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ON THE QUADRATIC VARIATION OF TWO-PARAMETER CONTINUOUS MARTINGALES

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Let $M = \{M(z),\ z \in [0, 1]^2\}$ be a two-parameter square integrable continuous martingale. We prove the sample continuity of the quadratic variation of M using an Itô's differentiation formula for M^2.

1. Introduction. The aim of this paper is to show some results concerning the quadratic variation of a two-parameter continuous martingale, which are well-known in the one-parameter case.

Suppose that $M = \{M_z, z \in [0, 1]^2\}$ is a square integrable continuous martingale with respect to an increasing family of σ-fields satisfying the usual conditions of R. Cairoli and J. B. Walsh [4]. The Doob-Meyer decomposition theorem (cf. [4] and [9]) assures the existence and uniqueness of a predictable, increasing process $\langle M \rangle$ vanishing on the axes, and such that $M^2 - \langle M \rangle$ is a weak martingale. The main result of this note is the sample continuity of the quadratic variation $\langle M \rangle$, which so far had only been proved for some special kinds of martingales, like path independent variation martingales or martingales with orthogonal increments in one direction (see [13]). If the martingale M is bounded in L^p for $p \geq 2$, then the process $\langle M \rangle$ is obtained as the $L^{p/2}$ limit of sums of the form $\sum_{i,j} M(\Delta_{ij})^2$. The method to show these results is based on the deduction of a two-parameter Itô's formula for M^2.

The construction of the quadratic variation of M and a more general Itô's formula have been obtained by L. Chevalier in [5], under the additional assumption that any square integrable martingale has a continuous version. Under this hypothesis any square integrable martingale can be approximated by continuous bounded martingales, as in the one-parameter case. As far as we know, this approximation is not allowed in the general case, because of the lack of stopping times, and we have replaced it by a more accurate application of martingale inequalities.

2. Notation and basic assumptions. The set of parameters will be $T = [0, 1]^2$, with the partial ordering $(s_1, t_1) \leq (s_2, t_2)$ if and only if $s_1 \leq s_2$ and $t_1 \leq t_2$. Then, $(s_1, t_1) < (s_2, t_2)$ means $s_1 < s_2$ and $t_1 < t_2$. Let $z_1 < z_2$, then $(z_1, z_2]$ denotes the rectangle $\{z \in T: z_1 < z \leq z_2\}$. Suppose that f is a mapping from T to R. The increment of f on a rectangle $(z_1, z_2]$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ will be $f((z_1, z_2]) = f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_z, z \in T\}$ be an increasing family of sub-σ-fields of $\mathcal{F}$. The σ-fields $\mathcal{F}_z$ are assumed to satisfy the

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usual conditions of [4]: (a) $\mathscr{F}_{00}$ includes the null sets of $\mathscr{F}$, (b) $\mathscr{F}_z$ is right-continuous, and (c) $\mathscr{F}_{s1}$ and $\mathscr{F}_{1t}$ are conditionally independent given $\mathscr{F}_{st}$.

Suppose that $M = \{M_z, z \in T\}$ is an integrable, $\mathscr{F}_z$-adapted process. Then: (a) M is a martingale if $E(M_{z'}/\mathscr{F}_z) = M_z$ for any $z \le z'$, and (b) M is a weak martingale if $E(M(z, z']/\mathscr{F}_z) = 0$ for $z \le z'$. For $p \ge 1$, let m_c^p be the class of all sample continuous martingales M such that $M_z = 0$ on the axes and $E(|M_z|^p) < \infty$ for all z. Given a martingale M of m_c^2, we will denote by $M_{\cdot t}$ and $M_{s\cdot}$ the one-parameter martingales $\{M_{st}, \mathscr{F}_{s1}, s \ge 0\}$ and $\{M_{st}\ \mathscr{F}_{1t}, t \ge 0\}$ respectively. A process $A = \{A_z, z \in T\}$ will be called increasing if it is adapted, $A_z = 0$ on the axes, and $A(\Delta) \ge 0$ for any rectangle $\Delta = (z_1, z_2]$.

A subset $\mathscr{S}$ of T will be called a grid if $\mathscr{S} = \mathscr{P}^1 \times \mathscr{P}^2$, where $\mathscr{P}^1$ and $\mathscr{P}^2$ are finite subsets of $[0, 1)$ containing the point zero. Suppose that $0 = s_1 < s_2 < \cdots < s_p$ are the points of $\mathscr{P}^1$, in increasing order, and $0 = t_1 < t_2 < \cdots < t_q$ are those of $\mathscr{P}^2$. For any $u = (s_i, t_j)$ in $\mathscr{S}$ we will write $\bar{u} = (s_{i+1}, t_{j+1})$, $\Delta_u = (u, \bar{u}]$, $\Delta_u^1 = (s_i, s_{i+1}] \times (0, t_j]$ and $\Delta_u^2 = (0, s_i] \times (t_j, t_{j+1}]$, with the convention $s_{p+1} = 1$ and $t_{q+1} = 1$. The class of all grids on T is ordered by inclusion. Given a grid $\mathscr{S}$ and an arbitrary point z of T, we denote by $\mathscr{S}_z$ the smallest grid containing $\mathscr{S}$ and z. We will write $\bar{\mathscr{S}}_z = \{z' \in \mathscr{S}_z : z' < z\}$. The norm of $\mathscr{S}$ is defined as $|\mathscr{S}| = \max\{|u - \bar{u}|, u \in \mathscr{S}\}$.

Throughout the paper, C_p will represent a constant, depending on p, which may be different from one formula to another one. In the same way, C will denote an arbitrary constant.

The next result about one-parameter continuous martingales will be needed in the following.

LEMMA 2.1. *Let $M = \{M_t, t \in [0, 1]\}$ be a square integrable continuous martingale with respect to an increasing family of σ-fields $\{\mathscr{F}_t, t \in [0, 1]\}$ satisfying the usual conditions. Suppose $M_0 = 0$ and denote by $\mathscr{P} = \{s_i\}$, $0 = s_1 < s_2 < \cdots < s_n < 1$ a finite set of points. Consider another finite set $\mathscr{P}' \supset \mathscr{P}$, whose points can always be written as σ_k^i, $i = 1, \cdots, n$; $k = 1, \cdots, r_i$, in such a way that $s_i = \sigma_1^i < \sigma_2^i < \cdots < \sigma_{r_i}^i < s_{i+1}$ for all i. Set $|\mathscr{P}| = \max_i |s_{i+1} - s_i|$, where $s_{n+1} = 1$. Then*

$$\lim_{|\mathscr{P}|\downarrow 0} \sup_{\mathscr{P}' \supset \mathscr{P}} E\Big(\sup_i \sum_{k=1}^{r_i} (M(\sigma_{k+1}^i) - M(\sigma_k^i))^2\Big) = 0. \tag{2.1}$$

By convention, we put $\sigma_{r_i+1}^i = s_{i+1}$.

