewal measures, renewal theory, regular variation, Blackwell theorem.

(b) *In case* $\alpha = -1$, *if* (i) *L is monotone and as* $x \to \infty$

$$1 - F(x) \sim Ka(x)(x \to \infty) \quad \text{for some} \quad K \geq 0 \tag{1.3}$$

or if (ii) $x^{1+\delta}(1 - F(x)) \to 0$ *for some* $\delta > 0$ *as* $x \to \infty$, *then* (1.2) *holds.*

(c) *In case* $\alpha < -1$, *if* (1.3) *is satisfied then* (1.2) *also holds.*

REMARK. For F lattice, a similar theorem is easily formulated and proved.

2. Proof of Theorem 1(a). The proof depends on the following renewal type of result. Let $\{b_n\}_{n \in N}$ be a sequence of nonnegative constants and $G(x) = \sum_{n=0}^{\infty} b_n P\{S_n \leq x\}$.

LEMMA 1. *Let* $\rho \geq 0$. *If*

$$\{\textstyle\sum_{k=0}^{n} b_k\}_n \in RV_\rho,$$

then

$$G(x) \sim \mu^{-\rho} \textstyle\sum_{k=0}^{[x]} b_k \quad \textit{as} \quad x \to \infty.$$

PROOF. Let $f(s) = E(e^{-sX_1})$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ and $g(s) = \int_0^{\infty} e^{-sx} dG(x)$. There exists a slowly varying function L such that $\sum_{k=0}^{n} b_k \sim n^\rho L(n)$ as $n \to \infty$. From Feller (1971, page 447), it follows that

$$B(z) \sim (1 - z)^{-\rho} L((1 - z)^{-1}) \Gamma(1 + \rho) \quad \text{as} \quad z \uparrow 1.$$

Since $g(s) = B(f(s))$ and $1 - f(s) \sim \mu s$ as $s \downarrow 0$, it follows that

$$g(s) \sim (\mu s)^{-\rho} L(s^{-1}) \Gamma(1 + \rho) \quad \text{as} \quad s \downarrow 0.$$

An application of Feller (1971, page 445) yields the conclusion. □

The following result is interesting in its own right.

LEMMA 2. *Let* $U(x) = \sum_{n=0}^{\infty} P\{S_n \leq x\}$ *be the renewal function and let* $R(x)$ *be a nondecreasing function such that* $R(0) = 0$ *and for all* $y \in \mathbb{R}$, $R(x + y) \sim R(x)$ *as* $x \to \infty$. *Then for all* $y \in \mathbb{R}$,

$$U{*}R(x + y) - U{*}R(x) \sim y\mu^{-1} R(x) \quad \textit{as} \quad x \to \infty.$$

PROOF. Take $y > 0$ and x_0 such that $0 < x_0 < x$, then

$$U{*}R(x + y) - U{*}R(x)$$

$$= \left(\int_0^{x-x_0} + \int_{x-x_0}^{x} \right) \{U(x + y - z) - U(x - z)\} dR(z)$$

$$+ \int_x^{x+y} U(x + y - z)\, dR(z) \equiv I_1 + I_2 + I_3,$$

say. First, since $0 \leq I_3 \leq U(y)\{R(x + y) - R(x)\}$, it follows that $I_3 = o(R(x))$ as $x \to \infty$.

In I_1 we have $x - z \geq x_0$. For any $\varepsilon > 0$, choose $x_0 = x_0(\varepsilon)$ large enough so that (by Blackwell's theorem), we get

$$\left(\frac{y}{\mu} - \varepsilon\right)\frac{R(x - x_0)}{R(x)} \leq \frac{I_1}{R(x)} \leq \left(\frac{y}{\mu} + \varepsilon\right)\frac{R(x - x_0)}{R(x)}.$$

Hence

$$\frac{y}{\mu} - \varepsilon \leq \liminf_{x\to\infty} \frac{I_1}{R(x)} \leq \limsup_{x\to\infty} \frac{I_1}{R(x)} \leq \frac{y}{\mu} + \varepsilon.$$

