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CONVERGENCE OF SUMS OF MIXING TRIANGULAR ARRAYS OF RANDOM VECTORS WITH STATIONARY ROWS[1]

BY JORGE D. SAMUR

Universidad Nacional de La Plata

This paper deals with the convergence in distribution to Gaussian, generalized Poisson and infinitely divisible laws of the row sums of certain ϕ or ψ-mixing triangular arrays of Banach space valued random vectors with stationary rows. Necessary and sufficient conditions for convergence in terms of individual r.v.'s are proved. These include sufficient conditions for the convergence to a stable law of the normalized sums of certain ϕ-mixing, stationary sequences. An invariance principle for stationary, ϕ-mixing triangular arrays is given.

0. Introduction. Several authors have studied the weak convergence of the laws of sums of random variables with the hypothesis of independence replaced by less restrictive properties which are expressed through certain dependence coefficients (see, for example, Ibragimov and Linnik [12], Billingsley [6], [7], Iosifescu and Theodorescu [13], Philipp [15]). In this paper we consider certain mixing conditions (the so-called ϕ and ψ-mixing) for triangular arrays of random vectors which take values in a separable Banach space and whose rows form stationary finite sequences (see Section 1 for the definitions). Our aim is to give necessary and sufficient conditions for the convergence of the laws of the row sums of such triangular arrays expressed in terms of the individual random vectors and, in principle, without moment assumptions. In order to do this, we depart to some extent from the usual paths in this area and follow the point of view developed by de Acosta, Araujo and Giné [3] for the case of row-wise independent infinitesimal triangular arrays. We use some results of that article through the technique, standard in the dependent case, of grouping random vectors in suitable blocks, an idea due to S. Bernstein. The framework that we present for the study of triangular arrays under dependence conditions and several of our specific results—for example, Corollaries 4.6, 5.8, 5.10 and 6.5—appear to be new even for the real-valued case.

Section 2 contains some basic inequalities, which are used in Section 3 to prove results about compactness and integrability.

In Sections 4, 5 and 6 we deal with necessary and sufficient conditions for convergence in a Banach space to a Gaussian, generalized Poisson or infinitely divisible law, respectively. In the first two cases, the ϕ-mixing condition is required to hold together with certain restrictions about contiguous random

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vectors; in Section 6, the ψ-mixing condition is added. For a Hilbert space and assuming some specified mixing rates, we give sufficient conditions for convergence expressed, as far as possible, in terms of individual random vectors (see Corollaries 4.5, 5.8 and 6.5; in Philipp [15] there are conditions in terms of blocks for convergence to certain infinitely divisible laws for ϕ-mixing triangular arrays of real random variables which satisfy different hypothesis from the ones given here).

From the sufficient conditions for convergence to a Gaussian law given in Section 4, we can derive a result (Corollary 4.7) which, essentially, is an infinite-dimensional generalization of a theorem of I. A. Ibragimov for real random variables (Theorem 18.5.2 in [12]). On the other hand, we show that by applying methods of de Acosta [2] we can obtain an almost sure invariance principle for stationary, ϕ-mixing triangular arrays (Theorem 4.8); from this, following de Acosta [2], we can deduce an invariance principle in distribution (Corollary 4.10) which generalizes a result of Eberlein [8].

Section 5 includes a direct theorem of the Poisson type (Theorem 5.6) and the proof that the classical conditions for convergence to a stable law of the normalized sums of a stationary sequence of independent random variables are still sufficient for certain ϕ-mixing sequences (Corollary 5.10).

1. Definitions and notations. Throughout the paper, B denotes a real separable Banach space and the random vectors we consider take values in B.

By a triangular array $\{X_{nj}\}$ we mean a doubly indexed family $\{X_{nj}: j = 1, \cdots, j_n, n \in N\}$ (N the set of nonzero natural numbers) of B-valued random vectors (r.v.'s) defined on a common probability space $(\Omega, \mathscr{A}, P)$; we will assume always that $j_n \to \infty$. Given $\{X_{nj}\}$, we define $\mathscr{M}_{hk}^{(n)} = \sigma(X_{nj}: h \le j \le k)$ (the σ-algebra generated by the indicated set of r.v.'s) for $n \in N$ and $1 \le h \le k \le j_n$. Analogously, we define for a sequence $\{X_j: j \in N\}$ of B-valued r.v.'s the σ-algebras $\mathscr{M}_{hk} (1 \le h \le k \le \infty)$ and also $\mathscr{M}_{hk} (1 \le h \le k \le n)$ for a finite set $\{X_1, \cdots, X_n\}$.

Given a triangular array $\{X_{nj}: j = 1, \cdots, j_n, n \in N\}$ we define the dependence coefficient

$$\phi(k) = \sup_{n \in N, j_n > k} \max_{1 \le h \le j_n - k} \sup \left\{ \left| \frac{P(E \cap F)}{P(E)} - P(F) \right| : E \in \mathscr{M}_{1h}^{(n)}, F \in \mathscr{M}_{h+k, j_n}^{(n)}, P(E) > 0 \right\}$$

($k \in N$); it follows that $\phi(1) \le 1$ and that $\{\phi(k)\}$ is a nonincreasing sequence. We say that $\{X_{nj}\}$ *is ϕ-mixing* if $\phi(k) \downarrow 0$ as $k \to \infty$ (the same letter is used to denote the coefficient and to name the property). For a sequence $\{X_j\}$ define

$$\phi(k) = \sup_{h \in N} \sup \left\{ \left| \frac{P(E \cap F)}{P(E)} - P(F) \right| : E \in \mathscr{M}_{1h}, F \in \mathscr{M}_{h+k, \infty}, P(E) > 0 \right\}$$

and then the ϕ-mixing property for $\{X_j\}$ is defined as above. Given a finite set $\{X_1, \cdots, X_n\}$ the numbers $\phi(1), \cdots, \phi(n-1)$ are defined in a similar way.

For a triangular array $\{X_{nj}\}$ we define

$$\psi(k) = \sup_{n\in N, j_n>k} \max_{1\le h\le j_n-k} \sup\left\{ \left| \frac{P(E\cap F)}{P(E)P(F)} - 1 \right| : E \in \mathscr{M}_{1h}^{(n)}, F \in \mathscr{M}_{h+k,j_n}^{(n)}, P(E)P(F) > 0 \right\}$$

($k \in N$); observe that $\psi(1) \le +\infty$ and that $\{\psi(k)\}$ is a nonincreasing sequence. We say that $\{X_{nj}\}$ *is* ψ*-mixing* if $\psi(k) \downarrow 0$ as $k \to \infty$. Also, we define these notions for a sequence and the coefficients $\psi(k)$ for a finite set of r.v.'s. Note that in any case $\phi(k) \le \psi(k)$.

The last coefficient we will consider for a triangular array $\{X_{nj}\}$ is

$$\psi^* = \sup_{n\in N, j_n>1} \max_{1\le h\le j_n-1} \sup\left\{ \frac{P(E\cap F)}{P(E)P(F)} : E \in \mathscr{M}_{1h}^{(n)}, F \in \mathscr{M}_{h+1,j_n}^{(n)}, P(E)P(F) > 0 \right\}$$

(this is not a standard notation); we have $\psi^* \le +\infty$ and $\psi^* \le 1 + \psi(1)$. It is defined analogously for sequences and finite sets of r.v.'s.

For examples of nonindependent sequences of random variables which are ϕ-mixing, ψ-mixing or satisfy $\psi^* < +\infty$ see Ibragimov and Linnik [12], Billingsley [6], [7], Iosifescu and Theodorescu [13]. There are examples with $\phi(n) = O(\rho^n)$ or $\psi(n) = O(\rho^n)$ where $0 < \rho < 1$.

We say that a finite set $\{X_1, \cdots, X_n\}$ of B-valued r.v.'s is *stationary* (*with stationary sums*) if $\mathscr{L}(X_1, \cdots, X_h) = \mathscr{L}(X_{k+1}, \cdots, X_{k+h})$ ($\mathscr{L}(X_1 + \cdots + X_h) = \mathscr{L}(X_{k+1} + \cdots + X_{k+h})$, respectively) for $1 \le h \le n$, $1 \le k \le n - h$ (if Z is a random vector, $\mathscr{L}(Z)$ denotes its distribution). A triangular array is *stationary* (*with stationary sums*) if each one of its rows has this property. We have similar definitions for a sequence of r.v.'s.

Let $\mathscr{B}$ denote the Borel σ-algebra of B. If $A \in \mathscr{B}$, I_A is the indicator function of A; for a B-valued r.v. X we write $X_\delta = XI_{[\|X\|\le\delta]}$, $X^\delta = X - X_\delta$ ($\delta > 0$, $\|\cdot\|$ is the norm of B). Sometimes we will denote $E[X; X\in A] = E[XI_{[X\in A]}]$. Given a triangular array $\{X_{nj}\}$ we write $S_{nk} = \sum_{j=1}^{k} X_{nj}$ if $1 \le k \le j_n$, $S_n = S_{nj_n}$, $S_{n,\delta} = \sum_{j=1}^{j_n} X_{nj\delta}$, $S_n^{(\delta)} = \sum_{j=1}^{j_n} X_{nj}^{\delta}$; if $\{X_{nj}\}$ has stationary sums and $\mu_n = \mathscr{L}(X_{n1})$ we write $\mu_n^{(k)} = \mathscr{L}(S_{nk})$ ($k = 1, \cdots, j_n$). For a probability measure μ on B and $k \in N$, μ^k denotes the kth convolution power of μ; if ν is infinitely divisible, $\{\nu^t\colon t \ge 0\}$ is the associated weakly continuous convolution semigroup. The symbols $\otimes$ and $*$ denote the product and convolution of measures, respectively.

We denote by $\to_w$ or w-lim the weak convergence of finite measures and by $\to_P$ the convergence in probability of random vectors. ρ is the Prohorov distance between probability measures on B and we write $\sigma(X) = E[\|X\|(1 + \|X\|)^{-1}]$ for a B-valued r.v. X.

For the notions and basic properties of infinitely divisible probability measure, Gaussian measure, Lévy measure and τ-centered Poisson measure in Banach spaces we refer to de Acosta et al [3] or Araujo and Giné [5]. If γ is a Gaussian measure on B, Φ_γ denotes its covariance. Given an infinitely divisible measure ν we will take as its Lévy measure μ that one which satisfies $\mu(\{0\}) = 0$.

If μ is a σ-finite measure on B we put $C(\mu) = \{r > 0: \mu(\{x: \|x\| = r\}) = 0\}$; if $A \in \mathscr{B}$ the measure $\mu \mid A$ is defined by $(\mu \mid A)(E) = \mu(A \cap E)(E \in \mathscr{B})$. δ_x denotes the point mass at $x \in B$. B' is the dual space of B and $B_r = \{x \in B: \|x\| \le r\}$ $(r > 0)$.

2. Some inequalities for sums of dependent random vectors. Let us state a simple extension of Lemma (3.5) of Eberlein [8]. The proof involves a monotone class argument and induction over k.

2.1 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s. *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k (k \in N)$ *be natural numbers such that* $1 \le a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k \le n$ *with* $a_i - b_{i-1} \ge q \in N(i = 2, \cdots, k)$ *and define* $\xi_h = \sum_{a_h \le j \le b_h} X_j (h = 1, \cdots, k)$. *Then*

$$| \mathscr{L}(\xi_1, \cdots, \xi_k)(A) - \mathscr{L}(\xi_1) \otimes \cdots \otimes \mathscr{L}(\xi_k)(A) | \le (k-1)\phi(q)$$

for every $A \in \mathscr{B}^k$ *(the k-fold product σ-algebra of B).*

The following version of Ottaviani's inequality can be proved as Lemma 1.1.6 of Iosifescu and Theodorescu [13] (note that it requires $\phi(1) < 1$).

2.2 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s *with* $\phi(1) < 1$ *and write* $S_k = \sum_{j=1}^k X_j$. *Suppose* $\phi(1) < \alpha < 1$ *and let* $V \in \mathscr{B}$ *be a symmetric convex set such that* $\max_{1 \le k \le n-1} P[S_n - S_k \notin (1/2) V] \le 1 - \alpha$. *Then*

$$P[S_k \notin V \text{ for some } k = 1, \cdots, n] \le (\alpha - \phi(1))^{-1} P[S_n \notin (1/2) V].$$

2.3 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s *with* $\phi(1) < 1$ *and write* $S_k = \sum_{j=1}^k X_j$. *Suppose* $\phi(1) < \alpha < 1$ *and let* $V \in \mathscr{B}$ *be a symmetric convex set such that* $\max_{1 \le k \le n-1} P[S_n - S_k \notin (1/4) V] \le 1 - \alpha$ *and* $P[S_n \notin (1/4) V] < (\alpha - \phi(1))(1 - \phi(1))$. *Then*

$$\sum_{j=1}^n P[X_j \notin V] \le \frac{P[S_n \notin (1/4) V]}{(\alpha - \phi(1))(1 - \phi(1)) - P[S_n \notin (1/4) V]}.$$

PROOF. Define $F_k = [X_{k+1} \in V, \cdots, X_n \in V]$ for $k = 0, \cdots, n-1$ and $F_n = \Omega$; then $F_k \in \mathscr{M}_{k+1,n}$ for $k < n$. It follows that

$$\begin{aligned} P[X_j \notin V \text{ for some } j = 1, \cdots, n] &= \textstyle\sum_{k=1}^n P([X_k \notin V] \cap F_k) \\ &\ge \textstyle\sum_{k=1}^n (P(F_k) - \phi(1)) P[X_k \notin V] \\ &\ge (P(F_0) - \phi(1)) \textstyle\sum_{j=1}^n P[X_j \notin V] \\ &= (1 - \phi(1) - P[X_j \notin V \text{ for some } j]) \textstyle\sum_{j=1}^n P[X_j \notin V]. \end{aligned}$$

Now it suffices to note that, writing $X_j = S_j - S_{j-1}$, one has $P[X_j \notin V$ for some $j] \leq P[S_k \notin (1/2)V$ for some $k] \leq (\alpha - \phi(1))^{-1}P[S_n \notin (1/4)V]$ by Proposition 2.2. □

The following generalization of Lemma 2, page 383 of Gihman and Skorohod [9] will be useful; the proof is similar to that given in [9] and uses Proposition 2.2.