PROOF. Notice that the random variables $\{\sum_i (M(s_{i+1}) - M(s_i))^2, \mathscr{P}$ finite subset of $[0, 1)\}$ are uniformly integrable. Indeed, this property can be shown by using Burkholder-Davis-Gundy's inequalities (cf. [2]) and the lemma due to de la Vallée-Poussin which gives necessary and sufficient conditions for the uniform integrability of a family of random variables. For every $\varepsilon > 0$, we choose a positive integer $h > 0$ such that $P(D_h^c) < \varepsilon$, where $D_h = \{\omega: \sup_s |M_s(\omega)| \le h\}$. Define $T_h = \inf\{s \ge 0: |M_s| > h\}$. Then, applying Burkholder's and maximal inequalities, we obtain, for all $\lambda > 0$

$$
\begin{aligned}
&P\{\sup_i \textstyle\sum_{k=1}^{r_i} (M(\sigma_{k+1}^i) - M(\sigma_k^i))^2 > \lambda\} \\
&\quad \le P(D_h^c) + P\{\sup_i \textstyle\sum_{k=1}^{r_i} (M(\sigma_{k+1}^i \wedge T_h) - M(\sigma_k^i \wedge T_h))^2 > \lambda\} \\
&\quad \le \varepsilon + \lambda^{-2} \textstyle\sum_i E((\sum_{k=1}^{r_i} (M(\sigma_{k+1}^i \wedge T_h) - M(\sigma_k^i \wedge T_h))^2)^2) \\
&\quad \le \varepsilon + \lambda^{-2} C \textstyle\sum_i E((M(s_{i+1} \wedge T_h) - M(s_i \wedge T_h))^4) \\
&\quad \le \varepsilon + \lambda^{-2} C \{E[(\textstyle\sum_i (M(s_{i+1} \wedge T_h) - M(s_i \wedge T_h))^2)^2] \\
&\qquad\qquad \cdot E[\sup_i (M(s_{i+1} \wedge T_h) - M(s_i \wedge T_h))^4]\}^{1/2} \\
&\le \varepsilon + \lambda^{-2} C\, h^3 \{E(\sup_{|s-s'| \le \mathscr{P}|} |M_s - M_{s'}|^2)\}^{1/2},
\end{aligned}
$$

and the proof follows easily. □

In the proof of our results we will often use the next inequality for a family of one-parameter martingales. The method to show this inequality is the same as that used in the proof of Theorem 1 of [8]:

LEMMA 2.2. *Let $\{M_j^i, j = 1, \cdots, m\}$, $i = 1, \cdots, n$, be a family of one-parameter martingales. Set $S_m^i = (\sum_{j=1}^m (M_j^i - M_{j-1}^i)^2)^{1/2}$, assuming $M_0^i = 0$. Then, there exists a universal constant C such that*

$$E[(\textstyle\sum_i (M_m^i)^2)^{1/2}] \le CE[(\sum_i (S_m^i)^2)^{1/2}].$$

PROOF. Denote by $\{r_i\}$ a family of Rademacher functions on [0, 1]. Using Khintchine and Davis inequalities, we have

$$
\begin{aligned}
E[(\textstyle\sum_i (M_m^i)^2)^{1/2}] &\le CE\left(\int_0^1 |\textstyle\sum_i M_m^i r_i(t)|\, dt\right) \\
&\le CE\left[\int_0^1 (\textstyle\sum_j (\sum_i (M_j^i - M_{j-1}^i) r_i(t))^2)^{1/2} dt\right] \\
&\le CE\left[\left(\textstyle\sum_j \int_0^1 (\sum_i (M_j^i - M_{j-1}^i) r_i(t))^2 dt\right)^{1/2}\right] \\
&\le CE[(\textstyle\sum_j \sum_i (M_j^i - M_{j-1}^i)^2)^{1/2}] \\
&= CE[(\textstyle\sum_i (S_m^i)^2)^{1/2}]. \quad \square
\end{aligned}
$$

3. Main results and proofs. Suppose that $M = \{M_z, z \in T\}$ is a martingale of m_c^2. We fix an increasing sequence of grids $\mathscr{S}^n$ whose norms tend to zero. Let $\mathscr{S}^n = \mathscr{P}_1^n \times \mathscr{P}_2^n$, $\mathscr{P}_1^n = \{0 = s_1^n < \cdots < s_{p_n}^n\}$ and $\mathscr{P}_2^n = \{0 = t_1^n < \cdots < t_{q_n}^n\}$. In order to simplify the notation, we omit the index n of the points of $\mathscr{P}_1^n$ and $\mathscr{P}_2^n$.

Then, for all $z = (s, t)$ in T the following equality holds

$$
\begin{aligned}
M_z^2 = {} & 2 \textstyle\sum_{u \in \mathcal{S}_z^n} M_u M(\Delta_u) + 2 \textstyle\sum_{u \in \mathcal{S}_z^n} M(\Delta_u^1) M(\Delta_u^2) \\
& + \textstyle\sum_{i=1}^{p_n} (M(s_{i+1} \wedge s, t) - M(s_i \wedge s, t))^2 \qquad (3.1) \\
& + \textstyle\sum_{j=1}^{q_n} (M(s, t_{j+1} \wedge t) - M(s, t_j \wedge t))^2 - \textstyle\sum_{u \in \mathcal{S}_z^n} M(\Delta_u)^2.
\end{aligned}
$$

Note that in the above expression the rectangles Δ_u, Δ_u^1 and Δ_u^2 are defined with respect to the grid $\mathcal{S}_z^n$. The proof of (3.1) is straightforward. It can also be viewed as a particular case of Lemma 1 of [5]. Next we are going to look over the behavior of each term of the right-hand side of (3.1) when n tends to infinity.