Finally for I_2 we have

$$0 \leq I_2 \leq U(y + x_0)\{R(x) - R(x - x_0)\},$$

so that $I_2 = o(R(x))$ as $x \to \infty$. Combining the estimates for I_1, I_2, I_3 and then $\varepsilon \downarrow 0$ we obtain the desired result. □

PROOF OF THEOREM 1(a). First we show that we can assume that $na(n)$ is nondecreasing. Since $\alpha > -1$, we know that $na(n)$ asymptotically equals a nondecreasing sequence, $nc(n)$ say. In this case, we have for any $\varepsilon > 0$

$$(1 - \varepsilon)c(n) \leq a(n) \leq (1 + \varepsilon)c(n)$$

for all $n \geq n_0(\varepsilon)$. Now, since $\mu = E(X_1) < \infty$, we have

$$\textstyle\sum_{n=1}^{n_0} a(n)P\{x < S_n \leq x + h\} = o(x^{-1})$$

and

$$\textstyle\sum_{n=1}^{n_0} c(n)P\{x < S_n \leq x + h\} = o(x^{-1}), \quad \text{as} \quad x \to \infty.$$

Hence if the result is true for $\{c(n)\}_n$, then it also holds for $\{a(n)\}_n$. From now on we assume that $\{na(n)\}_n$ is nondecreasing. Define,

$$G(x) = \textstyle\sum_{n=0}^{\infty} a(n)P\{S_n \leq x\}, \quad G_1(x) = \displaystyle\int_0^x y\,dG(y),$$

$$R(x) = \textstyle\sum_{n=0}^{\infty} b_nP\{S_n \leq x\}, \quad b_n = (n + 1)a(n + 1) - na(n),$$

$$Q(x) = \int_0^x y\,dF(y).$$

Then $G_1(x) = R * Q * U(x)$.

Now it follows from Lemma 1 and $xa(x) \in RV_{\alpha+1}$ that

$$R(x) \sim \mu^{-\alpha-1}xa(x) \quad \text{as} \quad x \to \infty.$$

By assumption, also $Q(x) \to \mu$ as $x \to \infty$. Hence

$$R * Q(x) \sim \mu^{-\alpha}xa(x) \quad \text{as} \quad x \to \infty.$$

Since $R * Q$ is nondecreasing and regularly varying, it follows from Lemma 2 that

$$\frac{G_1(x + y) - G_1(x)}{R * Q(x)} \to \frac{y}{\mu} \quad \text{as} \quad x \to \infty.$$

and hence also that

$$\frac{G_1(x+y) - G_1(x)}{xa(x)} \to \frac{y}{\mu^{\alpha+1}} \quad \text{as} \quad x \to \infty.$$

Now from the definition of G_1 it follows that (for $y > 0$)

$$x\{G(x+y) - G(x)\} \le G_1(x+y) - G_1(x) \le (x+y)\{G(x+y) - G(x)\}$$

therefore

$$\{G(x+y) - G(x)\}/a(x) \to \mu^{-\alpha-1}y \quad \text{as} \quad x \to \infty.$$

The proof of the Theorem is thus completed. □

3. Proof of Theorem 1(b) and (c). We prove Theorem 1(b) and (c) simultanously for fixed L and by an induction argument. We know from Theorem 1(a) that (1.2) holds for $\alpha > -1$. Suppose now (1.2) holds for $\alpha = \theta (\theta \le 0)$. We shall prove that it then also holds for $\alpha = \theta - 1$. Denote by $F^{(n)}$ the nth convolution of F with itself and write

$$\begin{aligned} \sum_{n=1}^{\infty} a(n)P\{x < S_n \le x+h\} &= \sum_{n=1}^{\infty} n^{\theta-1}L(n)F^{(n)}(]x, x+h]) \\ &= G_{\theta-1}(]x, x+h]). \end{aligned} \tag{3.1}$$

The following lemma will be needed.