2.4 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s *with* $\phi(1) < 1$; *write* $S_k = \sum_{j=1}^k X_j$. *Suppose* $\phi(1) < \alpha < 1$, $\| X_j \| \leq M$ a.s. $(j = 1, \cdots, n)$ *and let* $t > 0$, $\ell \in N$. *Then, if* $\max_{1\leq k\leq n-1} P[\| S_n - S_k \| > t/4] \leq 1 - \alpha$, *it holds that*

$$P[\max_{1\leq k\leq n} \| S_k \| > \ell t + (\ell - 1)M]$$
$$\leq (\phi(1) + (\alpha - \phi(1))^{-1}P[\| S_n \| > t/4])^{\ell-1}(\alpha - \phi(1))^{-1}P[\| S_n \| > t/2].$$

To close this section, we quote three moment inequalities (see Theorem 17.2.3 of Ibragimov and Linnik [12], Lemma 3 of Philipp [15] and page 27 of Billingsley [7]).

2.5 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s. *Let* $h, k \in N$ $h + k \leq n$ *and let* ξ, η *be real random variables which are* $\mathcal{M}_{1,h}$ *and* $\mathcal{M}_{h+k,n}$-*measurable, respectively. If* $E|\xi|^p < \infty$ *and* $E|\eta|^q < \infty$ *with* $p, q > 1$ *and* $p^{-1} + q^{-1} = 1$, *then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq 2\phi^{1/p}(k)(E|\xi|^p)^{1/p}(E|\eta|^q)^{1/q}.$$

2.6 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$, h, k, ξ *and* η *be as above but with the only assumption that* $E|\xi| < \infty$ *and* $E|\eta| < \infty$. *Then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq \psi(k)E|\xi|E|\eta|.$$

2.7 PROPOSITION. *Let* $\{X_1, \cdots, X_n\}$, ξ, η, h *be as in the previous proposition with* $k = 1$. *Then* $|E(\xi\eta)| \leq \psi^* E|\xi|E|\eta|$.

3. Preliminary results. In the following result we use some ideas which appear in Eberlein [8] (proof of Proposition (3.6)) which in turn is inspired in Kuelbs ([14], Lemma 1). The second part of the conclusion will be used later (see Theorem 4.8).

3.1 PROPOSITION. *Let* $\{X_{nj}\}$ *be a* ϕ-*mixing triangular array with stationary sums. Suppose that* $X_{n1} \to_P 0$ *and that* $\mathcal{L}(S_n) \to_w \nu$. *Then* ν *is infinitely divisible and for each* $p \in N$ *we have*

$$\mathcal{L}(\textstyle\sum_{j\in I(n,p,0)} X_{nj}, \cdots, \sum_{j\in I(n,p,p-1,} X_{nj}) \to_w (\nu^{1/p})^{\otimes p},$$

where $I(n, p, k) = \{j \in N: kj_n p^{-1} < j \leq (k + 1)j_n p^{-1}\}(k = 0, 1, \cdots, p - 1)$.

PROOF. Fix $p \in N$. Write $I(n, p, k) = [a_{nk}, b_{nk}]$ (interval of integer numbers) and note that it has $[j_n p^{-1}]$ or $[j_n p^{-1}] + 1$ elements (here [.] is the integer part of a real number).

By hypothesis, $\sigma_n = \sigma(X_{n1}) \to 0$ as $n \to \infty$. Take a sequence $\{d_n\} \subset N$ such that $d_n \to \infty$, $d_n \sigma_n \to 0$ and $d_n \le [j_n p^{-1}]$ for all sufficiently large n (for example: $d_n = \min\{[j_n p^{-1}], [\sigma_n^{-1/2}]\}$). Now define $b'_{nk} = a_{nk} + [j_n p^{-1}] - d_n$ and $\xi_{nk} = \sum_{a_{nk} \le j \le b'_{nk}} X_{nj} (k = 0, \cdots, p-1)$; by stationarity of sums we have $\mathscr{L}(\xi_{n0}) = \cdots = \mathscr{L}(\xi_{n,p-1}) = \lambda_n$ (say).

Since $\sigma(S_n - \sum_{k=0}^{p-1} \xi_{nk}) \le p d_n \sigma_n \to 0$ as $n \to \infty$, it follows that $\mathscr{L}(\sum_{k=0}^{p-1} \xi_{nk}) \to_w \nu$. On the other hand, from Proposition 2.1 we obtain for every $A \in \mathscr{B}^p$

$$(3.1) \qquad | \mathscr{L}(\xi_{n0}, \cdots, \xi_{n,p-1})(A) - \lambda_n^{\otimes p}(A) | \le (p-1)\phi(d_n)$$

and therefore, for every $C \in \mathscr{B}$,

$$| \mathscr{L}(\textstyle\sum_{k=0}^{p-1} \xi_{nk})(C) - \lambda_n^p(C) | \le (p-1)\phi(d_n)$$

which goes to zero as n goes to infinity. Then $\lambda_n^p \to_w \nu$. Hence, by well known properties of the weak convergence of probability measures, we conclude that there exists $\{x_n\} \subset B$ such that $\{\lambda_n * \delta_{x_n}\}$ is relatively compact and then we obtain the relative compactness of $\{\lambda_n^p * \delta_{px_n}\}$, $\{\delta_{px_n}\}$, $\{\delta_{x_n}\}$ and $\{\lambda_n\}$ successively. But if λ is a limit point of $\{\lambda_n\}$ then $\lambda^p = \nu$.

The arbitrariness of p above shows that ν is infinitely divisible. To obtain the second assertion of our statement, fix p and apply (3.1) observing that $\lambda_n \to_w \nu^{1/p}$ and $\sigma(\sum_{j \in I(n,p,k)} X_{nj} - \xi_{nk}) \le d_n \sigma_n$. □

For a triangular array $\{X_{nj}: j = 1, \cdots, j_n, n \in N\}$ with stationary sums we shall consider the following property:

$$(*) \qquad \{r_n\} \subset N, \quad r_n \le j_n, \quad r_n/j_n \to 0 \Rightarrow \textstyle\sum_{j=1}^{r_n} X_{nj} \to_P 0.$$

REMARK. Theorem 2.1 of de Acosta [2] shows that this property (which may be described as a strong form of infinitesimality) holds for a triangular array of B-valued r.v.'s which are row-wise independent and equidistributed and whose sums converge weakly.

This condition is an hypothesis in many of our statements but it is dropped in some results in which we give sufficient conditions for convergence (see Corollary 4.5, Theorems 5.6, 5.7 and Corollary 6.5); next we point out two cases in which it is verified.

1) Let $\{X_j\}$ be a ϕ-mixing stationary sequence and let $\{a_n\}$ be a sequence of real numbers tending to infinity such that $\{\mathscr{L}(a_n^{-1} \sum_{j=1}^n X_j)\}$ converges weakly. If $X_{nj} = a_n^{-1} X_j (j = 1, \cdots, n)$ then the triangular array $\{X_{nj}: j = 1, \cdots, n; n \in N\}$ satisfies (*)(It can be proved by using Theorem 2 of Philipp [16] and a theorem of Karamata [12, Theorem A.1.1]).

2) Let $\{X_{nj}\}$ be a ϕ-mixing triangular array with stationary sums such that $X_{n1} \to_P 0$ and $\mathscr{L}(X_{n1} + \cdots + X_{nk})$ is symmetric for $k = 1, \cdots, j_n$, $n \in N$. Then, if $\{\mathscr{L}(S_n)\}$ converges weakly, $\{X_{nj}\}$ has the property (*).

This is a consequence of the following fact: let $\{X_{nj}\}$ be a ϕ-mixing triangular array with stationary sums such that $X_{n1} \to_P 0$ and $\{\mathscr{L}(S_n)\}$ converges weakly; then, if $\{r_n\} \subset N$, $r_n \le j_n$ and $r_n/j_n \to 0$, there exists $\{x_n\} \subset B$ such that the sequence $\{\mu_n^{(r_n)} * \delta_{x_n}\}$ is relatively compact and all its limit points are point masses. To prove this, let ν be the limit of $\{\mathscr{L}(S_n)\}$ and take a sequence $\{r_n\}$ as indicated. Fix $p \in N$; by Proposition 3.1, $\mathscr{L}(\sum_{j=1}^{[j_n p^{-1}]} X_{nj}) \to_w \nu^{1/p}$. Let $\sigma_n = \sigma(X_{n1})$, $d_n = \min\{[j_n p^{-1}] - r_n, [\sigma_n^{-1/2}]\}$, $Y_n = \sum_{j=1}^{r_n} X_{nj}$, $Z_n = \sum_{r_n+d_n}^{[j_n p^{-1}]} X_{nj}$. We have $\sigma(\sum_{j=1}^{[j_n p^{-1}]} X_{nj} - (Y_n + Z_n)) \le d_n \sigma_n \to 0$ and, applying Proposition 2.1,

$$|\mathscr{L}(Y_n + Z_n)(A) - \mathscr{L}(Y_n) * \mathscr{L}(Z_n)(A)| \le \phi(d_n) \to 0$$

for every $A \in \mathscr{B}$. Then $\mu_n^{(r_n)} * \mathscr{L}(Z_n) \to_w \nu^{1/p}$. By a well known result, we deduce that there exists $\{x_n\} \subset B$ such that $\{\mu_n^{(r_n)} * \delta_{x_n}\}$ is relatively compact. Let α be a limit point of this sequence; then α^p is a factor of ν for every $p \in N$. From this we conclude that $\{\alpha^p * \delta_{y_p} \colon p \in N\}$ is relatively compact for some $\{y_p\} \subset B$, but this implies that $\alpha = \delta_z$ for some $z \in B$ (see [5, page 33]).

In view of the two cases described above and Theorem 2.1 of [2] it is natural to ask if in general: $\{X_{nj}\}$ stationary ϕ-mixing, $X_{n1} \to_P 0$, $\mathscr{L}(S_n) \to_w \cdot$ imply $\{X_{nj}\}$ satisfies (*). We have not been able to answer this question.

We shall need sequences of integers with the properties stated in the following result.

3.2 LEMMA. *Let $\{j_n\} \subset N$, $\{\sigma_n\} \subset [0, \infty)$ and $\{\phi(n)\} \subset [0, \infty)$ be sequences such that $j_n \to \infty$, $\sigma_n \to 0$ and $\phi(n) \to 0$ as $n \to \infty$. Then there exists sequences $\{p_n\}$, $\{q_n\}$ in N which tend to infinity such that $j_n(p_n + q_n)^{-1} \to \infty$, $\phi(q_n) j_n (p_n + q_n)^{-1} \to 0$, $q_n \sigma_n j_n (p_n + q_n)^{-1} \to 0$ and $q_n p_n^{-1} \to 0$.*

PROOF. Observe that if the last condition is verified, the remaining are equivalent to $j_n p_n^{-1} \to \infty$, $\phi(q_n) j_n p_n^{-1} \to 0$ and $q_n \sigma_n j_n p_n^{-1} \to 0$. First, we find sequences $\{q_n\} \subset N$ and $\{\beta_n\} \subset (0, \infty)$ such that $\beta_n \to 0$, $(j_n \beta_n)^{-1} q_n \to 0$, $\phi(q_n)\beta_n^{-1} \to 0$ and $q_n \sigma_n \beta_n^{-1} \to 0$. To do this, take $\{q_n\}$ such that $q_n \to \infty$, $q_n \sigma_n \to 0$ and $q_n j_n^{-1} \to 0$; for example, one can define $q_n = \min\{[\sigma_n^{-a}], [j_n^b]\}$ if $\sigma_n > 0$ and $q_n = [j_n^b]$ if $\sigma_n = 0$ ($[\cdot]$ is the integer part of a real number) with $0 < a < 1$, $0 < b < 1$. Now define $\beta_n = \max\{(q_n j_n^{-1})^u, (\phi(q_n))^v, (q_n \sigma_n)^w\}$ where u, v, w are real numbers in $(0, 1)$. Then $\{q_n\}$ and $\{\beta_n\}$ have the desired properties and it is sufficient to define $p_n = [j_n \beta_n] + 1$ in order to end the construction. □

Next, we prove a version of Theorem 2.1 of de Acosta [2] for the ϕ-mixing case; assertion (2) will be repeatedly used combined with some inequalities of Section 2.

3.3 THEOREM. *Let $\{X_{nj}\}$ be a ϕ-mixing triangular array with stationary sums which satisfies condition (*). Suppose that $\mathscr{L}(S_n) \to_w \nu$. Then*

(1) *if $\{r_n\} \subset N$, $r_n \le j_n$, $r_n/j_n \to t \in \mathbb{R}$, then $\mu_n^{(r_n)} \to_w \nu^t$,*

(2) *the set $\{\mu_n^{(k)} \colon k = 1, \cdots, j_n, n \in N\}$ is relatively compact and*

$$\lim_n \max_{1 \le k \le j_n} \rho(\mu_n^{(k)}, \nu^{k/j_n}) = 0.$$

PROOF. We only prove (1) because (2) can be deduced from it as in [2].