LEMMA 3.1. *Suppose that M belongs to m_c^p with $p \geq 2$. Then, there exists a martingale N in $m_c^{p/2}$ such that*

$$\lim_n \sup_{z \in T} E(|\textstyle\sum_{u \in \mathcal{S}_z^n} M_u M(\Delta_u) - N_z|^{p/2}) = 0. \qquad (3.2)$$

PROOF. For any natural n we define the martingale $N_z^n = \sum_{u \in \mathcal{S}_z^n} M_u M(\Delta_u)$. Fix $m > n$ and consider the difference

$$
\begin{aligned}
N_{11}^m - N_{11}^n &= \textstyle\sum_{u \in \mathcal{S}^m} M_u M(\Delta_u) - \textstyle\sum_{u \in \mathcal{S}^n} M_u M(\Delta_u) \\
&= \textstyle\sum_{u \in \mathcal{S}^m} (M_u - M_{u'}) M(\Delta_u),
\end{aligned}
$$

where $u' = \sup\{v \in \mathcal{S}^n : v \leq u\}$. The terms $(M_u - M_{u'})M(\Delta_u)$ are martingale differences with respect to the σ-fields $\{\mathcal{F}_{\bar{u}},\ u \in \mathcal{S}^m\}$. Therefore, using Burkholder-Davis-Gundy's inequalities extended to the case of two-parameters (cf. [8], [10]) we obtain

$$
\begin{aligned}
E(|N_{11}^m - N_{11}^n|^{p/2}) &\leq C_p E(|\textstyle\sum_{u \in \mathcal{S}^m} (M_u - M_{u'})^2 M(\Delta_u)^2|^{p/4}) \\
&\leq C_p E(\sup_{u \in \mathcal{S}^m} |M_u - M_{u'}|^{p/2} (\textstyle\sum_{u \in \mathcal{S}^m} M(\Delta_u)^2)^{p/4}) \\
&\leq C_p \{E(\sup_{u \in \mathcal{S}^m} |M_u - M_{u'}|^p) E((\textstyle\sum_{u \in \mathcal{S}^m} M(\Delta_u)^2)^{p/2})\}^{1/2} \\
&\leq C_p \{E(\sup_{|u - v| \leq |\mathcal{S}^n|} |M_u - M_v|^p) E(|M_{11}|^p)\}^{1/2}.
\end{aligned}
$$

In consequence, we have

$$\lim_n \sup_{m > n} \sup_z E(|N_z^m - N_z^n|^{p/2}) = 0,$$

and this implies the existence of a martingale N bounded in $L^{p/2}$ such that (3.2) holds. It remains to prove that N has a continuous version. For $p > 2$ this is an immediate consequence of Doob-Cairoli's maximal inequalities for two-parameter martingales. In fact, we have in this case

$$\lim_n E(\sup_z |\textstyle\sum_{u \in \mathcal{S}_z^n} M_u M(\Delta_u) - N_z|^{p/2}) = 0.$$

If $p = 2$, the continuity of the martingale N could be deduced from the properties of the stochastic integral $\int M\, dM$ (cf. [3]). However, we prefer to present here a direct proof of the existence of a continuous version of N. Fix a positive integer $h > 0$ and define $M_h(z) = (M(z) \wedge h) \vee (-h)$. Then, for any n and h, the process given by $N_h^n(z) = \sum_{u \in \mathcal{S}_z^n} M_h(u) M(\Delta_u)$ is a square integrable continuous martingale. Let $m > n$. Applying Doob-Cairoli's maximal inequality and Burkholder's inequality for two-parameter discrete martingales, we obtain

$$
\begin{aligned}
&E(\sup_z |N_h^m(z) - N_h^n(z)|^{3/2}) \\
&\qquad \le CE(|N_h^m(1,1) - N_h^n(1,1)|^{3/2}) \\
&\qquad = CE(|\textstyle\sum_{u\in \mathscr{S}^m} (M_h(u) - M_h(u'))M(\Delta_u)|^{3/2}) \\
&\qquad \le E(|\textstyle\sum_{u\in \mathscr{S}^m} (M_h(u) - M_h(u'))^2 M(\Delta_u)^2|^{3/4}) \\
&\qquad \le CE(\sup_{u\in \mathscr{S}^m} |M_h(u) - M_h(u')|^{3/2} (\textstyle\sum_{u\in \mathscr{S}^m} M(\Delta_u)^2)^{3/4}) \\
&\qquad \le C\{E(\sup_{|u-v|\le |\mathscr{S}^n|} |M_h(u) - M_h(v)|^6)\}^{1/4} (E(M_{11}^2))^{3/4}.
\end{aligned}
\tag{3.3}
$$

Set $D_h = \{\omega: \sup_z |M_z(\omega)| \le h\}$. Given an $\varepsilon > 0$ we take h in such a way that $P(D_h^c) < \varepsilon$. Then, for any positive number λ we will have

$$
\begin{aligned}
&P\{\sup_z |N^m(z) - N^n(z)| > \lambda\} \\
&\qquad \le P(D_h^c) + P(\{\sup_z |N^m(z) - N^n(z)| > \lambda\} \cap D_h) \\
&\qquad \le \varepsilon + P\{\sup_z |N_h^m(z) - N_h^n(z)| > \lambda\} \\
&\qquad \le \varepsilon + \lambda^{-3/2} E(\sup_z |N_h^m(z) - N_h^n(z)|^{3/2}).
\end{aligned}
\tag{3.4}
$$

Therefore, (3.3) and (3.4) imply $\lim_n \sup_{m>n} P\{\sup_z |N^m(z) - N^n(z)| > \lambda\} = 0$, which completes the proof. □

LEMMA 3.2. *Suppose that M is a martingale belonging to m_c^p with $p \ge 2$. Then, there exists a martingale S in $m_c^{p/2}$ such that*

$$
\lim_n \sup_z E(|\textstyle\sum_{u\in \mathscr{S}_z^n} M(\Delta_u^1)M(\Delta_u^2) - S_z|^{p/2}) = 0. \tag{3.5}
$$

PROOF. Define $S_z^n = \sum_{u\in \mathscr{S}_z^n} M(\Delta_u^1)M(\Delta_u^2)$. We are going to consider two different cases.