LEMMA 3. *Let $Q(x) = \int_0^x y\, dF(y)$ as before. Then for $n \ge 1$ and all $h > 0$,*

$$xF^{(n)}(]x, x+h]) \le nQ*F^{(n-1)}(]x, x+h]) \le (x+h)F^{(n)}(]x, x+h]).$$

PROOF. Let $W(x) = \int_0^x y\, dF^{(n)}(y)$. Using Laplace-Stieltjes transforms we easily see that

$$W(x) = nQ*F^{(n-1)}(x).$$

Hence

$$\begin{aligned} &nQ*F^{(n-1)}(]x, x+h]) \\ &= \int_x^{x+h} y\, dF^{(n)}(y) = (x+h)F^{(n)}(x+h) - xF^{(n)}(x) - \int_x^{x+h} F^{(n)}(y)\, dy \\ &\le (x+h)F^{(n)}(x+h) - xF^{(n)}(x) - hF^{(n)}(x) \\ &= (x+h)F^{(n)}(]x, x+h]). \end{aligned}$$

The left hand side inequality follows similarly. □

Using this lemma in (3.1), we have

$$G_{\theta-1}(]x, x+h]) \le x^{-1} \sum_{n=1}^{\infty} n^{\theta}L(n)F^{(n-1)}*Q(]x, x+h]) \tag{3.2}$$

and

$$G_{\theta-1}(]x, x+h]) \geq (x+h)^{-1} \sum_{n=1}^{\infty} n^{\theta}L(n)F^{(n-1)} * Q(]x, x+h]).$$

Let

$$V(]x, x+h]) \equiv \sum_{n=1}^{\infty} n^{\theta}L(n)F^{(n-1)}(]x, x+h])$$
$$= \sum_{n=0}^{\infty} (n+1)^{\theta}L(n+1)F^{(n)}(]x, x+h]).$$

Now $(n+1)^{\theta}L(n+1) \sim n^{\theta}L(n)$ as $n \to \infty$ and $\sum_{n=1}^{\infty} n^{\theta}L(n)F^{(n)}(]x, x+h]) \sim h\mu^{-\theta-1}x^{\theta}L(x)$ by the induction hypothesis. Using for fixed n_0 the estimates $\sum_{n=1}^{n_0} n^{\theta}L(n)F^{(n)}(]x, x+h]) = o(a(x))$ and $\sum_{n=1}^{n_0} (n+1)^{\theta}L(n+1)F^{(n)}(]x, x+h]) = o(a(x))$, we also have

$$V(]x, x+h]) \sim \mu^{-\theta-1}hx^{\theta}L(x) \quad \text{as} \quad x \to \infty. \tag{3.3}$$

To prove the theorem it remains to show (cf. (3.2)) that

$$\frac{V * Q(]x, x+h])}{x^{\theta}L(x)} \to h\mu^{-\theta} \quad \text{as} \quad x \to \infty.$$

Now

$$\begin{aligned} &V * Q(]x, x+h]) \\ &= \int_0^x [V(x+h-y) - V(x-y)]\, dQ(y) + \int_x^{x+h} V(x+h-y)\, dQ(y) \qquad (3.4)\\ &\equiv I_1 + I_2, \end{aligned}$$

say. We consider two cases: $\theta = 0$ or $\theta < 0$.

(I) Case $\theta < 0$. In I_2, since $x \leq y \leq x+h$, we have

$$0 \leq I_2 \leq V(h) \int_x^{x+h} dQ(y) \leq V(h)(x+h)[F(x+h) - F(x)].$$

Hence

$$\lim_{x\to\infty} I_2/(x^{\theta}L(x)) = 0 \tag{3.5}$$

by (1.3). As to I_1 in (3.4), we write for some $0 < \varepsilon < 1$,

$$I_1 = \int_0^{\varepsilon x} + \int_{\varepsilon x}^{x} \equiv I_{11} + I_{12},$$

say. We first consider I_{12}. If $\theta < 0$, $V(x+h-y) - V(x-y) \leq c$ for some positive constant c, so that

$$I_{12} \leq c \int_{x\varepsilon}^{x} dQ(y) = c \int_{x\varepsilon}^{x} y dF(y) \leq cx[F(x) - F(x\varepsilon)].$$