Let $\{r_n\}$ and t be as in (1). We may suppose that $t \in (0, 1)$; otherwise, the result follows easily from (*). Let $\{p_n\}$, $\{q_n\}$ be as in Lemma 3.2 where we have taken $\sigma_n = \sigma(X_{n1})$ and write $k'_n = [r_n(p_n + q_n)^{-1}]$, $k''_n = [(j_n - r_n)(p_n + q_n)^{-1}]$ ([.] denotes the integer part of a real number); then $k'_n \to \infty$, $k''_n \to \infty$ as $n \to \infty$. Define

$$\xi'_{nk} = \sum_{j=(k-1)(p_n+q_n)+1}^{(k-1)(p_n+q_n)+p_n} X_{nj} \quad \text{and} \quad \eta'_{nk} = \sum_{j=(k-1)(p_n+q_n)+p_n+1}^{k(p_n+q_n)} X_{nj}$$

for $k = 1, \cdots, k'_n$,

$$\eta'_{n,k'_n+1} = \sum_{j=k'_n(p_n+q_n)+1}^{r_n} X_{nj},$$

$$\xi''_{nk} = \sum_{j=r_n+(k-1)(p_n+q_n)+1}^{r_n+(k-1)(p_n+q_n)+p_n} X_{nj} \quad \text{and} \quad \eta''_{nk} = \sum_{j=r_n+(k-1)(p_n+q_n)+p_n+1}^{r_n+k(p_n+q_n)} X_{nj}$$

for $k = 1, \cdots, k''_n$,

$$\eta''_{n,k''_n+1} = \sum_{j=r_n+k''_n(p_n+q_n)+1}^{j_n} X_{nj};$$

note that $0 \le r_n - k'_n(p_n + q_n) < p_n + q_n$ and $0 \le j_n - r_n - k''_n(p_n + q_n) < p_n + q_n$.

The inequalities

$$r_n j_n^{-1} - (p_n + q_n) j_n^{-1} < k'_n(k'_n + k''_n)^{-1} < r_n j_n^{-1}(1 - 2(p_n + q_n) j_n^{-1})^{-1}$$

show that $k'_n(k'_n + k''_n)^{-1} \to t$ as $n \to \infty$. On the other hand,

$$\sigma(S_n - \sum_{k=1}^{k'_n} \xi'_{nk} - \sum_{k=1}^{k''_n} \xi''_{nk}) \le (k'_n + k''_n) q_n \sigma_n + \sigma(\eta'_{n,k'_n+1}) + \sigma(\eta''_{n,k''_n+1})$$

which goes to zero as $n \to \infty$ by the preceding construction and condition (*). Then $\mathscr{L}(\sum_{k=1}^{k'_n} \xi'_{nk} + \sum_{k=1}^{k''_n} \xi''_{nk}) \to_w \nu$. Since $\mathscr{L}(\xi'_{nk}) = \mathscr{L}(\xi''_{nk}) = \mu_n^{(p_n)}$ for each k, Proposition 2.1 gives

$$| \mathscr{L}(\sum_{k=1}^{k'_n} \xi'_{nk} + \sum_{k=1}^{k''_n} \xi''_{nk})(A) - (\mu_n^{(p_n)})^{k'_n+k''_n}(A) | \le j_n(p_n + q_n)^{-1}\phi(q_n)$$

for every $A \in \mathscr{B}$. Hence $(\mu_n^{(p_n)})^{k'_n+k''_n} \to_w \nu$ and then, by Theorem 2.1 of [2], $(\mu_n^{(p_n)})^{k'_n} \to_w \nu^t$.

We can argue as above to prove that $\sum_{j=1}^{r_n} X_{nj} - \sum_{k=1}^{k'_n} \xi'_{nk} \to_P 0$ and that $\mathscr{L}(\sum_{k=1}^{k'_n} \xi'_{nk}) \to_w \nu^t$; then $\mu_n^{(r_n)} \to_w \nu^t$. □

The following result is a version for the stationary ϕ-mixing case of a theorem of Le Cam [3, Theorem 2.2].

3.4 THEOREM. *Let $\{X_{nj}\}$ be a triangular array with stationary sums which is ϕ-mixing with $\phi(1) < 1$ and satisfies condition (*). Suppose that $\{\mathscr{L}(S_n)\}$ is relatively compact. Then for every $\varepsilon > 0$ the set $\{j_n \mathscr{L}(X_{n1}) \mid B_\varepsilon^c\}$ is relatively compact.*

PROOF. By an argument with subsequences we may suppose that $\mathscr{L}(S_n) \to_w \nu$. We will show that (a) $\sup_n j_n P[\| X_{n1} \| > \varepsilon] < \infty$ for every $\varepsilon > 0$, and (b) for every $\varepsilon > 0$ there exists a compact set K_ε such that $\sup_n j_n P[X_{n1} \in K_\varepsilon^c] \le \varepsilon$.

To prove (a), fix $\varepsilon > 0$, take α such that $\phi(1) < \alpha < 1$ and let $\eta = \min\{1 - \alpha, (\frac{1}{2})(\alpha - \phi(1))(1 - \phi(1)), \varepsilon/4\}$. Choose $t_0 \in (0, 1)$ such that $\sup_{0 \le t \le t_0} \nu^t(B^c_{4^{-1}\varepsilon - 2^{-1}\eta}) < \eta/2$ and $n_0 \in N$ such that $\max_{1 \le k \le j_n} \rho(\mu_n^{(k)}, \nu^{k/j_n}) < \eta/2$ for $n \ge n_0$ (the choices of n_0 and t_0 are possible by Theorem 3.3 and the fact that $\nu_t \to_w \delta_0$ as $t \to 0$); let $\{r_n\} \subset N$ such that $r_n/j_n \le t_0$ and $r_n/j_n \to t_0$. By the definition of ρ we have for $n \ge n_0$ and $k = 1, \cdots, r_n$

$$\mu_n^{(k)}(B^c_{\varepsilon/4}) < \nu^{k/j_n}(B^c_{4^{-1}\varepsilon - 2^{-1}\eta}) + \eta/2 < \eta;$$

therefore by Proposition 2.3 we obtain

$$r_n P[X_{n1} \in B^c_\varepsilon] \le \eta((\alpha - \phi(1))(1 - \phi(1)) - \eta)^{-1} \le 1$$

for $n \ge n_0$. Choosing $n_1 \ge n_0$ such that $t_0/2 \le r_n/j_n$ if $n \ge n_1$, we have $j_n P[X_{n1} \in B^c_\varepsilon] \le 2t_0^{-1}$ for $n \ge n_1$. Then (a) is proved.

In order to prove (b), let $\varepsilon \in (0, 1)$ and take α as above. Theorem 3.3 implies that there exists a compact, convex, symmetric set K_ε such that

$$\sup_{1 \le k \le j_n, n \in N} \mu_n^{(k)}((4^{-1}K_\varepsilon)^c) \le \min\{1 - \alpha, \varepsilon 2^{-1}(\alpha - \phi(1))(1 - \phi(1))\}.$$

Then by Proposition 2.3 we have $j_n P[X_{n1} \in K^c_\varepsilon] \le \varepsilon$ for every n. □

3.5 PROPOSITION. *Let $\{X_{nj}\}$ be a triangular array with stationary sums which is ϕ-mixing with $\phi(1) < 1$ and satisfies condition (*). If $\{\mathscr{L}(S_n)\}$ is relatively compact and there exists M such that $\| X_{nj} \| \le M$ a.s. (for all n, j) then $\sup_n E \exp(\lambda \| S_n \|) < \infty$ for some $\lambda > 0$.*

PROOF. From the relative compactness of $\{\mathscr{L}(S_n)\}$ we deduce that of $\{\mu_n^{(k)}: k = 1, \cdots, j_n, n \in N\}$ by an argument with subsequences and Theorem 3.3. Fix α such that $\phi(1) < \alpha < 1$ and choose $t_0 > 0$ such that

$$\sup_{1 \le k \le j_n, n \in N} \mu_n^{(k)}(B^c_{t_0/4}) \le \min\{1 - \alpha, (\alpha - \phi(1))^2\}.$$

By Proposition 2.4 one has, for $\ell, n \in N$, $P[\| S_n \| > \ell(t_0 + M)] \le \alpha^\ell$. Write $c = t_0 + M$ and take $\lambda > 0$ such that $\alpha e^{\lambda c} < 1$; then, we have for every n

$$E \exp(\lambda \| S_n \|) = 1 + \int_0^\infty \lambda e^{\lambda t} P[\| S_n \| > t]\, dt \le e^{\lambda c} \textstyle\sum_{\ell=0}^\infty (\alpha e^{\lambda c})^\ell < \infty. \quad \square$$

4. Gaussian limits. From now on, given a ϕ-mixing triangular array $\{X_{nj}\}$ with stationary sums which satisfies condition (*), we consider sequences $\{p_n\}$, $\{q_n\}$ with the properties of Lemma 3.2 where we take $\sigma_n = \sigma(X_{n1})$; also, we write:

$$k_n = [j_n(p_n + q_n)^{-1}] \quad ([.] \text{ denotes the integer part of a real number}),$$

$$P(n, k) = \{j \in N: (k - 1)(p_n + q_n) + 1 \le j \le (k - 1)(p_n + q_n) + p_n\}$$

and

$Q(n, k) = \{j \in N: (k-1)(p_n + q_n) + p_n + 1 \le j \le k(p_n + q_n)\}$ if $k = 1, \cdots, k_n$,

$$Q(n, k_n + 1) = \{j \in N: k_n(p_n + q_n) + 1 \le j \le j_n\},$$

$$\xi_{nk} = \sum_{j \in P(n,k)} X_{nj} \quad \text{if} \quad k = 1, \cdots, k_n,$$

$$\eta_{nk} = \sum_{j \in Q(n,k)} X_{nj} \quad \text{if} \quad k = 1, \cdots, k_n + 1.$$

This grouping in blocks will be used (always with this meaning) in some proofs, the first of which is that of the following result.

4.1 THEOREM. *Let $\{X_{nj}\}$ be a triangular array with stationary sums which is ϕ-mixing with ϕ (1) < 1 and satisfies condition (*). Suppose that $\mathscr{L}(S_n) \to_w \nu$. Then ν is Gaussian if and only if $j_n P[\| X_{n1} \| > \varepsilon] \to 0$ for every $\varepsilon > 0$.*

PROOF. *Necessity.* Assume that ν is Gaussian. Arguing as in the proof of Theorem 3.3(1), we obtain that $\mathscr{L}(\xi_{n1})^{k_n} \to_w \nu$ and then ([3, Corollary 2.11]) $k_n P[\| \xi_{n1} \| > \varepsilon] \to 0$ for every $\varepsilon > 0$.

Fix $\varepsilon > 0$. Choose α such that $\phi(1) < \alpha < 1$ and let $\eta = \min\{1 - \alpha, 2^{-1}(\alpha - \phi(1))(1 - \phi(1)), \varepsilon/4\}$; take $t_0 \in (0, 1)$ such that $\sup_{0 \le t \le t_0} \nu^t(B^c_{4^{-1}\varepsilon - 2^{-1}\eta}) < \eta/2$ and $n_0 \in N$ such that if $n \ge n_0$ then $p_n/j_n \le t_0$ and $\max_{1 \le k \le j_n} \rho(\mu_n^{(k)}, \nu^{k/j_n}) < \eta/2$. Therefore if $n \ge n_0$ and $1 \le k \le p_n$ we have $\mu_n^k(B^c_{\varepsilon/4}) < \eta$ and Proposition 2.3 gives, writing $c = 2((\alpha - \phi(1))(1 - \phi(1)))^{-1}$, the inequality $p_n P[\| X_{n1} \| > \varepsilon] \le cP[\| \xi_{n1} \| > \varepsilon/4]$; then for n large enough we have $j_n P[\| X_{n1} \| > \varepsilon] \le 2k_n p_n P[\| X_{n1} \| > \varepsilon] \le 2\, ck_n P[\| \xi_{n1} \| > \varepsilon/4]$. Hence $\lim_n j_n P[\| X_{n1} \| > \varepsilon] = 0$.

Sufficiency. We may suppose that $B = \mathbb{R}$ (apply functionals $f \in B'$ to deduce the general case from this). Let μ be the Lévy measure of ν and assume that $j_n P[| X_{n1} | \ge \varepsilon] \to 0$ for every $\varepsilon > 0$.

For a fixed $M > 0$, consider the triangular array $\{X_{njM}\}$; define $\tilde{\xi}_{nk} = \sum_{j \in P(n,k)} X_{njM}$ for $k = 1, \cdots, k_n$ and $\tilde{\eta}_{nk} = \sum_{j \in Q(n,k)} X_{njM}$ for $k = 1, \cdots, k_n + 1$. As in the proof of Theorem 3.3(1) we can obtain that $S_{n,M} - \sum_{k=1}^{k_n} \tilde{\xi}_{nk} \to_P 0$ because $\sigma(X_{n1M}) \le \sigma(X_{n1})$ and $\{X_{njM}\}$ has the property (*) (write $\sum_{j=1}^{r_n} X_{njM} = \sum_{j=1}^{r_n} X_{nj} - \sum_{j=1}^{r_n} X_{nj}^M$ and observe that $P[| \sum_{j=1}^{r_n} X_{nj}^M | > 0] \le r_n P[| X_{n1} | > M]$). Since $S_n^{(M)} \to_P 0$ we have also that $\mathscr{L}(\tilde{\xi}_{n1}) * \cdots * \mathscr{L}(\tilde{\xi}_{nk_n}) \to_w \nu$; moreover, $\{\mathscr{L}(\tilde{\xi}_{nk})\}$ is infinitesimal (given $\varepsilon > 0$, write $\max_{1 \le k \le k_n} P[| \tilde{\xi}_{nk} | > \varepsilon] \le P[| \xi_{n1} | > \varepsilon] + p_n P[| X_{n1} | > M]$ and note that $p_n/j_n \to 0$). Now, we may apply the converse central limit theorem of the independent case [3, Theorem 2.10] to conclude that, for every $\tau \in C(\mu)$, $\sum_{k=1}^{k_n} \mathscr{L}(\tilde{\xi}_{nk}) | B^c_\tau \to_w \mu | B^c_\tau$.