(a) If $p > 2$, the assertion of the lemma will follow from the convergence

$$
\lim_n \sup_{m>n} E(|S_{11}^m - S_{11}^n|^{p/2}) = 0. \tag{3.6}
$$

Assume that $\mathscr{S}$ is a grid on T which contains $\mathscr{S}^n$ and has the same projection on the "t" axis. If $\{u = (s_i, t_j), 1 \le i \le p_n, 1 \le j \le q_n\}$ are the points of $\mathscr{S}^n$, those of $\mathscr{S}$ will be of the form $u' = (\sigma_{i'}, t_j)$, $1 \le i' \le p$, $1 \le j \le q_n$. For any $i = 1, \cdots, p_n$, we denote by I_i the set $\{i': \sigma_{i'} \in [s_i, s_{i+1})\}$. Put $\tilde{S}_z = \sum_{u\in \mathscr{S}_z} M(\Delta_u^1)M(\Delta_u^2)$. In order to show (3.6) it suffices to prove that $\lim_n \sup_{\mathscr{S}} E(|\tilde{S}_{11} - S_{11}^n|^{p/2}) = 0$, and a similar result for grids with the same projection than $\mathscr{S}^n$ on the "s" axis. Using Burkholder-Davis-Gundy's inequality for two-parameter discrete martingales, we obtain

$$
\begin{aligned}
E(|\tilde{S}_{11} - S_{11}^n|^{p/2}) &= E(|\textstyle\sum_{u'\in \mathscr{S}} M(\Delta_{u'}^1)M(\Delta_{u'}^2) - \sum_{u\in \mathscr{S}^n} M(\Delta_u^1)M(\Delta_u^2)|^{p/2}) \\
&= E(|\textstyle\sum_{u=(s_i,t_j)\in \mathscr{S}^n} \sum_{i'\in I_i} M(\Delta_{u'}^1)M(\Delta_{u'}^2 - \Delta_u^2)|^{p/2}) \\
&\le C_p E(|\textstyle\sum_{u\in \mathscr{S}^n} \sum_{i'\in I_i} M(\Delta_{u'}^1)^2 M(\Delta_{u'}^2 - \Delta_u^2)^2|^{p/4}),
\end{aligned}
$$

where $u = (s_i, t_j)$, $u' = (\sigma_{i'}, t_j)$, $\Delta^1_{u'} = (\sigma_{i'}, \sigma_{i'+1}] \times (0, t_j]$ and $\Delta^2_{u'} - \Delta^2_u = (s_i, \sigma_{i'}] \times (t_j, t_{j+1}]$. Therefore,

$$E(|\tilde{S}_{11} - S^n_{11}|^{p/2}) \le C_p\{E(|\textstyle\sum_i \sum_{i' \in I_i} \sup_j M(\Delta^1_{u'})^2|^{p/2}) \cdot E(|\sup_i \sup_{i' \in I_i} \sum_j M(\Delta^2_{u'} - \Delta^2_u)^2|^{p/2})\}^{1/2}. \tag{3.7}$$

i) We will show that the first factor of the right-hand side of (3.7) is bounded by some constant. To do this consider the continuous increasing and $\mathscr{F}_{1t}$-adapted process defined by

$$A_t = \textstyle\sum_i \sum_{i' \in I_i} \sup_{\tau \le t}(M(\sigma_{i'+1}, \tau) - M(\sigma_{i'}, \tau))^2.$$

Then $E(|\sum_i \sum_{i' \in I_i} \sup_j M(\Delta^1_{u'})^2|^{p/2}) \le E(A_1^{p/2})$.

Next we compute the potential Z_t associated to A_t,

$$\begin{aligned} Z_t &= E(A_1 - A_t | \mathscr{F}_{1t}) \\ &= E(\textstyle\sum_i \sum_{i' \in I_i} (\sup_\tau(M(\sigma_{i'+1}, \tau) - M(\sigma_{i'}, \tau))^2 \\ &\qquad - \sup_{\tau \le t}(M(\sigma_{i'+1}, \tau) - M(\sigma_{i'}, \tau))^2) | \mathscr{F}_{1t}) \\ &\le E(\textstyle\sum_i \sum_{i' \in I_i} \sup_{\tau \ge t}(M(\sigma_{i'+1}, \tau) - M(\sigma_{i'}, \tau))^2 / \mathscr{F}_{1t}) \\ &\le C \textstyle\sum_i \sum_{i' \in I_i} E((M(\sigma_{i'+1}, 1) - M(\sigma_{i'}, 1))^2 | \mathscr{F}_{1t}) = m_t, \end{aligned}$$

where m_t is an $\mathscr{F}_{1t}$-adapted martingale. Then, from Garsia-Neveu's inequality (cf. [7]) we obtain

$$\begin{aligned} E(A_1^{p/2}) \le C_p E(m_1^{p/2}) &= C_p E(|\textstyle\sum_i \sum_{i' \in I_i} (M(\sigma_{i'+1}, 1) - M(\sigma_{i'}, 1))^2|^{p/2}) \\ &\le C_p E(|M_{11}|^p). \end{aligned}$$

ii) The second factor converges to zero when n tends to infinity, uniformly with respect to $\mathscr{S}$. Indeed, applying Doob's maximal inequality and Burkholder's inequality for discrete martingales, we deduce

$$\begin{aligned} &E(|\sup_i \sup_{i' \in I_i} \textstyle\sum_j M(\Delta^2_{u'} - \Delta^2_u)^2|^{p/2}) \\ &\qquad \le \textstyle\sum_i E(\sup_{i' \in I_i}(\sum_j M(\Delta^2_{u'} - \Delta^2_u)^2)^{p/2}) \\ &\qquad \le C_p \textstyle\sum_i E(|\sum_j M(\Delta_u)^2|^{p/2}) \\ &\qquad \le C_p \textstyle\sum_i E(|M(s_{i+1}, 1) - M(s_i, 1)|^p) \\ &\qquad \le C_p\{E(|\textstyle\sum_i (M(s_{i+1}, 1) - M(s_i, 1))^2|^{p/2})\}^{2/p} \\ &\qquad\quad \cdot \{E(\sup_i |M(s_{i+1}, 1) - M(s_i, 1)|^p)\}^{1-(2/p)} \\ &\qquad \le C_p\{E(|M_{11}|^p)\}^{2/p} \\ &\qquad\quad \cdot \{E(\sup_{|z_1 - z_2| \le 1/n} |M(z_1) - M(z_2)|^p)\}^{1-(2/p)}. \end{aligned}$$

(b) Suppose $p = 2$. With the same assumptions as above on the grids $\mathscr{S}$ and

$\mathscr{S}^n$, we will first show that

$$\lim_n \sup_{\mathscr{S}} E(\sup_s |\tilde{S}_{s1} - S^n_{s1}|) = 0. \tag{3.8}$$

By Davis inequality in the case of continuous martingales, we have

$$\begin{aligned} &E(\sup_s |\tilde{S}_{s1} - S^n_{s1}|) \\ &\quad = E(\sup_s |\textstyle\sum_{u=(s_i,t_j)\in \mathscr{S}^n} \sum_{i'\in I_i} M(\Delta^2_{u'} - \Delta^2_u)(M(\sigma_{i'+1} \wedge s, t_j) - M(\sigma_{i'} \wedge s, t_j))|) \\ &\quad \le C\, E(|\textstyle\sum_i \sum_{i'\in I_i} \langle \sum_j M(\Delta^2_{u'} - \Delta^2_u)(M(\sigma_{i'+1} \wedge \cdot, t_j) - M(\sigma_{i'} \wedge \cdot, t_j))\rangle_1|^{1/2}). \end{aligned}$$

For any i' we consider a partition of the interval $[\sigma_{i'}, \sigma_{i'+1}]$ determined by the finite set $\mathscr{P}_{i'} = \{\sigma^{i'}_k\}$, $\sigma_{i'} = \sigma^{i'}_1 < \sigma^{i'}_2 < \cdots < \sigma^{i'}_{r_{i'}} = \sigma_{i'+1}$. Set $|\mathscr{P}_{i'}| = \max_k(\sigma^{i'}_{k+1} - \sigma^{i'}_k)$.