Therefore

$$\limsup_{x\to\infty} \frac{I_{12}}{x^\theta L(x)}$$

$$\leq c\left[\lim_{x\to\infty} \frac{x^{1-\theta}}{L(x)} \{(1 - F(\varepsilon x)) - (1 - F(x))\}\right] \tag{3.6}$$

$$\leq c\left[\lim_{x\to\infty} \frac{x^{1-\theta}L(\varepsilon x)}{(\varepsilon x)^{1-\theta}L(x)} \cdot \frac{(\varepsilon x)^{1-\theta}(1 - F(\varepsilon x))}{L(\varepsilon x)} - K\right] = c \cdot K(\varepsilon^{\theta-1} - 1).$$

Finally we consider I_{11}. Since in I_{11}, $(1 - \varepsilon)x \leq x - y \leq x$, we have from (3.3), for some constant $c > 0$,

$$\frac{V(x + h - y) - V(x - y)}{x^\theta L(x)} \leq c \frac{(x - y)^\theta L(x - y)}{x^\theta L(x)} \tag{3.7}$$

$$\leq c(1 - \varepsilon)^\theta \sup_{(1-\varepsilon)\leq t\leq 1} \frac{L(tx)}{L(x)},$$

which is bounded independently of x if x is large. So, using Lebesgue's theorem and (3.3) we obtain

$$\lim_{x\to\infty} \frac{I_{11}}{x^\theta L(x)} = \mu^{-\theta-1} h \int_0^\infty Q(dy) = \mu^{-\theta}h. \tag{3.8}$$

Now combine (3.5), (3.6) and (3.8) to see that

$$\limsup_{x\to\infty} \left| \frac{V*Q(]x, x + h])}{x^\theta L(x)} - \mu^{-\theta}h \right| \leq cK(\varepsilon^{\theta-1} - 1).$$

Letting $\varepsilon \uparrow 1$ yields

$$\frac{V * Q(]x, x + h])}{x^\theta L(x)} \to \mu^{-\theta}h.$$

This proves the case $\theta < 0$.

(II) Case $\theta = 0$. First suppose L is monotone and (1.3) is satisfied. If L is nonincreasing, the argument of (I) applies. Next suppose L is nondecreasing. As in (I), $I_2 = o(L(x))$. Write, for some x_0 with $0 < x_0 < x$,

$$I_1 = \int_0^{x-x_0} + \int_{x-x_0}^x \equiv I_{11}^\times + I_{12}^\times,$$

say. In $I_{11}^\times$ we have $x - y \geq x_0$, hence for large x_0 and some constant $c > 0$,

$$\frac{V(x - y + h) - V(x - y)}{L(x)} \leq c \frac{L(x - y)}{L(x)} \leq c,$$

since L is nondecreasing. Hence by Lëbesgue's theorem, $I_{11}^\times/L(x) \to h\mu$. In $I_{12}^\times$, we have

$$V(x+h-y) - V(x-y) \le V(x_0+h)$$

so that

$$I_{12}^{\times} \le V(x_0+h)[Q(x) - Q(x+h)] = o(L(x)).$$

Combining these estimates yields

$$\frac{V * Q(]x, x+h])}{L(x)} \to h\mu.$$

Next suppose L is not monotone, but $x^{1+\delta}(1-F(x)) \to 0$ for some $\delta > 0$. We write

$$V * Q(]x, x+h]) = \left(\int_0^{x/2} + \int_{x/2}^{x-x_0} + \int_{x-x_0}^{x}\right)\{V(x+h-y) - V(x-y)\}\, dQ(y) + \int_x^{x+h} V(x+h-y)\, dQ(y) \equiv J_1 + J_2 + J_3 + J_4,$$