We will prove that $\mu(B^c_\varepsilon) = 0$ for every $\varepsilon > 0$; this will show that ν is Gaussian. Fix $\varepsilon > 0$. Let α be such that $\phi(1) < \alpha < 1$ and take an integer $\ell \ge 2$; put $M = \varepsilon(2(\ell - 1))^{-1}$, $t = \varepsilon(2\ell)^{-1}$. Choose $n_0 = n_0(\varepsilon, \alpha, \ell) \in N$ such that

$$\max_{1 \le k \le k_n} \max_{i \in P(n,k)} P[| \textstyle\sum_{j \in P(n,k), j \le i} X_{njM} | > t/4] \le 1 - \alpha$$

if $n \ge n_0$ (the left member is less than or equal to $\max_{1 \le i \le p_n} P[| S_{ni} | > t/4] + p_n P[| X_{n1} | > M]$ which goes to zero as $n \to \infty$ by the hypothesis and the property

(*) of $\{X_{nj}\}$). Now, let ξ_{nk} be the r.v.'s associated to M as above; applying Proposition 2.4 to the r.v.'s X_{njM} and writing $a = (\alpha - \phi(1))^{-1}$, one has, for $n \geq n_0$ and $1 \leq k \leq k_n$,

$$P[|\xi_{nk}| > \varepsilon] \\ \leq p_n P[|X_{n1}| > M] + (\phi(1) + aP[|\tilde{\xi}_{nk}| > t/4])^{\ell-1} aP[|\tilde{\xi}_{nk}| > t/2].$$

Arguing again as in the proof of Theorem 3.3 we obtain that $\mathscr{L}(\xi_{n1})^{k_n} \to_w \nu$; then [3, Theorem 2.10] shows that $k_n \mathscr{L}(\xi_{n1}) | B_\tau^c \to_w \mu | B_\tau^c$ for every $\tau \in C(\mu)$. Hence, by hypothesis and the preceding arguments, we have

$$\begin{aligned} \mu(B_\varepsilon^c) &\leq \liminf_n k_n P[|\xi_{n1}| > \varepsilon] \\ &\leq \limsup_n k_n p_n P[|X_{n1}| > M] \\ &\quad + (\phi(1) + a \limsup_n \max_{1 \leq k \leq k_n} P[|\tilde{\xi}_{nk}| > t/4])^{\ell-1} \\ &\quad \cdot a \limsup_n \textstyle\sum_{k=1}^{k_n} P[|\tilde{\xi}_{nk}| > t/2] \\ &\leq \phi(1)^{\ell-1} a \mu((\mathring{B}_{t/2})^c) \end{aligned}$$

($\mathring{A}$ denotes the interior of A). We have proved that for every integer $\ell \geq 2$ it holds that

$$\mu(B_\varepsilon^c) \leq a\phi(1)^{\ell-1} \mu((\mathring{B}_{\varepsilon/4\ell})^c).$$

It follows that for any $r > 0$ and for each integer $\ell \geq 2$

$$\mu(B_\varepsilon^c) \leq a \left\{ \phi(1)^{\ell-1} \ell^2 (4/\varepsilon)^2 \int_{B_r} x^2 \mu(dx) + \mu(B_r^c) \right\}.$$

First, since $\phi(1) < 1$ and $\int_{B_r} x^2 \mu(dx) < \infty$ ([3, Theorem 1.4]) we obtain, letting $\ell \to \infty$, that $\mu(B_\varepsilon^c) \leq a\mu(B_r^c)$ for each $r > 0$. Then, letting $r \to \infty$, we conclude that $\mu(B_\varepsilon^c) = 0$ because $\mu(B_\varepsilon^c) < \infty$ ([3, Theorem 1.4]). □

Next, we give necessary conditions for convergence to a Gaussian measure.

4.2 THEOREM. *Let $\{X_{nj}\}$ be a stationary triangular array which is ϕ-mixing with $\phi(1) < 1$ and satisfies condition (*). Suppose that $\mathscr{L}(S_n) \to_w \delta_z * \gamma$, where $z \in B$ and γ is a centered Gaussian measure. Then for every $\delta > 0$,*

(a) $j_n P[\|X_{n1}\| > \delta] \to 0$,
(b) $\lim_n Ef^2(S_{n,\delta} - ES_{n,\delta}) = \Phi_\gamma(f, f)$ *for each* $f \in B'$,
(c) $\mathscr{L}(S_n - ES_{n,\delta}) \to_w \gamma$, $S_n^{(\delta)} \to_P 0$, $\mathscr{L}(S_{n,\delta} - ES_{n,\delta}) \to_w \gamma$ *and* $ES_{n,\delta} \to z$ *in* B.

PROOF. The previous theorem gives (a) which in turn implies that $S_n^{(\delta)} \to_P 0$ for every $\delta > 0$ (write $P[\|S_n^{(\delta)}\| > 0] \leq j_n P[\|X_{n1}\| > \delta]$).

Fix $\delta > 0$. Since $S_n = S_{n,\delta} + S_n^{(\delta)}$ one has $\mathscr{L}(S_{n,\delta}) \to_w \delta_z * \gamma$. On the other hand, $\{X_{nj\delta}\}$ satisfies the hypotheses of Proposition 3.5 (to verify (*) write

$P[\| \sum_{j=1}^{r_n} X_{nj\delta} \| > \varepsilon] \leq P[\| \sum_{j=1}^{r_n} X_{nj} \| > \varepsilon] + r_n P[\| X_{n1} \| > \delta]$); by standard arguments we have then that $\lim_n ES_{n,\delta} = \int x\, \delta_z * \gamma(dx) = z$ in B and $\lim_n Ef^2(S_{n,\delta}) = \int f^2\, d(\delta_z * \gamma) = f^2(z) + \Phi_\gamma(f, f)$ for each $f \in B'$. From these facts we can deduce the remaining conclusions. □

Given a stationary triangular array $\{X_{nj}: j = 1, \cdots, j_n, n \in N\}$, $\delta > 0$ and $f \in B'$ we write

$$V_n(\delta, f) = j_n Ef^2(X_{n1\delta} - EX_{n1\delta}) + 2j_n \sum_{j=1}^{j_n - 1} E[f(X_{n1\delta} - EX_{n1\delta}) f(X_{n,j+1,\delta} - EX_{n,j+1,\delta})].$$

4.3 COROLLARY. *Let $\{X_{nj}\}$ be as in Theorem 4.2. If $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$ and there exists $\delta > 0$ such that for every $f \in B'$*

$$C_{\delta,f} = \sup_n j_n Ef^2(X_{n1\delta} - EX_{n1\delta}) < \infty$$

then (b′) $\lim_n V_n(\delta, f) = \Phi_\gamma(f, f)$ *for each* $f \in B'$.

PROOF. Fix δ and f as in the statement and put $Y_{nj} = f(X_{nj\delta} - EX_{nj\delta})$. By stationarity, we have the equalities (see, for example, Iosifescu and Theodorescu [13, page 24])

$$Ef^2(S_{n,\delta} - ES_{n,\delta}) = E(\sum_{j=1}^{j_n} Y_{nj})^2 = j_n EY_{n1}^2 + 2 \sum_{j=1}^{j_n - 1} (j_n - j) EY_{n1} Y_{n,j+1} = V_n(\delta, f) - 2 \sum_{j=1}^{j_n - 1} jEY_{n1} Y_{n,j+1}.$$

But Proposition 2.5 gives (note that $EY_{n1} = 0$)

$$| \sum_{j=1}^{j_n - 1} jEY_{n1} Y_{n,j+1} | \leq 2(j_n^{-1} \sum_{j=1}^{j_n - 1} j\phi^{1/2}(j)) C_{\delta,f}$$

which goes to zero as $n \to \infty$ by the convergence of the series $\sum \phi^{1/2}(j)$. The desired conclusion now follows from Theorem 4.2. □

REMARK. Let $\{X_{nj}\}$ be as in Theorem 4.2. If $\sum_{j=1}^{\infty} \phi^{1/2}(j) < 1/4$ then $C_{\delta,f}$ (defined as in the corollary) is finite for each $\delta > 0$ and $f \in B'$; hence, assertion (b′) holds.

In fact, fixing δ and f and writing $Y_{nj} = f(X_{nj\delta} - EX_{nj\delta})$ one has by Proposition 2.5 (see above)

$$Ef^2(S_{n,\delta} - ES_{n,\delta}) \geq j_n EY_{n1}^2 - 4 \sum_{j=1}^{j_n - 1} (j_n - j) \phi^{1/2}(j) EY_{n1}^2 \geq \{1 - 4 \sum_{j=1}^{\infty} \phi^{1/2}(j)\} j_n EY_{n1}^2;$$

to conclude the proof observe that $\sup_n Ef^2(S_{n,\delta} - ES_{n,\delta}) < \infty$.

In the following results, we shall give sufficient conditions for convergence to a Gaussian law. For any subspace F of B we write $d_F(x) = \inf\{\| x - y \| : y \in F\}$. If B is a separable Hilbert space we denote $d_k = d_{F_k}$ the distance to the subspace F_k spanned by $\{e_1, \cdots, e_k\}$, where $\{e_i: i \in N\}$ is a fixed (but arbitrary) orthonormal basis of B, when B is infinite-dimensional; if the dimension of B is finite we have an orthonormal basis $\{e_1, \cdots, e_n\}(n \in N)$ and we put $d_k = 0$ for $k \geq n$.

4.4 THEOREM. *Let* $\{X_{nj}\}$ *be a stationary triangular array which is* ϕ*-mixing with* $\phi(1) < 1$ *and such that*

(1) *for some* $\alpha > 0$, *the triangular array* $\{X_{nj\alpha} - EX_{nj\alpha}\}$ *satisfies* (*),

(2) *for every* $\varepsilon > 0$, $j_n P[\| X_{n1} \| > \varepsilon] \to 0$,

(3) *there exists a sequentially* w^**-dense subset* W *of* B' *and* $\delta > 0$ *such that*

$$\Phi(f) = \lim_n Ef^2(S_{n,\delta} - ES_{n,\delta})$$

exists for every $f \in W$,

(4) *there exist* $\beta > 0$, $p > 0$ *and a sequence* $\{F_k\}$ *of finite-dimensional subspaces of* B *such that*

$$\lim_k \sup_n Ed^p_{F_k}(S_{n,\beta} - ES_{n,\beta}) = 0.$$

Then (a) *there exists a centered Gaussian measure* γ *such that* $\Phi_\gamma(f, f) = \Phi(f)$ *for every* $f \in W$, (b) *for every* $\tau > 0$, $\mathscr{L}(S_n - ES_{n,\tau}) \to_w \gamma$.

PROOF. We may assume that $\alpha = \delta = \beta$ (this fact is a consequence of (2)).

Given $f \in W$, by (3) we have $C_f = \sup_n Ef^2(S_{n,\delta} - ES_{n,\delta}) < \infty$ and by Chebyshev's inequality we obtain

$$P[|f(S_{n,\delta} - ES_{n,\delta})| > t] \le t^{-2} C_f$$

for each $t > 0$; then $\{\mathscr{L}(f(S_{n,\delta} - ES_{n,\delta}))\}$ is relatively compact. On the other hand, (4) and Chebyshev's inequality imply that

$$\lim_k \sup_n P[d_{F_k}(S_{n,\delta} - ES_{n,\delta}) > s] = 0$$

for every $s > 0$. Therefore an application of [1, Theorem 2.3] shows that $\{\mathscr{L}(S_{n,\delta} - ES_{n,\delta})\}$ is relatively compact.

Write $Y_{nj} = X_{nj\delta} - EX_{nj\delta}$. The triangular array $\{Y_{nj}\}$ is stationary, ϕ-mixing with $\phi(1) < 1$ and satisfies (*) by (1). We will prove now that $j_n P[\| Y_{n1} \| > \varepsilon] \to 0$ for every $\varepsilon > 0$. To do this, note first that $EX_{n1\delta} \to 0$ in B (we have $\| EX_{n1\delta} \| \le \eta + \delta P[\| X_{n1} \| > \eta]$ for each $\eta > 0$); next observe that, given $\varepsilon > 0$, if n is large enough to have $\| EX_{n1\delta} \| \le \varepsilon/2$, one has $j_n P[\| Y_{n1} \| > \varepsilon] \le j_n P[\| X_{n1} \| > \varepsilon/2]$ and it suffices to apply (2).

Let $\{n'\}$ be a sequence of integers such that $\{\mathscr{L}(S_{n',\delta} - ES_{n',\delta})\}$ converges weakly. By Theorems 4.1 and 4.2 applied to $\{Y_{nj}\}$, its limit is a Gaussian measure γ with zero expectation whose covariance satisfies $\Phi_\gamma(f, f) = \Phi(f)$ for every $f \in W$ (observe that $Y_{nj,2\delta} = Y_{nj}$).

In view of the preceding argument, the compactness of $\{\mathscr{L}(S_{n,\delta} - ES_{n,\delta})\}$ implies the existence of the desired γ and the convergence to it of the whole sequence. Since, by (2), $S_n^{(\delta)} \to_P 0$ we have $\mathscr{L}(S_n - ES_{n,\delta}) \to_w \gamma$ and then, using (2) again, we deduce that $\mathscr{L}(S_n - ES_{n,\tau}) \to_w \gamma$ for every $\tau > 0$ (if, for example, $\delta < \tau$ we have $\| ES_{n,\tau} - ES_{n,\delta} \| \le \tau j_n P[\| X_{n1} \| > \delta]$). □

REMARK. If B is finite-dimensional, hypothesis (4) of the previous theorem may be omitted; a similar remark applies to the next results (and to Theorems 5.7 and 6.4 below).

The following corollaries give sufficient conditions for convergence expressed in terms of the individual random vectors and pairs of them. As an additional hypothesis, it is required that the dependence coefficient $\phi(j)$ converge to zero at a certain speed.

4.5 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_{nj}\}$ be a stationary triangular array which is ϕ-mixing with $\phi(1) < 1$ and $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume*

(1) *for every $\varepsilon > 0$, $j_n P[\| X_{n1} \| > \varepsilon] \to 0$,*

(2) *there exists $\delta > 0$ such that for every $f \in B'$*

$$C_{\delta,f} = \sup_n j_n E f^2(X_{n1\delta} - EX_{n1\delta}) < \infty$$

and the limit

$$\Phi(f) = \lim_n V_n(\delta, f) \quad \textit{exists},$$

(3) *there exists $\beta > 0$ such that*

$$\lim_k \sup_n j_n E d_k^2(X_{n1\beta} - EX_{n1\beta}) = 0.$$

Then there exists a centered Gaussian measure γ with covariance $\Phi_\gamma(f, f) = \Phi(f)(f \in B')$ such that $\mathscr{L}(S_n - ES_{n,\tau}) \to_w \gamma$ for every $\tau > 0$.