Then, using Fatou's lemma, we obtain

$$\begin{aligned} &E(\sup_s |\tilde{S}_{s1} - S^n_{s1}|) \\ &\quad \le CE(|\textstyle\sum_i \sum_{i'\in I_i} \lim_{|\mathscr{P}_{i'}|\downarrow 0} \sum_k (\sum_j M(\Delta^2_{u'} - \Delta^2_u)(M(\sigma^{i'}_{k+1}, t_j) - M(\sigma^{i'}_k, t_j)))^2|^{1/2}) \\ &\quad \le C \sup_{\mathscr{P}} E(|\textstyle\sum_i \sum_{i'\in I_i} \sum_k (\sum_j M(\Delta^2_{u'} - \Delta^2_u) M(\Delta^j_{i'k}))^2|^{1/2}), \end{aligned}$$

where $\Delta^j_{i'k} = (\sigma^{i'}_k, \sigma^{i'}_{k+1}] \times (0, t_j]$ and the supremum is taken over all finite sets $\mathscr{P} = \{\sigma^{i'}_k\}$ which contain the points $\sigma_{i'}$.

Applying Lemma 2.2 to the martingale differences (with respect to the index j) $M(\Delta^2_{u'} - \Delta^2_u)M(\Delta^j_{i'k})$, we have

$$\begin{aligned} &E(\sup_s |\tilde{S}_{s1} - S^n_{s1}|) \\ &\quad \le C \sup_{\mathscr{P}} E(|\textstyle\sum_i \sum_{i'\in I_i} \sum_k \sum_j M(\Delta^2_{u'} - \Delta^2_u)^2 M(\Delta^j_{i'k})^2|^{1/2}) \\ &\quad \le C\{E(\textstyle\sum_{i,j} \sup_{i'\in I_i} M(\Delta^2_{u'} - \Delta^2_u)^2)\cdot \sup_{\mathscr{P}} E(\sup_{i,j} \sum_{i'\in I_i} \sum_k M(\Delta^j_{i'k})^2)\}^{1/2}. \end{aligned} \tag{3.9}$$

The first factor of the above expression is bounded by $E(M^2_{11})$ because of Doob's maximal inequality. The process $(\sup_i \sum_{i',k} M(\Delta^j_{i'k})^2)^{1/2}$, appearing in the second factor, is a submartingale with respect to the index j. In fact, it can be regarded as a convex function of the martingales $M(\Delta^j_{i'k})$. Then, we apply Doob's maximal inequality, obtaining

$$\begin{aligned} &E(\sup_{i,j} \textstyle\sum_{i'\in I_i} \sum_k M(\Delta^j_{i'k})^2) \\ &\qquad \le CE(\sup_i \textstyle\sum_{i'\in I_i} \sum_k (M(\sigma^{i'}_{k+1}, 1) - M(\sigma^{i'}_k, 1))^2). \end{aligned} \tag{3.10}$$

Therefore, from (3.9) and (3.10) it follows that

$$\begin{aligned} &\sup_{\mathscr{S}} E(\sup_s |\tilde{S}_{s1} - S^n_{s1}|) \\ &\qquad \le C\{E(M^2_{11}) \cdot \sup_{\mathscr{P}} E(\sup_i \textstyle\sum_{i'\in I_i} (M(\sigma_{i'+1}, 1) - M(\sigma_{i'}, 1))^2)\}^{1/2}. \end{aligned} \tag{3.11}$$

Now, from (3.11) and Lemma 2.1 applied to the martingale $M_{\cdot 1}$, we see that (3.8) holds. Notice that for the convergence to zero in (3.8) we only need that $\lim_n |\mathscr{P}^n_1| = 0$.

We could obtain a similar result for grids $\mathscr{S}$ with the same projection than

$\mathscr{S}^n$ on the "s" axis. That means

$$\lim_n \sup_{\nearrow} E(\sup_t |\tilde{S}_{1t} - S^n_{1t}|) = 0. \tag{3.12}$$

Then, from (3.8) and (3.12) we deduce the existence of a martingale S such that (3.5) holds. It remains to show that S has a continuous version. For any $m > n$ denote by $\mathscr{S}^{mn}$ the grid on T with the same projection on the "t" axis than $\mathscr{S}^n$ and with the same projection on the "s" axis than $\mathscr{S}^m$. Set $S^{mn}_z = \sum_{u \in \mathscr{T}^{mn}_z} M(\Delta^1_u)M(\Delta^2_u)$. Then by maximal inequality, for all $\lambda > 0$ we have

$$\begin{aligned} P\{\sup_{s,t} |S^m_{st} - S^n_{st}| > \lambda\} &\le P\left\{\sup_{s,t} |S^m_{st} - S^{mn}_{st}| > \frac{\lambda}{2}\right\} + P\left\{\sup_{s,t} |S^{mn}_{st} - S^n_{st}| > \frac{\lambda}{2}\right\} \\ &\le \frac{2}{\lambda} E(\sup_t |S^m_{1t} - S^{mn}_{1t}|) + \frac{2}{\lambda} E(\sup_s |S^{mn}_{s1} - S^n_{s1}|), \end{aligned}$$

which converges to zero when $n \to \infty$, uniformly with respect to n, because of (3.8) and (3.12). □

The next result states the continuity in both arguments of the quadratic variation in one direction of a two-parameter continuous square integrable martingale.