say. As before,

$$J_4 \le V(h)(x+h)(F(x+h) - F(x)) \le V(h)(x+h)(1-F(x)) = o(L(x)),$$

since $x^{1+\delta}(1-F(x)) \to 0$. In J_1, we have $x - y \ge x/2$ so that

$$\frac{V(x+h-y) - V(x-y)}{L(x)} \le \text{const. } \{L(x-y)/L(x)\}$$

$$\le \text{const. } \sup_{1/2 \le t \le 1}\{L(tx)/L(x)\} \to \text{const.}$$

as $x \to \infty$. Hence we can use Lebesgue's theorem

$$\{J_1/L(x)\} \to h\int_0^{\infty} dQ(y) = h\mu.$$

For J_2, if x_0 is large, it follows from (3.3) that

$$\{J_2/L(x)\} \le \text{const.} \int_{x/2}^{x-x_0} \{L(x-y)/L(x)\}\, dQ(y).$$

Note that $x^{-\delta/2} \le L(x) \le x^{\delta/2}$ for large x. Then we have

$$\{J_2/L(x)\} \le \text{const. } x^{\delta/2}\int_{x/2}^{x-x_0} (x-y)^{\delta/2} dQ(y)$$

$$\le \text{const. } x^{\delta}2^{-\epsilon}\{Q(x-x_0) - Q(x/2)\}$$

$$\le \text{const. } x^{1+\delta}\{1 - F(x/2)\}$$

which tends to zero as $x \to \infty$ by assumption. As to J_3, we have

$$J_3 = \int_{x-x_0}^{x} \{V(x+h-y) - V(x-y)\}\, dQ(y)$$
$$\leq V(x_0+h)\{Q(x) - Q(x-x_0)\} \leq V(x_0+h)\{1 - F(x-x_0)\}$$

from which we get $J_3 = o(L(x))$ as $x \to \infty$. This proves the case $\theta = 0$. Hence we have proved the statement for noninteger α under assumption (1.3) and for integer α under the extra assumption that $x^{1+\delta}\{1 - F(x)\} \to 0$ as well as (1.3). However, if α is an integer less than or equal to 2, the condition $x^{1+\delta}\{1 - F(x)\} \to 0$ is implied by (1.3). So, we don't have to impose this extra assumption in the case $\alpha \leq 2$. The conclusions in Theorem 1(b) and (c) are thus proved. □

4. Concluding remarks.

(a) If $\alpha = -1$ and L is a positive constant, the generalized renewal measure $G(x) = \sum_{n=1}^{\infty} (1/n)P\{S_n \leq x\}$ is called the harmonic renewal measure which was well studied by Greenwood et al (1982). Let us compare our Theorem 1(b) with their results. In our paper, we always assume $\mu < \infty$. Then by Theorem 2 in Greenwood et al (1982),

$$\log x - G(x) = D + o(1) \quad \text{as} \quad x \to \infty, \tag{4.1}$$

where D is determined by $\mu = \exp\{\gamma + D\}$, γ being Euler's constant. From (4.1), we only get

$$G(x+h) - G(x) = o(1) \quad \text{as} \quad x \to \infty. \tag{4.2}$$

However, since μ is finite, $1 - F(x) = o(1/x)$, which is (1.3) with $K = 0$. Hence our Theorem 1(b) gives us the rate of convergence in (4.2):

$$G(x+h) - G(x) \sim (h/x) \text{ as } x \to \infty.$$

If we assume

$$1 - F(x) \sim x^{-\beta}L(x), \ 1 < \beta < 2, \tag{4.3}$$

then it follows from Theorem 3 in Greenwood et al (1982) that

$$G(x+h) - G(x) \sim (h/x) \quad \text{as} \quad x \to \infty, \tag{4.4}$$

which is the same as our conclusion. However, in our Theorem 1, to get (4.4), we do not have to assume (4.3) and only need $1 - F(x) = o(x^{-\beta})$ for some $\beta > 1$. Furthermore, our Theorem 1 assures that similar results also hold for the case where L is not necessarily constant.