PROOF. We suppose that B is infinite-dimensional (otherwise the proof is simpler). Let $\langle \cdot, \cdot \rangle$ denote the inner product of B and let $\{e_i : i \in N\}$ be an orthonormal basis.

We will show that the hypotheses of Theorem 4.4 are verified. In view of (1) we may assume that $\delta = \beta$. Let $Y_{nj} = X_{nj\delta} - EX_{nj\delta}$. First, we prove that $\{Y_{nj}\}$ satisfies (*). For this purpose, take $\{r_n\} \subset N$ such that $r_n \le j_n$ and $r_n / j_n \to 0$; by stationarity and Proposition 2.5, writing $U_{nji} = \langle Y_{nj}, e_i \rangle$, we have

$$\begin{aligned} E d_k^2\left(\textstyle\sum_{j=1}^{r_n} Y_{nj}\right) &= E\left(\textstyle\sum_{i=k+1}^{\infty} \langle \sum_{j=1}^{r_n} Y_{nj}, e_i \rangle^2\right) = \textstyle\sum_{i=k+1}^{\infty} E\left(\sum_{j=1}^{r_n} U_{nji}\right)^2 \\ &= \textstyle\sum_{i=k+1}^{\infty} \left(r_n E U_{n1i}^2 + 2 \sum_{j=1}^{r_n} (r_n - j) E(U_{n1i} U_{n,j+1,i})\right) \\ &\le \left(1 + 4 \textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j)\right) r_n E d_k^2(Y_{n1}) \end{aligned}$$

and

$$\begin{aligned} E f^2\left(\textstyle\sum_{j=1}^{r_n} Y_{nj}\right) &= r_n E f^2(Y_{n1}) + 2 \textstyle\sum_{j=1}^{r_n - 1} (r_n - j) E(f(Y_{n1}) f(Y_{n,j+1})) \\ &\le r_n j_n^{-1} \left(1 + 4 \textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j)\right) C_{\delta,f} \end{aligned}$$

for every $f \in B'$. Then, applying Chebyshev's inequality twice and [1, Theorem 2.3], it follows from our hypotheses that $\sum_{j=1}^{r_n} Y_{nj} \to_P 0$.

Similarly, we obtain the inequality

$$E d_k^2(S_{n,\delta} - ES_{n,\delta}) \le \left(1 + 4 \textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j)\right) j_n E d_k^2(Y_{n1})$$

which shows that (4) of Theorem 4.4 holds with $p = 2$. Finally, in order to prove that assumption (3) of that result also holds we observe that, for each $f \in B'$,

$$|Ef^2(S_{n,\delta} - ES_{n,\delta}) - V_n(\delta, f)| = |2\sum_{j=1}^{j_n-1} jE(f(Y_{n1})f(Y_{n,j+1}))|$$
$$\leq 4(j_n^{-1} \sum_{j=1}^{j_n-1} j\phi^{1/2}(j))C_{\delta,f}$$

which goes to zero as $n \to \infty$. □

4.6 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_{nj}\}$ be a stationary triangular array which is ϕ-mixing with $\phi(1) < 1$ and $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume*

(1) $E\|X_{n1}\|^2 < \infty$, $EX_{n1} = 0$,
(2) *for every* $\varepsilon > 0$, $\lim_n j_n E[\|X_{n1}\|^2 I_{[\|X_{n1}\| > \varepsilon]}] = 0$,
(3) *for every* $f \in B'$,

$$C_f = \sup_n j_n Ef^2(X_{n1}) < \infty$$

and $\Phi(f) = \lim_n \{j_n Ef^2(X_{n1}) + 2j_n \sum_{j=1}^{j_n-1} E(f(X_{n1})f(X_{n,j+1}))\}$ *exists,*
(4) $\lim_k \sup_n j_n Ed_k^2(X_{n1}) = 0$.

Then there exists a centered Gaussian measure γ with covariance $\Phi_\gamma(f, f) = \Phi(f)(f \in B')$ such that $\mathscr{L}(S_n) \to_w \gamma$.

PROOF. We will show that $\{X_{nj}\}$ satisfies the hypotheses of Corollary 4.5. Condition (1) of Corollary 4.5 follows from the inequality $j_n P[\|X_{n1}\| > \varepsilon] \leq \varepsilon^{-2} j_n E[\|X_{n1}\|^2; \|X_{n1}\| > \varepsilon]$, valid for every $\varepsilon > 0$.

Fix now any $\delta > 0$. With the notation of (2) of the previous result, we have $C_{\delta,f} \leq C_f$ for every $f \in B'$ and this implies the first part of that condition. To verify the second, fix $f \in B'$. Since $Ef(X_{n1}) = 0$ we have

$$j_n Ef^2(X_{n1}) - j_n Ef^2(X_{n1\delta} - EX_{n1\delta}) = j_n Ef^2(X_{n1}^\delta) + j_n(Ef(X_{n1}^\delta))^2$$
$$\leq 2\|f\|^2 j_n E[\|X_{n1}\|^2; \|X_{n1}\| > \delta],$$

which tends to zero as $n \to \infty$, and

$$\begin{aligned} &j_n \sum_{j=1}^{j_n-1} E(f(X_{n1})f(X_{n,j+1})) \\ &\quad - j_n \sum_{j=1}^{j_n-1} E(f(X_{n1\delta} - EX_{n1\delta})f(X_{n,j+1,\delta} - EX_{n,j+1,\delta})) \\ &\quad = j_n \sum_{j=1}^{j_n-1} (E(f(X_{n1})f(X_{n,j+1})) - E(f(X_{n1\delta})f(X_{n,j+1,\delta}))) \\ &\quad\quad + j_n(j_n - 1)(Ef(X_{n1\delta}))^2 \\ &\quad = a_n + b_n \quad \text{(say)}. \end{aligned}$$

We will prove that $\{a_n\}$ and $\{b_n\}$ both converge to zero. Since $Ef(X_{n1}) = 0$ we have

$$b_n \leq (j_n Ef(X_{n1}^\delta))^2 \leq (\delta^{-1}\|f\| j_n E[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^2$$

which goes to zero as $n \to \infty$. On the other hand, observe that

$$|E(f(X_{n1})f(X_{n,j+1})) - E(f(X_{n1\delta})f(X_{n,j+1,\delta}))|$$
$$= |E[f(X_{n1})f(X_{n,j+1}); \|X_{n1}\| > \delta \text{ or } \|X_{n,j+1}\| > \delta]|$$
$$\leq 6\|f\|\phi^{1/2}(j)(E[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^{1/2}(Ef^2(X_{n1}))^{1/2}$$
$$+ 3\delta^{-2}\|f\|^2(E[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^2$$

since, for example, one has by Proposition 2.5 (recall that $Ef(X_{n1}) = 0$)

$$|E[f(X_{n1})f(X_{n,j+1}); \|X_{n1}\| > \delta, \|X_{n,j+1}\| \leq \delta]|$$
$$\leq 2\phi^{1/2}(j)(E[f^2(X_{n1}); \|X_{n1}\| > \delta])^{1/2}(Ef^2(X_{n1}))^{1/2}$$
$$+ |E[f(X_{n1}); \|X_{n1}\| > \delta]|\,|E[f(X_{n1}); \|X_{n1}\| \leq \delta]|$$
$$\leq 2\|f\|\phi^{1/2}(j)(E[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^{1/2}(Ef^2(X_{n1}))^{1/2}$$
$$+ (\delta^{-1}\|f\|E[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^2$$

and the other two terms which are involved have the same bound; then

$$a_n \leq 6\|f\|(\textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j))(j_nE[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^{1/2}C_f^{1/2}$$
$$+ 3\delta^{-2}\|f\|^2(j_nE[\|X_{n1}\|^2; \|X_{n1}\| > \delta])^2$$

which tends to zero as $n \to \infty$. This implies that assumption (2) of Corollary 4.5 holds with the Φ given in our hypothesis (3).

In order to prove (3) of the previous corollary it is sufficient to remark that $Ed_k^2(X_{n1\delta} - EX_{n1\delta}) = Ed_k^2(X_{n1\delta}) - d_k^2(EX_{n1\delta}) \leq Ed_k^2(X_{n1})$ (to prove it write down the first member in terms of coordinates).

Now, Corollary 4.5 proves the existence of the desired γ and that $\mathscr{L}(S_n - ES_{n,\tau}) \to_w \gamma$ for each $\tau > 0$; but for such a τ one has $\|ES_{n,\tau}\| = \|j_nEX_{n1}^{\tau}\| \leq \tau^{-1}j_nE[\|X_{n1}\|^2; \|X_{n1}\| > \tau]$ which tends to zero. This completes the proof. □

We can deduce easily the following

4.7 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_j: j \in N\}$ be a stationary sequence which is ϕ-mixing with $\phi(1) < 1$ and $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume $E\|X_1\|^2 < \infty$ and $EX_1 = 0$. Then for every $f \in B'$ the sum*

$$\Phi(f) = Ef^2(X_1) + 2\textstyle\sum_{j=1}^{\infty} E(f(X_1)f(X_{j+1}))$$

converges and defines the covariance of a centered Gaussian measure γ which satisfies $\mathscr{L}(n^{-1/2}\sum_{j=1}^{n} X_j) \to_w \gamma$.

REMARK. In the case $B = \mathbb{R}$ and without the restriction $\phi(1) < 1$, Corollary 4.7 was proved by Ibragimov (Ibragimov and Linnik [12, Theorem 18.5.2]) by different methods. Let us point out, omitting the proof, that by using the result of this author and de Acosta [1, Theorem 2.3] one can obtain Corollary 4.7 without the assumption $\phi(1) < 1$. (The referee has informed us that this result has appeared in an article by V. V. Malt'tsev and E. I. Ostrovskii (*Teor. Veroj.* **27** 2, June 1982).)

As an application, let us observe that from this result in the Hilbert space case it is possible to calculate, using an argument in Araujo and Giné [5, page 180], the limit distribution of the Cramér-von Mises statistic of certain ϕ-mixing stationary sequences of random variables; let us observe that in Billingsley [6, Theorem 22.1] the limit distribution of the whole empirical process of such sequences is given under the stronger assumption $\sum_{j=1}^{\infty} j^2\phi^{1/2}(j) < \infty$. The result that we can derive is this: Let $\{X_j\}$ be a stationary sequence of real random variables which is ϕ-mixing with $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume that X_1 has a continuous distribution function F; denote by F_n the nth empirical distribution function of $\{X_j\}$. Then

$$\mathscr{L}\left(n \int_{-\infty}^{+\infty} (F_n(x) - F(x))^2\, dF(x)\right) \to_w \mathscr{L}(\textstyle\sum_{k=1}^{\infty} \eta_k^2)$$

where $\{\eta_k : k \in N\}$ is a sequence of Gaussian real random variables with $E\eta_k = 0$ and

$$E\eta_h\eta_k = 2(hk\pi^2)^{-1}\{2^{-1}\delta_{hk} + \textstyle\sum_{j=1}^{\infty} E[\cos h\pi F(X_1).\ \cos k\pi F(X_{j+1})]$$
$$+ \textstyle\sum_{j=1}^{\infty} E[\cos h\pi F(X_{j+1}).\ \cos k\pi F(X_1)]\}$$

where $\delta_{hk} = 1$ if $h = k$, $= 0$ if $h \neq k$.

Next, we give an almost sure invariance principle. Its proof is carried out by first obtaining from our Proposition 3.1 an invariance principle in probability and then deriving from this the desired result; in both steps we use arguments of de Acosta [2]. The remark that one can deduce, in the independent case, an almost sure invariance principle from the invariance principle in probability of de Acosta [2, Theorem 3.1] is due to H. Dehling and W. Philipp. An invariance principle in probability for stationary, ϕ-mixing sequences is given in Philipp [16, Theorem 4].

4.8 THEOREM. *Let $\{X_{nj}\}$ be a triangular array with stationary sums which is ϕ-mixing with $\phi(1) < 1$. Assume*

(1) $X_{n1} \to_P 0$,

(2) *for every $\varepsilon > 0$ there exists $a > 0$ such that*

$$\limsup_n \max_{1\le k\le [aj_n]} P[\| S_{nk} \| > \varepsilon] < 1 - \phi(1),$$

(3) $\mathscr{L}(S_n) \to_w \gamma$ *for some Gaussian measure γ.*

Then there exist a probability space and two triangular arrays $\{X'_{nj}\}$ and $\{Y_{nj}\}$ defined on it such that

(a) $\mathscr{L}(X'_{n1}, \cdots, X'_{nj_n}) = \mathscr{L}(X_{n1}, \cdots, X_{nj_n})$ *for each $n \in N$,*

(b) $Y_{n1}, \cdots, Y_{nj_n}$ *are independent, identically distributed with $\mathscr{L}(Y_{n1}) = \gamma^{1/j_n}$ for each $n \in N$,*

(c) $\max_{1\le k\le j_n}\| S'_{nk} - T_{nk}\| \to 0$ a.s. *as* $n \to \infty$
where $S'_{nk} = \sum_{j=1}^{k} X'_{nj}$, $T_{nk} = \sum_{j=1}^{k} Y_{nj}$.

SKETCH OF PROOF. As mentioned above, it is sufficient to prove the result with $\to_P$ in place of $\to$ a.s. in assertion (c) (see [2, Addendum]). Call this statement (c′).

For the moment, we shall consider for a given $p \in N$ the product space B^p endowed with the norm $\| x \|_1 = \sum_{j=0}^{p-1} \| x_j \|$ for $x = (x_0, x_1, \cdots, x_{p-1}) \in B^p$ and we shall denote ρ_p the Prohorov distance between probability measures on B^p. We use the notation $I(n, p, k)$ of Proposition 3.1 and write $c(n, p, k) = \operatorname{card} I(n, p, k)$.