THEOREM 3.3. *Let M be a martingale of m^2_c. Then the processes $\langle M_{s\cdot}\rangle_t$ and $\langle M_{\cdot t}\rangle_s$ have continuous versions in both coordinates.*

PROOF. We will show the existence of a continuous modification of $\langle M_{s\cdot}\rangle_t$. Consider an increasing sequence of finite sets $\mathscr{P}^n_2 = \{0 = t_1 < t_2 < \cdots < t_{q_n} < 1\}$ such that $|\mathscr{P}^n_2| = \max_j |t_{j+1} - t_j|$ tends to zero when $n \to \infty$. Define

$$P^n_{st} = \textstyle\sum_j (M(s, t_{j+1} \wedge t) - M(s, t_j \wedge t))^2.$$

We know that $\lim_n E(|P^n_{st} - \langle M_{s\cdot}\rangle_t|) = 0$. For any $m > n$ the difference $P^m_{s\cdot} - P^n_{s\cdot}$ is a martingale and by the maximal inequality we will have $P\{\sup_{s,t} |P^m_{st} - P^n_{st}| > \lambda\} \le (1/\lambda)E(\sup_s |P^m_{s1} - P^n_{s1}|)$ for all $\lambda > 0$. Therefore, the theorem will follow from the convergence

$$\lim_n \sup_{m>n} E(\sup_s |P^m_{s1} - P^n_{s1}|) = 0. \tag{3.13}$$

In order to prove (3.13) we make the decomposition $P^n_{st} = 2R^n_s + T^n_s$, where

$$R^n_s = \textstyle\sum_j \displaystyle\int_0^s (M(\sigma, t_{j+1}) - M(\sigma, t_j))\partial(M(\sigma, t_{j+1}) - M(\sigma, t_j))$$

(here the symbol ∂ denotes a one-parameter stochastic integral with respect to the first index) and $T^n_s = \sum_j \langle M_{\cdot t_{j+1}} - M_{\cdot t_j}\rangle_s$. Then the proof of (3.13) will be done in several steps.

i) First we will show that

$$\lim_n \sup_{m>n} E(\sup_s |R^m_s - R^n_s|) = 0. \tag{3.14}$$

Set

$$
\begin{aligned}
&E(\sup_s |R_s^m - R_s^n|) \\
(3.15)\quad &\leq E\left(\sup_s \left| \sum_{j=1}^{q_m} \int_0^s (M(\sigma, t_{j+1}) - M(\sigma, t_j))\partial(M(\sigma, t_j) - M(\sigma, t_{\nu(j)})) \right|\right) \\
&+ E\left(\sup_s \left| \sum_{j=1}^{q_m} \int_0^s (M(\sigma, t_j) - M(\sigma, t_{\nu(j)}))\partial(M(\sigma, t_{j+1}) - M(\sigma, t_j)) \right|\right),
\end{aligned}
$$

where $t_{\nu(j)} = \sup\{t \in \mathscr{P}^n : t \leq t_j\}$, for any $j = 1, \cdots, q_m$. The method we will use to show (3.14) is similar to the demonstration of (3.8). That means, we apply Davis inequality to the first term of the right-hand side of (3.15) in order to obtain

$$
\begin{aligned}
&E\left(\sup_s \left| \sum_{j=1}^{q_m} \int_0^s (M(\sigma, t_{j+1}) - M(\sigma, t_j))\partial(M(\sigma, t_j) - M(\sigma, t_{\nu(j)})) \right|\right) \\
&\quad \leq CE\left\{ \left| \sum_{j,j'=1}^{q_m} \int_0^1 (M(s, t_{j+1}) - M(s, t_j))(M(s, t_{j'+1}) \right.\right. \\
&\qquad \left.\left. - M(s, t_{j'}))\, d\langle M_{\cdot t_j} - M_{\cdot t_{\nu(j)}}, M_{\cdot t_{j'}} - M_{\cdot t_{\nu(j')}}\rangle_s \right|^{1/2} \right\} \\
&\quad = CE\{\lim_{\mathscr{\ell}\to\infty} | \sum_{i=1}^{p_\ell} \sum_{j,j'=1}^{q_m} (M(s_i, t_{j+1}) - M(s_i, t_j)) \\
&\qquad \cdot (M(s_i, t_{j'+1}) - M(s_i, t_{j'})) \\
&\qquad \cdot (\langle M_{\cdot t_j} - M_{\cdot t_{\nu(j)}}, M_{\cdot t_{j'}} - M_{\cdot t_{\nu(j')}}\rangle_{s_{i+1}} \\
&\qquad - \langle M_{\cdot t_j} - M_{\cdot t_{\nu(j)}}, M_{\cdot t_{j'}} - M_{\cdot t_{\nu(j')}}\rangle_{s_i}) |^{1/2}\}.
\end{aligned}
$$

The limit is taken with respect to an increasing sequence of finite sets $\mathscr{P}_1^n = \{0 = s_1 < s_2 < \cdots < s_{p_n} < 1\}$ such that $\lim_n |\mathscr{P}_1^n| = 0$. By Fatou's lemma and using the same arguments that in the proof of (3.8), the above expression is bounded by

$$
(3.16)\quad C \sup_{\ell} \sup_{\mathscr{P}} E(| \sum_{i=1}^{p_\ell} \sum_k (\sum_{j=1}^{q_m} (M(s_i, t_{j+1}) - M(s_i, t_j))M(\bar{\Delta}_{ijk}))^2 |^{1/2}),
$$

where $\bar{\Delta}_{ijk} = (\sigma_k^i, \sigma_{k+1}^i] \times (t_{\nu(j)}, t_j]$ and the points σ_k^i, $k = 1, \cdots, r_i$, form a partition of the interval $[s_i, s_{i+1}]$. In (3.16) the supremum is taken over all finite sets $\mathscr{P} = \{\sigma_k^i\}$ which contain $\mathscr{P}_1^\ell$. Put $\bar{\Delta}_{ij} = (s_i, s_{i+1}] \times (t_{\nu(j)}, t_j]$. Then, applying Lemma 2.2 we obtain that (3.16) is less or equal than

$$
\begin{aligned}
&C \sup_{\ell} \sup_{\mathscr{P}} E(| \sum_{i=1}^{p_\ell} \sum_k \sum_{j=1}^{q_m} (M(s_i, t_{j+1}) - M(s_i, t_j))^2 M(\bar{\Delta}_{ijk})^2 |^{1/2}) \\
(3.17)\quad &\leq C \sup_{\ell} \sup_{\mathscr{P}} \{E(\sup_i \sup_j \sum_{j' \in I_j} (M(s_i, t_{j+1}) - M(s_i, t_j))^2) \\
&\qquad \cdot E(\sum_{i=1}^{p_\ell} \sum_k \sum_{j=1}^{q_n} \sup_{j' \in I_j} M(\bar{\Delta}_{ij'k})^2)\}^{1/2},
\end{aligned}
$$

where for any $j = 1, \cdots, q_n$, $I_j = \{j' : t_{j'}$ is a point of $\mathscr{P}_2^m$ belonging to the interval $[t_j, t_{j+1})\}$. The second factor of (3.17) is bounded by $\{E(M_{11}^2)\}^{1/2}$, and the first one converges to zero when $n \to \infty$, uniformly with respect to m, as in the proof of part b) of Lemma 3.2.