(b) In our theorems, if we replace $a(x) = x^{\alpha}L(x)$ by other $a(x)$, decreasing more rapidly than a power say, then in general, the order of growth of

$$\textstyle\sum_{n=1}^{\infty} a(n)P\{x < S_n \leq x+h\}$$

will be different from $a(x/\mu)$. The following example illustrates this situation.

Let $a(x) = e^{-cx}$, $c > 0$ and $1 - F(x) = e^{-bx}$, $b > 0$. Then

$$P\{x < S_n \le x + h\} = \int_x^{x+h} \{b^n t^{n-1} e^{-bt}/(n-1)!\}\, dt.$$

Hence

$$\sum_{n=1}^{\infty} a(n) P\{x < S_n \le x + h\}$$
$$= e^{-c}(1 - e^{-c})^{-1}(\exp(-\mu^{-1}(1-e^{-c})h)\exp(-(1-e^{-c})x\mu^{-1}),$$

where we have used $EX_1 = \mu = b^{-1}$. If $c > 0$, $c > 1 - e^{-c}$, so that the order of growth of $\sum a(n)P\{x < S_n \le x + h\}$ as $x \to \infty$ is slower than that of $a(x/\mu)$.

(c) Using the notation $G(x) = \sum_{n=0}^{\infty} a(n)P\{S_n \le x\}$ we can restate Theorem 1(a) ((1.2)) as

(4.5) $$\{G(x + h) - G(x)\}/a(x) \to \mu^{-\alpha-1}h \quad \text{as} \quad x \to \infty.$$

Hence, if $H(x) = G(\log x)$, $A(x) = a(\log x)$ and $x = \log x'$, $h = \log h'$ then (4.5) becomes

(4.6) $$\{H(x'h') - H(x')\}/A(x') \to c \log h' \quad \text{as} \quad x' \to \infty$$

where $c = \mu^{-\alpha-1}$ is strictly positive. Hence $H \in \Pi(A)$, therefore (by de Haan, 1970) $A(x')$ is slowly varying, which is equivalent to

$$\forall\, y \in \mathbb{R}: a(x + y) \sim a(y) \quad \text{as} \quad x \to \infty.$$

This means that whenever a result of type (4.5) holds, the sequence $\{a(n)\}_n$ is essentially nonexponential. This explains the main reason why $a(x) \in RV$ is a natural condition in our theorems.

(d) It is also worth noticing that the convergence in (4.5) holds uniformly in $[0, A]$ for all A finite. This fact, even in the ordinary Blackwell theorem, is seldomly stated explicitly. See for instance Chan (1976), however without explicit proof. The uniform convergence follows directly from (4.6) and Seneta (1976, Theorem 2.12 page 79).

Acknowledgement. The authors would like to thank Professor T. Mori for some very valuable discussions regarding an earlier draft of the paper. This paper was written while the second author was visiting the Katholieke Universiteit Leuven. He would like to thank the Department of Mathematics for its financial support and kind hospitality.

REFERENCES

[1] CHAN, Y. K. (1976). A constructive renewal theorem. *Ann. Probab.* **4** 644–655.

[2] EMBRECHTS, P. and OMEY, E. (1983). On subordinated distributions and random record processes. *Math. Proc. Camb. Philos. Soc.* **93** 339–353.

[3] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications, Vol. 2*, 2nd ed. Wiley, New York.

[4] GREENWOOD, P., OMEY, E. and TEUGELS, J. L. (1982). Harmonic renewal measures. *Z. Wahrsch. verw. Gebiete* **59** 391–409.
[5] HEYDE, C. C. (1966). Some renewal theorems with applications to a first passage problem. *Ann. Math. Statist.* **37** 699–710.
[6] KALMA, J. M. (1972). Generalized renewal measures. Thesis, Groningen University.
[7] KAWATA, T. (1961). A theorem of renewal type. *Kodai Math. Sem. Rep.* **13** 185–194.
[8] SENETA, E. (1976). Regularly varying functions. *Lecture Notes in Math.* **508**. Springer, Berlin.
[9] SMITH, W. L. (1964). On the elementary renewal theorem for non-identically distributed random variables. *Pacific J. Math.* **14** 673–699.

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