For each $p \in N$ choose $n_p \in N$ such that $n_p \uparrow \infty$ as $p \to \infty$ and such that $n \ge n_p$ implies

$$\rho_p(\mathscr{L}(\textstyle\sum_{j\in I(n,p,0)}X_{nj}, \cdots, \sum_{j\in I(n,p,p-1)}X_{nj}), (\gamma^{1/p})^{\otimes p}) < 1/p^2$$

and

$$\rho_p(\otimes_{k=0}^{p-1} \gamma^{c(n,p,k)/j_n} (\gamma^{1/p})^{\otimes p}) < 1/p^2.$$

This choice is possible by Proposition 3.1.

Fix $p \in N$ and $n \in N$ such that $n_p \le n < n_{p+1}$. Then, by a theorem of Strassen [17], there exists a probability measure $\lambda_{n,p}$ on $B^p \times B^p$ such that

$$\lambda_{n,p}(\{(x, y) \in B^p \times B^p \colon \| x - y \|_1 > 2/p^2\}) < 2/p^2,$$

$$\lambda_{n,p}\circ\pi_1^{-1} = \mathscr{L}(\textstyle\sum_{j\in I(n,p,0)}X_{nj}, \cdots, \sum_{j\in I(n,p,p-1)}X_{nj})$$

and

$$\lambda_{n,p}\circ\pi_2^{-1} = \otimes_{k=0}^{p-1} \gamma^{c(n,p,k)/j_n}$$

(π_1, π_2 are the canonical projections defined on $B^p \times B^p$). Let $\alpha_n = \mathscr{L}(X_{n1}, \cdots, X_{nj_n})$, $\beta_n = (\gamma^{1/j_n})^{\otimes j_n}$ and define $\zeta_{n,p}\colon B^{j_n} \to B^p$ by $\zeta_{n,p}(y_1, \cdots, y_{j_n}) = (\sum_{j\in I(n,p,0)} y_j, \cdots, \sum_{j\in I(n,p,p-1)} y_j)$; one has $\alpha_n\circ\zeta_{n,p}^{-1} = \lambda_{n,p}\circ\pi_1^{-1}$ and $\beta_n\circ\zeta_{n,p}^{-1} = \lambda_{n,p}\circ\pi_2^{-1}$. By Theorem A.1 of de Acosta [2] there exist a probability space $(\Omega_n, \mathscr{A}_n, P_n)$ and random vectors $X'_n = (X'_{n1}, \cdots, X'_{nj_n})\colon \Omega_n \to B^{j_n}$, $Y_n = (Y_{n1}, \cdots, Y_{nj_n})\colon \Omega_n \to B^{j_n}$ with $\mathscr{L}(X'_n) = \alpha_n$, $\mathscr{L}(Y_n) = \beta_n$ and $\mathscr{L}(\zeta_{n,p}(X'_n), \zeta_{n,p}(Y_n)) = \lambda_{n,p}$.

We may consider the triangular arrays $\{X'_{nj}\}$, $\{Y_{nj}\}$ defined on the product space of the spaces $(\Omega_n, \mathscr{A}_n, P_n)$. By construction, (a) and (b) hold. Finally, the proof of (c′) is similar to step V of the proof of [2, Theorem 3.1]; Proposition 2.2 must be used and this is possible by our hypothesis (2). □

REMARK. If a triangular array satisfies (*) then (1) and (2) of the previous result are verified. In particular (see 1 of the remark following Proposition 3.1), one has: if $\{X_j\}$ is a stationary ϕ-mixing sequence with $\phi(1) < 1$ and $\{a_n\}$ is a sequence of real numbers tending to infinity such that $\{\mathscr{L}(a_n^{-1} \sum_{j=1}^{n} X_j)\}$ converges weakly to a Gaussian measure then the conclusion of Theorem 4.8 is true for $\{a_n^{-1} X_j : j = 1, \cdots, n, n \in N\}$.

Let $C = C([0, 1], B)$ be the Banach space of continuous functions of $[0, 1]$ into B endowed with the supremum norm and let $D = D([0, 1], B)$ be the space of functions of $[0, 1]$ into B which are right-continuous on $[0, 1)$ and have left limits on $(0, 1]$ equipped with the Skorohod topology ([6, Chapter 3]). Given a Gaussian measure γ on B, we shall denote by W_γ the associated Wiener measure on (the Borel σ-algebra of) C or D. As in de Acosta [2] the following two results can be deduced from Theorem 4.8.

4.9 COROLLARY. *Let $\{X_{nj}\}$ and γ be as in Theorem 4.8. Then there exist a probability space $(\Omega, \mathscr{A}, P)$, a triangular array $\{X'_{nj}\}$ defined on Ω and a stochastic process $Z = \{Z(t): t \in [0, 1]\}: \Omega \to C$ (resp., $Z: \Omega \to D$) such that*

(a) $\mathscr{L}(X'_{n1}, \cdots, X'_{nj_n}) = \mathscr{L}(X_{n1}, \cdots, X_{nj_n})$,
(b) $\mathscr{L}(Z) = W_\gamma$,
(c) $\max_{1 \le k \le j_n} \| S'_{nk} - Z(k/j_n) \| \to_P 0$, *as* $n \to \infty$,

where $S'_{nk} = \sum_{j=1}^{k} X'_{nj}$.

If $a_1, \cdots, a_n \in B$, define $p_n(a_1, \cdots, a_n) \in C$ and $r_n(a_1, \cdots, a_n) \in D$ by $p_n(a_1, \cdots, a_n)(t) = a_{[nt]} + (nt - [nt])(a_{[nt]+1} - a_{[nt]})$ if $0 \le t \le 1$, $r_n(a_1, \cdots, a_n)(t) = a_{[nt]+1}$ if $0 \le t < 1$, $= a_n$ if $t = 1$.

4.10 COROLLARY. *Let $\{X_{nj}\}$ and γ be as in Theorem 4.8. Then*

$$\mathscr{L}(p_{j_n}(S_{n1}, \cdots, S_{nj_n})) \to_w W_\gamma \quad in \quad C$$

and

$$\mathscr{L}(r_{j_n}(S_{n1}, \cdots, S_{nj_n})) \to_w W_\gamma \quad in \quad D.$$

REMARK. The first part of this result generalizes an invariance principle in distribution of Eberlein [8, Theorem 3.1]. Condition (4) there is our hypothesis (2) and is a version for the dependent case of condition (3.3) in Kuelbs [14] (which always holds in the independent identically distributed case as it can be deduced from [2, Theorem 2.1]).

To close this section, we state a version for random vectors with values in a Hilbert space of Theorem 20.1 of Billingsley [6] (it can be proved combining Corollaries 4.7 and 4.10 with the remark following Theorem 4.8) and an arc-sine law for stationary, ϕ-mixing triangular arrays (it follows from the second conclusion of Corollary 4.10 and P. Lévy's arc-sine law for Brownian Motion).

4.11 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_j\}$ be a stationary, ϕ-mixing sequence with $\phi(1) < 1$ and $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume $E\|X_1\|^2 < \infty$ and $EX_1 = 0$. Then for every $f \in B'$ the sum $\Phi(f) = Ef^2(X_1) + 2\sum_{j=1}^{\infty} E(f(X_{j+1}))$ converges and defines the covariance of a centered Gaussian* *measure* γ *which satisfies*

$$\mathcal{L}(r_n(n^{-1/2}S_1, n^{-1/2}S_2, \cdots, n^{-1/2}S_n)) \to_w W_\gamma \quad in \quad D,$$

where $S_k = \sum_{j=1}^{k} X_j$.

4.12 COROLLARY. *Let* $B = \mathbb{R}$ *and let* $\{X_{nj}\}$ *be a triangular array which satisfies the hypotheses of Theorem* 4.8 *with a centered, nondegenerate Gaussian measure* γ. *Let* $L_n = \text{card}\{k \le j_n : S_{nk} > 0\}$. *Then*

$$\mathcal{L}(L_n/j_n) \to_w \alpha$$

where $\alpha(dx) = \pi^{-1}(x(1-x))^{-1/2}I_{(0,1)}(x)\,dx$.

5. Generalized Poisson limits. Proposition 3.1 gives conditions under which the limit of the row sums of a triangular array is infinitely divisible; as in the independent case, we want to relate the Lévy measure of the limit with the laws of the individual random vectors (under suitable assumptions). We need a modification of an inequality in Hoffmann-Jorgensen [11, proof of Theorem 3.1].

5.1 LEMMA. *Let* $\{X_1, \cdots, X_n\}$ *be a set of B-valued* r.v.'s *with stationary sums such that* $\phi(1) < 1$; *write* $S_k = \sum_{j=1}^{k} X_j$. *Suppose* $\phi(1) < \alpha < 1$ *and let* $s > 0$, $t > 0$, $u > 0$ *be such that* $t > s + u$, $\max_{1\le k\le n} P[\| S_k \| > (t-s-u)/2] \le 1 - \alpha$ *and* $\max_{1\le k\le n} P[\| S_k \| > u/2] \le 1 - \alpha$. *Then*

$$P[\| S_n \| > t] \le P[\max_{1\le j\le n} \| X_j \| > s] + (\alpha - \phi(1))^{-2}\psi^* P[\| S_n \| > (t-s-u)/2] P[\| S_n \| > u/2].$$

PROOF. Let $M = \max_{1\le j\le n} \| X_j \|$, $A_1 = [\| S_1 \| > u]$, $A_k = [\max_{1\le h\le k-1} \| S_h \| \le u, \| S_k \| > u]$ $(k = 2, \cdots, n)$. We have

$$\begin{aligned} P[\| S_n \| > t] &\le P[M > s] + \textstyle\sum_{k=1}^{n} P(A_k \cap [\| S_n - S_k \| > t-s-u]) \\ &\le P[M > s] + \textstyle\sum_{k=1}^{n} \psi^* P(A_k) P[\| S_n - S_k \| > t-s-u] \\ &\le P[M > s] + \psi^* \max_{1\le k\le n} P[\| S_n - S_k \| > t-s-u] \\ &\quad \cdot P[\max_{1\le k\le n} \| S_k \| > u]. \end{aligned}$$

Now it suffices to apply Proposition 2.2. □

5.2 THEOREM. *Let* $\{X_{nj}\}$ *be a* ϕ*-mixing triangular array with stationary sums which satisfies* $\phi(1) < 1$, $\psi^* < \infty$ *and condition* $(*)$. *Suppose that* $\mathcal{L}(S_n) \to_w \nu$ *and that* μ *is the Lévy measure of* ν. *Then, for every* $\tau \in C(\mu)$,

$$j_n \mathcal{L}(X_{n1}) \mid B_\tau^c \to_w \mu \mid B_\tau^c.$$

PROOF. First observe that, arguing as in the proof of (1) of Theorem 3.3, it follows that $\mathcal{L}(\xi_{n1})^{k_n} \to_w \nu$ and, by the general converse central limit theorem of the independent case [3, Theorem 2.10], $k_n \mathscr{L}(\xi_{n1}) | B_\tau^c \to_w \mu | B_\tau^c$ for every $\tau \in C(\mu)$.

We will prove that if $0 < s < t$ then

$$\mu(B_t^c) \le \liminf_n j_n P[\| X_{n1} \| > s]. \tag{5.1}$$

In order to do this, take u and α such that $0 < u < t - s$ and $\phi(1) < \alpha < 1$. Property (*) implies that for n large enough we have

$$\max_{1 \le k \le p_n} P[\| S_{nk} \| > (t - s - u)/2] \le 1 - \alpha$$

and

$$\max_{1 \le k \le p_n} P[\| S_{nk} \| > u/2] \le 1 - \alpha;$$

then Lemma 5.1 gives for such an n that

$$P[\| \xi_{n1} \| > t] \le p_n P[\| X_{n1} \| > s] + (\alpha - \phi(1))^{-2} \psi^* P[\| \xi_{n1} \| > (t - s - u)/2] P[\| \xi_{n1} \| > u/2].$$

Therefore

$$\begin{aligned} \mu(B_t^c) &\le \liminf_n k_n P[\| \xi_{n1} \| > t] \le \liminf_n k_n p_n P[\| X_{n1} \| > s] \\ &\quad + (\alpha - \phi(1))^{-2} \psi^* (\sup_n k_n P[\| \xi_{n1} \| > (t - s - u)/2]) \\ &\quad \cdot \limsup_n P[\| \xi_{n1} \| > u/2] \\ &= \liminf_n j_n P[\| X_{n1} \| > s] \end{aligned}$$

by the independent case and the finiteness of ψ^*.