The same arguments can be used to treat the second term of the right-hand side of (3.15), obtaining that it is bounded by

$$C\{E(\sup_{|t-t'|\le|\mathscr{P}_2^n|}|M_{1t}-M_{1t'}|^2)E(M_{11}^2)\}^{1/2}.$$

So, (3.14) holds.

ii) We want to prove that

$$\lim_{\delta\downarrow 0}\sup_n E(\sup_{|s-s'|<\delta}|T_s^n - T_{s'}^n|) = 0. \tag{3.18}$$

The processes T_s^n are continuous and increasing. Thus, if we consider a finite set $\mathscr{P} = \{0 = s_1 < s_2 < \cdots < s_\ell < 1\}$ such that $|\mathscr{P}| < \delta$, we obtain

$$\begin{aligned} E(\sup_{|s-s'|<\delta}|T_s^n - T_{s'}^n|) \\ &\le 2E(\sup_i|T_{s_{i+1}}^n - T_{s_i}^n|) \\ &= 2E(\sup_i \textstyle\sum_{j=1}^{q_n}(\langle M_{\cdot t_{j+1}} - M_{\cdot t_j}\rangle_{s_{i+1}} - \langle M_{\cdot t_{j+1}} - M_{\cdot t_j}\rangle_{s_i})) \\ &= 2E(\sup_i\lim_{|\mathscr{P}_i|\to 0}\textstyle\sum_{j=1}^{q_n}\sum_{\sigma_k^i\in\mathscr{P}_i} M(\Delta_{ijk})^2), \end{aligned} \tag{3.19}$$

where $\Delta_{ijk} = (\sigma_k^i, \sigma_{k+1}^i] \times (t_j, t_{j+1}]$ and $\mathscr{P}_i = \{s_i = \sigma_1^i < \sigma_2^i < \cdots < \sigma_{r_i}^i < s_{i+1}\}$ determines a partition of the interval $[s_i, s_{i+1}]$ for any $i = 1, \cdots, \ell$. Put $\overline{\Delta}_{ijk} = (\sigma_k^i, \sigma_{k+1}^i] \times (0, t_j]$. Then, (3.19) is bounded by

$$\begin{aligned} 2E(\sup_i(\langle M_{\cdot 1}\rangle_{s_{i+1}} - \langle M_{\cdot 1}\rangle_{s_i})) \\ + 4E(\sup_i\lim_{|\mathscr{P}_i|\to 0}|\textstyle\sum_{j=1}^{q_n}\sum_k M(\Delta_{ijk})M(\overline{\Delta}_{ijk})|). \end{aligned} \tag{3.20}$$

The first term of (3.20) does not depend on n and converges to zero when $\delta \downarrow 0$. The second term can be bounded by

$$4E\{\lim_{|\mathscr{P}_i|\to 0}|\textstyle\sum_i(\sum_j\sum_k M(\Delta_{ijk})M(\overline{\Delta}_{ijk}))^2|^{1/2}\},$$

and using Lemma 2.2, this quantity is less than or equal to

$$\begin{aligned} C\sup_{\mathscr{P}_i}E(|\textstyle\sum_j\sum_i(\sum_k M(\Delta_{ijk})M(\overline{\Delta}_{ijk}))^2|^{1/2}) \\ \le C\{E(M_{11}^2)\sup_{\mathscr{P}_i}E(\sup_j\sup_i\textstyle\sum_k M(\overline{\Delta}_{ijk})^2)\}^{1/2}. \end{aligned}$$

This expression converges to zero when $\delta \downarrow 0$, uniformly with respect to n, as in the proof of part b) of Lemma 3.2.

iii) We will show that

$$\lim_n\sup_{m>n}\sup_s E(|T_s^m - T_s^n|) = 0. \tag{3.21}$$

Fix $m > n$. With the same notation as above we have

$$\begin{aligned} E(|T_s^m - T_s^n|) &= E(|\textstyle\sum_{j=1}^{q_m}\langle M_{\cdot t_{j+1}} - M_{\cdot t_j}\rangle_s - \sum_{j=1}^{q_n}\langle M_{\cdot t_{j+1}} - M_{\cdot t_j}\rangle_s|) \\ &= 2E(|\textstyle\sum_{j=1}^{q_m}\langle M_{\cdot t_{j+1}} - M_{\cdot t_j}, M_{\cdot t_j} - M_{\cdot t_{\nu(j)}}\rangle_s|) \\ &\le 2\sup_{\mathscr{P}}E(|\textstyle\sum_{j=1}^{q_m}\sum_i M(\Delta_{ij})M(\overline{\Delta}_{ij})|), \end{aligned} \tag{3.22}$$

where $\Delta_{ij} = (s_i, s_{i+1}] \times (t_j, t_{j+1}]$, $\overline{\Delta}_{ij} = (s_i, s_{i+1}] \times (t_{\nu(j)}, t_j]$, and the supremum is taken over all finite sets $\mathscr{P} = \{0 = s_1 < s_2 < \cdots < s_\ell < s_{\ell+1} = s\}$. By Davis inequality, (3.22) is bounded by

$$C \sup_{\mathscr{P}} E(|\textstyle\sum_{j=1}^{q_m} (\sum_i M(\Delta_{ij})M(\overline{\Delta}_{ij}))^2|^{1/2})$$
$$\leq C \sup_{\mathscr{P}} E(|\textstyle\sum_{j=1}^{q_m} (\sum_i M(\Delta_{ij})^2)(\sum_i M(\overline{\Delta}_{ij})^2)|^{1/2})$$
$$\leq C \sup_{\mathscr{P}}\{E(\sup_j \textstyle\sum_{j'\in I_j} \sum_i M(\Delta_{ij'})^2)E(\sum_{j=1}^{q_n} \sup_{j'\in I_j} \sum_i M(\overline{\Delta}_{ij'})^2)\}^{1/2}.$$

Now we apply Doob's maximal inequality, obtaining that (3.23) is less than or equal to

$$C\{E(M_{11}^2) \sup_{\mathscr{P}} E(\sup_j \textstyle\sum_{j'\in I_j} \sum_i M(\Delta_{ij'})^2)\}^{1/2}.$$