Now we claim that

$$\mu(F) \ge \limsup_n j_n \mathscr{L}(X_{n1})(F) \tag{5.2}$$

for every closed set F such that $d(0, F) > 0$. To prove this, take such an F and let $\varepsilon > 0$. For $n \in N$, $i = 1, \cdots, p_n$ let $\xi_{n1}^{(i)} = \xi_{n1} - X_{ni}$, $C_i = C_{ni} = [X_{ni} \in F]$, $D_i = D_{ni} = [\xi_{n1}^{(i)} \in B_\varepsilon]$. We have

$$\begin{aligned} P[\xi_{n1} \in F + B_\varepsilon] &\ge P(\cup_{i=1}^{p_n} (C_i \cap D_i)) \\ &= \textstyle\sum_{i=1}^{p_n} P((C_i \cap D_i) \cap [\cap_{1 \le j < i} (C_j \cap D_j)^c]) \\ &\ge \textstyle\sum_{i=1}^{p_n} P((C_i \cap D_i) \cap [\cap_{1 \le j < i} C_j^c]) \\ &= \textstyle\sum_{i=1}^{p_n} \{P(C_i \cap D_i) - P(C_i \cap D_i \cap [\cup_{1 \le j < i} C_j])\} \end{aligned}$$

and

$$\begin{aligned} P(C_i \cap D_i \cap [\cup_{1 \le j < i} C_j]) &\le P(C_i \cap [\cup_{1 \le j < i} C_j]) \\ &\le \psi^* P(C_i) P(\cup_{1 \le j < i} C_j) \le \psi^* p_n (P(C_1))^2. \end{aligned}$$

Next, fix $h \in N$. If n is such that $p_n > h$ one has for $i = 2, \cdots, p_n - h$, writing

$$U_i = U_{ni} = \textstyle\sum_{j=1}^{i-1} X_{nj}, \quad V_i' = V_{ni}' = \sum_{j=i+1}^{i+h-1} X_{nj}, \quad V_i'' = V_{ni}'' = \sum_{j=i+h}^{p_n} X_{nj},$$

that

$$\begin{aligned} P(C_i \cap D_i) &\geq P(C_i \cap [U_i \in B_{\varepsilon/3}] \cap [V_i' \in B_{\varepsilon/3}] \cap [V_i'' \in B_{\varepsilon/3}]) \\ &= P(C_i \cap [V_i'' \in B_{\varepsilon/3}]) - P(C_i \cap [V_i'' \in B_{\varepsilon/3}] \\ &\qquad \cap ([U_i \in B_{\varepsilon/3}^c] \cup [V_i' \in B_{\varepsilon/3}^c])) \\ &\geq P(C_i)P[V_i'' \in B_{\varepsilon/3}] - \phi(h)P(C_i) - P(C_i \cap [U_i \in B_{\varepsilon/3}^c]) \\ &\quad - P(C_i \cap [V_i' \in B_{\varepsilon/3}^c]) \\ &\geq P(C_1)\{1 - \phi(h) - P[V_i'' \in B_{\varepsilon/3}^c] - \psi^* P[U_i \in B_{\varepsilon/3}^c] - \psi^* P[V_i' \in B_{\varepsilon/3}^c]\} \\ &\geq P(C_1)\{1 - \phi(h) - (1 + 2\psi^*)\delta_n\} \end{aligned}$$

where $\delta_n = \max_{1 \leq k \leq p_n} P[\| S_{nk} \| > \varepsilon/3]$; therefore

$$\begin{aligned} &P[\xi_{n1} \in F + B_\varepsilon] \\ &\qquad \geq (p_n - h - 1)P(C_{n1})\{1 - \phi(h) - (1 + 2\psi^*)\delta_n - \psi^* p_n P(C_{n1})\}. \end{aligned}$$

Now $\delta_n \to 0$ as $n \to \infty$ by the property (*) (recall that $p_n/j_n \to 0$) and $p_n P(C_{n1}) \leq p_n P[\| X_{n1} \| \geq d(0, F)]$, which goes to zero as a consequence of Theorem 3.4; hence

$$\begin{aligned} \mu(F + B_\varepsilon) &\geq \limsup_n k_n P[\xi_{n1} \in F + B_\varepsilon] \\ &\geq (\limsup_n k_n (p_n - h - 1)P(C_{n1}))\{1 - \phi(h)\} \\ &= (\limsup_n j_n P[X_{n1} \in F])\{1 - \phi(h)\} \end{aligned}$$

for all $h \in N$. By the ϕ-mixing condition we deduce that

$$\mu(F + B_\varepsilon) \geq \limsup_n j_n P[X_{n1} \in F]$$

for every $\varepsilon > 0$, but this implies (5.2) since F is closed.

To conclude the proof, fix $\tau \in C(\mu)$ and observe that it is sufficient to prove that every sequence $M \subset N$ contains a subsequence M' such that $w - \lim_{n \in M'} j_n \mathscr{L}(X_{n1}) \mid B_\tau^c = \mu \mid B_\tau^c$. Let $M \subset N$ be a sequence; using Theorem 3.4 and a diagonal procedure we obtain a subsequence M' of M and a σ-finite measure μ' with $\mu'(\{0\}) = 0$ such that $\text{w-}\lim_{n \in M'} j_n \mathscr{L}(X_{n1}) \mid B_{\tau'}^c = \mu' \mid B_{\tau'}^c$ for every $\tau' \in C(\mu')$. Now it is enough to show that $\text{w-}\lim_{n \in M'} j_n \mathscr{L}(X_{n1}) \mid B_{\tau'}^c = \mu \mid B_{\tau'}^c$ for every $\tau' \in C(\mu) \cap C(\mu')$ (since this implies that $\mu' = \mu$ and then the desired result follows). To prove this, take such a τ' and observe that by (5.2) we have that $\lim_n j_n P[\| X_{n1} \| = \tau'] = 0$ and then

$$\begin{aligned} &\limsup_{n \in M'} (j_n \mathscr{L}(X_{n1}) \mid B_{\tau'}^c)(F) \\ &\qquad = \limsup_{n \in M'} j_n \mathscr{L}(X_{n1})((\mathring{B}_{\tau'})^c \cap F) \leq (\mu \mid B_{\tau'}^c)(F) \end{aligned}$$

for each closed set F. It remains to show that $\lim_{n \in M'} j_n \mathscr{L}(X_{n1})(B_{\tau'}^c) = \mu(B_{\tau'}^c)$; since $\tau' \in C(\mu')$ the limit in the left member exists and coincides with $\mu'(B_{\tau'}^c)$. By the preceding inequality we only need to prove that $\mu'(B_{\tau'}^c) \geq \mu(B_{\tau'}^c)$; but if

$0 < \delta < \tau'$ and $\delta \in C(\mu')$, (5.1) gives

$$\mu(B^c_{\tau'}) \le \liminf_{n \in M'} j_n \mathscr{L}(X_{n1})(B^c_\delta) = \mu'(B^c_\delta)$$

and the desired inequality follows taking a sequence of such δ's increasing to τ' because $\tau' \in C(\mu')$. □

The following lemma and its proof were communicated to us by A. de Acosta. Given a subset A of B, ∂A denotes the boundary of A and $A^\varepsilon = \{x \in B: d(x, A) \le \varepsilon\}$ if $\varepsilon > 0$.

5.3 LEMMA. *Let $\{X_1, \cdots, X_n\}$ be a set of B-valued* r.v.'s. *If A is a subset of B and $\varepsilon > 0$ then*

$$\|(\textstyle\sum_{j=1}^n X_j) I_A(\sum_{j=1}^n X_j) - \sum_{j=1}^n X_j I_A(X_j)\| \le 2 \sum_{j=1}^n \|X_j\| \{I_{(\partial A)^\varepsilon}(X_j) + I_{B^c_\varepsilon}(\sum_{i=1, i\ne j}^n X_i)\}.$$

PROOF. Fix j with $1 \le j \le n$ and write $Z_j = \sum_{i=1, i \ne j}^n X_i$. We have

$$X_j I_A(Z_j + X_j) - X_j I_A(X_j) = X_j\{I_A(Z_j + X_j) I_{A^c}(X_j) - I_A(X_j) I_{A^c}(X_j + Z_j)\};$$

moreover

$$\begin{aligned} I_A(Z_j + X_j) I_{A^c}(X_j) &\le I_A(Z_j + X_j) I_{(A^\varepsilon)^c}(X_j) + I_A(Z_j + X_j) I_{A^c \cap A^\varepsilon}(X_j) \\ &\le I_{B^c_\varepsilon}(Z_j) + I_{(\partial A)^\varepsilon}(X_j) \end{aligned}$$

(note that since B is a normed linear space, $d(x, A) = d(x, \partial A)$ if $x \in A^c$) and, analogously,

$$I_A(X_j) I_{A^c}(X_j + Z_j) \le I_{B^c_\varepsilon}(Z_j) + I_{(\partial A)^\varepsilon}(X_j).$$

Then $\|X_j I_A(Z_j + X_j) - X_j I_A(X_j)\| \le 2\|X_j\| \{I_{(\partial A)^\varepsilon}(X_j) + I_{B^c_\varepsilon}(Z_j)\}$ for $j = 1, \cdots, n$. □

5.4 THEOREM. *Let $\{X_{nj}\}$ be a stationary, ϕ-mixing triangular array which satisfies $\phi(1) < 1$, $\psi^* < \infty$ and condition (*). Suppose that, for each $n \in N$, $\mathscr{L}(X_{n1}) = (1 - \lambda_n(B))\delta_0 + \lambda_n$, where λ_n is a finite positive measure such that $\lambda_n(B) \le 1$ and $\lambda_n(B_t) = 0$ for some $t > 0$ independent of n. Then, if $\mathscr{L}(S_n) \to_w \nu$ and μ is the Lévy measure of ν, we have $\mu(\mathring{B}_t) = 0$ and $\nu =$* Pois μ; *moreover, $\mathscr{L}(S_n^{(\tau)}) \to_w$* Pois$(\mu \mid B^c_\tau)$ *for every $\tau \in C(\mu)$.*

PROOF. Assume for the moment that we have proved that

$$\mathscr{L}(S_n^{(\tau)}) \to_w \text{Pois}(\mu \mid B^c_\tau) \tag{5.3}$$

for every $\tau \in C(\mu)$. If $\tau \in C(\mu)$, $\tau < t$, we will have that $\mathscr{L}(S_n) = \mathscr{L}(S_n^{(\tau)}) \to_w$ Pois$(\mu \mid B^c_\tau)$ (observe that $P[S_n \ne S_n^{(\tau)}] \le j_n P[X_{n1} \ne X^\tau_{n1}] \le j_n P[0 < \|X_{n1}\| \le \tau] = 0$) and then $\nu =$ Pois$(\mu \mid B^c_\tau)$. One can deduce that $\mu \mid B^c_\tau \cap B_{\tau'} = 0$ if $\tau, \tau' \in C(\mu)$ with $\tau < \tau' < t$ (use the uniqueness of the Lévy-Khintchine representation);

this implies that $\mu(\mathring{B}_t) = 0$ and $\nu = \text{Pois}\mu$. Hence the proof of the theorem will be done.

Fix $\tau \in C(\mu)$. By similar arguments to those used in the proof of (1) of Theorem 3.3 and an application of [3, Theorem 2.10] we can deduce from the weak convergence of $\{\mathscr{L}(S_n)\}$ to ν that $\mathscr{L}(\sum_{k=1}^{k_n} \xi_{nk}^\tau) \to_w \text{Pois}(\mu \mid B_\tau^c)$. Also, we can prove that

$$\sum_{k=1}^{k_n+1} \sum_{j \in Q(n,k)} X_{nj}^\tau \to_P 0$$

since $\sigma(X_{n1}^\tau) \le \sigma(X_{n1})$ and the triangular array $\{X_{nj}^\tau\}$ is stationary and satisfies (*) (by the inequality $P[\| \sum_{j=1}^{r_n} X_{nj} \| > 0] \le (r_n/j_n) j_n P[\| X_{n1} \| > \tau]$, (*) is a consequence of Theorem 3.4). Then (5.3) will follow if we prove that

$$\sum_{k=1}^{k_n} (\xi_{nk}^\tau - \sum_{j \in P(n,k)} X_{nj}^\tau) \to_P 0. \tag{5.4}$$

Take ε such that $0 < \varepsilon < \tau$; Lemma 5.3 gives for each $n \in N$

$$\begin{aligned} &\| \sum_{k=1}^{k_n} (\xi_{nk}^\tau - \sum_{j \in P(n,k)} X_{nj}^\tau) \| \\ &\qquad \le 2 \sum_{k=1}^{k_n} \sum_{j \in P(n,k)} \| X_{nj} \| I_{\{x : \tau - \varepsilon \le \|x\| \le \tau + \varepsilon\}}(X_{nj}) \\ &\qquad\quad + 2 \sum_{k=1}^{k_n} \sum_{j \in P(n,k)} \| X_{nj} \| I_{B_\varepsilon^c}(\sum_{i \in P(n,k), i \ne j} X_{ni}) \\ &\qquad = 2Y_{\varepsilon,n} + 2Z_{\varepsilon,n} \quad \text{(say)}. \end{aligned}$$

This shows that (5.4) holds if we prove that

$$\lim_{\varepsilon \downarrow 0} \limsup_n P[Y_{\varepsilon,n} > 0] = 0 \tag{5.5}$$

and

$$\text{for every} \quad \varepsilon > 0, \quad Z_{\varepsilon,n} \to_P 0. \tag{5.6}$$

Observe that if $0 < \varepsilon < \tau$

$$\begin{aligned} &\limsup_n P[Y_{\varepsilon,n} > 0] \\ &\quad \le \limsup_n j_n P[\tau - \varepsilon \le \| X_{n1} \| \le \tau + \varepsilon] \le \mu(\{x : \tau - \varepsilon \le \| x \| \le \tau + \varepsilon\}) \end{aligned}$$

by Theorem 5.2 and this implies (5.5) since $\tau \in C(\mu)$.

If $\varepsilon > 0$, $\eta > 0$, $s > 0$ and $n \in N$ write

$$\begin{aligned} P[Z_{\varepsilon,n} > \eta] &\le P[\max_{1 \le j \le j_n} \| X_{nj} \| > s] \\ &\quad + P[\sum_{k=1}^{k_n} \sum_{j \in P(n,k)} \| X_{njs} \| I_{B_\varepsilon^c}(\sum_{i \in P(n,k), i \ne j} X_{ni}) > \eta] \\ &\le P[\max_{1 \le j \le j_n} \| X_{nj} \| > s] \\ &\quad + \eta^{-1} k_n \sum_{j=1}^{p_n} E[\| X_{njs} \| I_{B_\varepsilon^c}(\sum_{i=1, i \ne j}^{p_n} X_{ni})]; \end{aligned}$$

then we have (5.6) if we prove the following two claims:

$$\lim_{s \to \infty} \limsup_n P[\max_{1 \le j \le j_n} \| X_{nj} \| > s] = 0, \tag{5.7}$$

$$\text{given } \varepsilon > 0 \text{ and } s > 0, \ \lim_n k_n \sum_{j=1}^{p_n} E[\| X_{njs} \| I_{B_\varepsilon^c}(\sum_{i=1, i \ne j}^{p_n} X_{ni})] = 0. \tag{5.8}$$

To prove (5.7), fix α such that $\phi(1) < \alpha < 1$. Given $\delta > 0$ write $\eta = \min\{1 - \alpha, \delta(\alpha - \phi(1))\}$, take $s > 0$ such that

$$\sup_{0\le t\le 1}\nu^t(B^c_{4^{-1}s-2^{-1}\eta}) < \eta/2$$

and $n_0 \in N$ such that

$$\max_{1\le k\le j_n}\rho(\mu_n^{(k)}, \nu^{k/j_n}) < \eta/2$$

if $n \ge n_0$ (possible by Theorem 3.3(2)). Then if $n \ge n_0$ one has by the definition of ρ that $\max_{1\le k\le j_n}P[\| S_{nk}\| > s/4] < \eta$ and hence $P[\max_{1\le j\le j_n}\| X_{nj}\| > s] \le P[\max_{1\le k\le j_n}\| S_{nk}\| > s/2] \le \delta$ by Proposition 2.2. This implies (5.7).