Next we set

$$E(\sup_j \textstyle\sum_{j'\in I_j} \sum_i M(\Delta_{ij'})^2)$$
$$\leq E(\sup_j \textstyle\sum_{j'\in I_j} (M(1, t_{j'+1}) - M(1, t_{j'}))^2) \tag{3.24}$$
$$+ 2E(\sup_j |\textstyle\sum_{j'\in I_j} \sum_i (M(s_i, t_{j'+1}) - M(s_i, t_{j'}))M(\Delta_{ij'})|).$$

The first term of (3.24) converges to zero when n tends to infinity, uniformly with respect to m, by Lemma 2.1 applied to the martingale $M_{1\cdot}$. This convergence holds too for the second term. Indeed, applying Lemma 2.2, this term is bounded by

$$2E(|\textstyle\sum_{j=1}^{q_n} (\sum_{j'\in I_j} \sum_i (M(s_i, t_{j'+1}) - M(s_i, t_{j'}))M(\Delta_{ij'}))^2|^{1/2})$$
$$\leq CE(|\textstyle\sum_i \sum_{j=1}^{q_n} (\sum_{j'\in I_j} (M(s_i, t_{j'+1}) - M(s_i, t_{j'}))M(\Delta_{ij'}))^2|^{1/2})$$
$$\leq C\{E(M_{11}^2)E(\sup_{i,j} \textstyle\sum_{j'\in I_j} (M(s_i, t_{j'+1}) - M(s_i, t_{j'}))^2)\}.^{1/2}$$

Then we apply Doob's maximal inequality to the positive submartingale (with respect to the coordinate s) $(\sup_j \sum_{j'\in I_j} (M(s, t_{j'+1}) - M(s, t_{j'}))^2)^{1/2}$, as in the proof of (3.10), obtaining that the above expression is majored by

$$C\{E(M_{11}^2)E(\sup_j \textstyle\sum_{j'\in I_j} (M(1, t_{j'+1}) - M(1, t_{j'}))^2)\}^{1/2}.$$

iv) Finally we will prove that

$$\lim_n \sup_{m>n} E(\sup_s |T_s^m - T_s^n|) = 0. \tag{3.25}$$

This convergence together with (3.14) will imply the theorem. In the deduction of (3.14) we have essentially used Davis inequality applied to the one-parameter continuous martingales $R_s^m - R_s^n$. Here we substitute this inequality by the uniform continuity of the processes T_s^n with respect to n, which has been obtained in part ii). Given a real number $\varepsilon > 0$ we fix $\delta > 0$ such that $E(\sup_{|s-s'|<\delta} |T_s^n - T_{s'}^n|) < \varepsilon/3$ for all n. Let $\mathscr{P} = \{0 = s_1 < s_2 < \cdots < s_\ell < 1\}$ a finite set with $|\mathscr{P}| < \delta$. Then

$$E(\sup_s |T_s^m - T_s^n|) \leq E(\sup_i \sup_{s\in[s_i,s_{i+1}]} |T_s^m - T_{s_i}^m|) + \textstyle\sum_i E(|T_{s_i}^m - T_{s_i}^n|)$$
$$+ E(\sup_i \sup_{s\in[s_i,s_{i+1}]} |T_{s_i}^n - T_s^n|)$$
$$\leq \frac{2\varepsilon}{3} + \textstyle\sum_i E(|T_{s_i}^m - T_{s_i}^n|) \leq \varepsilon,$$

for any $n \geq n_0$ and for all $m > n$, because of (3.21). □

Now we can state the main result.

THEOREM 3.4. *Let M be a martingale of m_c^p with $p \geq 2$. There exists a continuous increasing process $\langle M \rangle$ such that*

$$\lim_n \sup_z E(|\textstyle\sum_{u \in \mathcal{T}_z^n} M(\Delta_u)^2 - \langle M \rangle_z|^{p/2}) = 0, \tag{3.26}$$

and the following Itô's formula holds

$$M_{st}^2 = 2N_{st} + 2S_{st} + \langle M_{s\cdot} \rangle_t + \langle M_{\cdot t} \rangle_s - \langle M \rangle_{st}, \tag{3.27}$$

where N and S are the martingales of $m_c^{p/2}$ given by Lemmas 3.1 *and* 3.2.

PROOF. The following convergences are well-known from the results in the one-parameter case

$$\lim_n \sup_{s,t} E(|\textstyle\sum_i (M(s_{i+1} \wedge s, t) - M(s_i \wedge s, t))^2 - \langle M_{\cdot t} \rangle_s|^{p/2}) = 0, \tag{3.28}$$

$$\lim_n \sup_{s,t} E(|\textstyle\sum_j (M(s, t_{j+1} \wedge t) - M(s, t_j \wedge t))^2 - \langle M_{s\cdot} \rangle_t|^{p/2}) = 0. \tag{3.29}$$

Then, applying these convergences and Lemmas 3.1 and 3.2 to the decomposition given in (3.1), we obtain an adapted and integrable process $\langle M \rangle_z$ for which (3.26) holds. It is easy to see that this process has a right-continuous and increasing modification. Finally the sample continuity of $\langle M \rangle_z$ follows from Lemma 3.1, 3.2 and Theorem 3.3.

REMARKS.

1. A sequence of continuous processes $X_n = \{X_n(z), z \in T\}$ is said to converge uniformly in probability to a process $X = \{X(z), z \in T\}$ if $\lim_n P\{\sup_z |X_n(z) - X(z)| > \varepsilon\} = 0$ for any $\varepsilon > 0$. Suppose that M is a martingale of m_c^2. Then, the preceding results imply that the five terms appearing in the right-hand side of (3.1) converge uniformly in probability to the continuous processes N_{st}, S_{st}, $\langle M_{\cdot t} \rangle_s$, $\langle M_{s\cdot} \rangle_t$ and $\langle M \rangle_{st}$, respectively.

2. Let M be a martingale of m_c^2. A limit argument in Burkholder's inequalities for two-parameter discrete martingales leads to the following inequalities for all $p > 1$

$$C_p E(\sup_z |M_z|^p) \leq E(\langle M \rangle_{11}^{p/2}) \leq C_p^1 E(\sup_z |M_z|^p),$$

provided that, for $p > 2$, the expectation $E(|M_{11}|^p)$ is finite. For $p = 1$, we can only affirm that

$$E(\sup_s |M_{s1}|) \leq CE(\langle M \rangle_{11}^{1/2}),$$

because $E(\sup_s |M_{s1}|) \leq CE(\langle M_{\cdot 1} \rangle_1^{1/2})$ by Davis inequality, and moreover $E(\langle M_{\cdot 1} \rangle_1^{1/2}) \leq CE(\langle M \rangle_{11}^{1/2})$, by a limit argument in Lemma 2.2.

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