Now fix $\varepsilon > 0$ and $s > 0$. If $n \in N$ we have for $j = 1, \cdots, p_n$, writing $U_{nj} = \sum_{1\le i<j}X_{ni}$, $V_{nj} = \sum_{j<i\le p_n}X_{ni}$, that

$$E[\| X_{njs}\| I_{B^c_\varepsilon}(U_{nj} + V_{nj})] \le E[\| X_{njs}\| I_{B^c_{\varepsilon/2}}(U_{nj})] + E[\| X_{njs}\| I_{B^c_{\varepsilon/2}}(V_{nj})]$$

$$\le 2\,\psi^*\delta_n E\| X_{n1s}\|,$$

where $\delta_n = \max_{1\le k\le p_n}P[\| S_{nk}\| > \varepsilon/2]$, by Proposition 2.7. Hence

$$k_n \sum_{j=1}^{p_n} E[\| X_{njs}\| I_{B^c_\varepsilon}(\textstyle\sum_{i=1,i\ne j}^{p_n} X_{ni})] \le 2\psi^*\delta_n j_n E\| X_{n1s}\|$$

which tends to zero since $\delta_n \to 0$ by the property (*) and $\sup_n j_n E\| X_{n1s}\| < \infty$ as a consequence of Theorem 3.4 (note the inequality $j_n E\| X_{n1s}\| \le s j_n P[\| X_{n1}\| > t]$). Then (5.8) is proved. □

We will need the following result (de Acosta et al [3, Lemma 2.4]); note that there is no dependence assumption in its statement.

5.5 LEMMA. *Let $\{X_{nj}\}$ be a triangular array. Assume that $\{\sum_{j=1}^{j_n} \mathscr{L}(X_{nj}) \mid B^c_\delta\}$ is relatively compact for some $\delta > 0$. Then $\{\mathscr{L}(S_n^{(\tau)})\}$ is relatively compact for every $\tau \ge \delta$.*

Now we can give sufficient conditions for convergence to certain compound Poisson measures.

5.6 THEOREM. *Let $\{X_{nj}\}$ be a stationary, ϕ-mixing triangular array such that $\phi(1) < 1$ and $\psi^* < \infty$. Suppose that, for each $n \in N$, $\mathscr{L}(X_{n1}) = (1 - \lambda_n(B))\delta_0 + \lambda_n$, where λ_n is a finite positive measure such that $\lambda_n(B) \le 1$ and $\lambda_n(B_t) = 0$ for some $t > 0$ independent of n. Assume that there exists a finite measure μ such that $j_n\lambda_n \to_w \mu$. Then $\mathscr{L}(S_n) \to_w$ Poisμ.*

PROOF. Observe that $\{j_n\mathscr{L}(X_{n1}) \mid B^c_t\} = \{j_n\lambda_n\}$ is relatively compact by hypothesis. Then Lemma 5.5 implies that $\{\mathscr{L}(S_n)\} = \{\mathscr{L}(S_n^{(t)})\}$ is relatively compact ($P[S_n \ne S_n^{(t)}] \le j_n\lambda_n(B_t) = 0$).

To conclude the proof, it suffices to show that each convergent subsequence of $\{\mathscr{L}(S_n)\}$ has the desired limit. Assume that $\mathscr{L}(S_{n'}) \to_w \nu$ and let μ' be the Lévy measure of ν; by Theorem 5.4 (the inequality $P[\| \sum_{j=1}^{r_n} X_{nj}\| > 0] \le (r_n/j_n)j_n\lambda_n(B)$ and the hypothesis imply that $\{X_{nj}\}$ satisfies (*)) we have that

$\mu'(\mathring{B}_t) = 0$ (hence μ' is finite) and $\nu = \text{Pois}(\mu')$. On the other hand, if $\tau \in C(\mu')$ with $\tau < t$, Theorem 5.2 gives that $j_{n'}\lambda_{n'} = j_{n'}\mathscr{L}(X_{n'1}) \mid B_\tau^c \to_w \mu' \mid B_\tau^c = \mu'$. Therefore $\mu = \mu'$ and $\nu = \text{Pois}\,\mu$. □

The following two results give sufficient conditions for convergence to a generalized Poisson measure.

5.7 THEOREM. *Let $\{X_{nj}\}$ be a stationary, ϕ-mixing triangular array such that $\phi(1) < 1$ and $\psi^* < \infty$. Assume*

(1) *there exists a σ-finite measure μ such that, for every $\tau \in C(\mu)$,*

$$j_n\mathscr{L}(X_{n1}) \mid B_\tau^c \to_w \mu \mid B_\tau^c,$$

(2) *there exist $r > 0$ and a sequence $\{\delta_k\} \subset C(\mu)$ such that $\delta_k \downarrow 0$ and*

$$\lim_k \lim\sup_n E\,\| S_{n,\delta_k} - ES_{n,\delta_k} \|^r = 0,$$

(3) *there exist $\beta > 0$, $p > 0$ and a sequence $\{F_k\}$ of finite-dimensional subspaces of B such that*

$$\lim_k \sup_n E d_{F_k}^p (S_{n,\beta} - ES_{n,\beta}) = 0.$$

Then (a) *μ is a Lévy measure,* (b) *for every $\tau \in C(\mu)$, $\mathscr{L}(S_n - ES_{n,\tau}) \to_w c_\tau \text{Pois}\,\mu$.*

PROOF. As a consequence of assumption (1) we have that $\mathscr{L}(S_n^{(\delta)}) \to_w \text{Pois}(\mu \mid B_\delta^c)$ for every $\delta \in C(\mu)$ and $\lim_n(ES_{n,\tau} - ES_{n,\delta}) = \lim_n ES_{n,\tau}^{(\delta)} = \int_{B_\tau} x(\mu \mid B_\delta^c)\,(dx)$ if $\delta, \tau \in C(\mu)$ and $\delta < \tau$. The first assertion follows applying Theorem 5.6 to $\{X_{nj}^\delta\}$ (with $\lambda_n = \mathscr{L}(X_{n1}) \mid B_\delta^c$) and the second is deduced by a standard argument (write $ES_{n,\tau}^{(\delta)} = \int x I_{B_\tau}(x)(j_n\mathscr{L}(X_{n1}) \mid B_\delta^c)\,(dx)$ and note that the set of discontinuities of the bounded, Borel measurable function (from B into B) $xI_{B_\tau}(x)$ has $\mu \mid B_\delta^c$-measure zero). Then we have (see [3, Section 1])

(5.9) if $\delta, \tau \in C(\mu)$ and $\delta < \tau$, $\mathscr{L}(S_n^{(\delta)} + ES_{n,\delta} - ES_{n,\tau}) \to_w c_\tau\text{Pois}(\mu \mid B_\delta^c)$.

Fix $\tau \in C(\mu)$. We will show first that $\{\mathscr{L}(S_n - ES_{n,\tau})\}$ is relatively compact. Choose $\delta > 0$ such that $\delta < \tau$, $\delta < \beta$ and $\sup_n E\,\| S_{n,\delta} - ES_{n,\delta} \|^r < \infty$ (this is possible by (2)). Hence we have (Chebyshev's inequality) that for every $f \in B'$, $\{\mathscr{L}(f(S_{n,\delta} - ES_{n,\delta}))\}$ and then $\{\mathscr{L}(f(S_{n,\beta} - ES_{n,\beta}))\}$ are relatively compact; the second assertion follows from the first by the equality $S_{n,\beta} - ES_{n,\beta} = (S_{n,\delta} - ES_{n,\delta}) + S_{n,\beta}^{(\delta)} - ES_{n,\beta}^{(\delta)}$ since assumption (1) implies that $\{\mathscr{L}(S_{n,\beta}^{(\delta)})\}$ is relatively compact (write $S_{n,\beta}^{(\delta)} = S_n^{(\delta)} - S_n^{(\beta)}$ and use Lemma 5.5) and then $\{ES_{n,\beta}^{(\delta)}\}$ is relatively compact in B (Proposition 3.5 ensures the uniform integrability of $\{\| S_{n,\beta}^{(\delta)} \|\}$). Now (3) and [1, Theorem 2.3] imply that $\{\mathscr{L}(S_{n,\beta} - ES_{n,\beta})\}$ and then $\{\mathscr{L}(S_{n,\delta} - ES_{n,\delta})\}$ are relatively compact. On the other hand, arguing as above starting from (1) we can obtain that $\{\mathscr{L}(S_n^{(\delta)} + ES_{n,\delta} - ES_{n,\tau})\}$ is relatively compact. Hence the equality

(5.10) $$S_n - ES_{n,\tau} = (S_{n,\delta} - ES_{n,\delta}) + (S_n^{(\delta)} + ES_{n,\delta} - ES_{n,\tau})$$

shows that $\{\mathscr{L}(S_n - ES_{n,\tau})\}$ is relatively compact.

Suppose that $\mathscr{L}(S_{n'} - ES_{n',\tau}) \to_w \nu$. There is a subsequence $\{n_k\}$ of $\{n'\}$ such that if $\delta_k < \tau$

$$E \| S_{n_k,\delta_k} - ES_{n_k,\delta_k} \|^r < k^{-1} + \limsup_n E \| S_{n,\delta_k} - ES_{n,\delta_k} \|^r$$

and $\rho(\mathscr{L}(S_{n_k}^{(\delta_k)} + ES_{n_k,\delta_k} - ES_{n_k,\tau}), c_\tau \mathrm{Pois}(\mu \mid B_{\delta_k}^c)) < k^{-1}$ (this is possible by (5.9)). By (2) we have that $S_{n_k,\delta_k} - ES_{n_k,\delta_k} \to_P 0$ as $k \to \infty$ and then (apply (5.10) with $n = n_k$, $\delta = \delta_k$) $\mathscr{L}(S_{n_k}^{(\delta_k)} + ES_{n_k,\delta_k} - ES_{n_k,\tau}) \to_w \nu$. Hence $c_\tau \mathrm{Pois}(\mu \mid B_{\delta_k}^c) \to_w \nu$ by the choice of $\{n_k\}$. An application of [3, Theorem 1.6] gives that μ is a Lévy measure and $\nu = c_\tau \mathrm{Pois}\,\mu$.

The relative compactness of $\{\mathscr{L}(S_n - ES_{n,\tau})\}$ and the above argument imply the desired conclusions. □

5.8 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_{nj}\}$ be a stationary, ϕ-mixing triangular array such that $\phi(1) < 1$, $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$ and $\psi^* < \infty$. Assume*:

(1) *there exists a σ-finite measure μ such that, for every $\tau \in C(\mu)$,*

$$j_n \mathscr{L}(X_{n1}) \mid B_\tau^c \to_w \mu \mid B_\tau^c,$$

(2) *there exists a sequence $\{\delta_k\} \subset C(\mu)$ such that $\delta_k \downarrow 0$ and*

$$\lim_k \limsup_n j_n E \| X_{n1\delta_k} - EX_{n1\delta_k} \|^2 = 0,$$

(3) *there exists $\beta > 0$ such that*

$$\lim_k \sup_n j_n E d_k^2(X_{n1\beta} - EX_{n1\beta}) = 0$$

(*the d_k's are as in Corollary* 4.5).

Then (a) *μ is a Lévy measure,* (b) *for every $\tau \in C(\mu)$, $\mathscr{L}(S_n - ES_{n,\tau}) \to_w c_\tau \mathrm{Pois}\,\mu$.*

PROOF. As in the proof of Corollary 4.5 we can obtain the inequalities

$$E \| S_{n,\delta} - ES_{n,\delta} \|^2 \le (1 + 4 \textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j)) j_n E \| X_{n1\delta} - EX_{n1\delta} \|^2,$$

$$E d_k^2(S_{n,\beta} - ES_{n,\beta}) \le (1 + 4 \textstyle\sum_{j=1}^{\infty} \phi^{1/2}(j)) j_n E d_k^2(X_{n1\beta} - EX_{n1\beta})$$

which show that the result follows from Theorem 5.7. □

Let $\{X_j : j \in N\}$ be a stationary sequence of B-valued random vectors, $\{a_n\}$ a sequence of real numbers tending to infinity and $\{b_n\} \subset B$; it is known that if $\{\mathscr{L}(a_n^{-1}(X_1 + \cdots + X_n) - b_n)\}$ converges weakly and $\{X_j\}$ is ϕ-mixing (or even under a weaker assumption) then the limit is a stable measure (see Ibragimov and Linnik [12, Theorem 18.1.1] and Philipp [16, Theorem 2]). The following two consequences of the previous result give sufficient conditions for that behavior (with a nonGaussian limit).

5.9 COROLLARY. *Suppose that B is a Hilbert space. Let $\alpha \in (0, 2)$ and let σ be a finite measure on $S = \{x \in B : \| x \| = 1\}$; denote $\mu_{\alpha,\sigma}$ the measure on B induced* *by the product measure* $r^{-1-\alpha}dr\otimes\sigma$ *through the map* $(r, x) \to rx$ *from* $[0, \infty) \times S$ *onto* B. *Assume that* $\{X_j : j \in N\}$ *is a stationary,* ϕ*-mixing sequence such that* $\phi(1) < 1$, $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$, $\psi^* < \infty$ *and such that it satisfies*

$$n\mathscr{L}(n^{-1/\alpha}X_1) \mid B_\tau^c \to_w \mu_{\alpha,\sigma} \mid B_\tau^c$$

for every $\tau > 0$. *Then* $\mu_{\alpha,\sigma}$ *is a Lévy measure and*

$$\mathscr{L}(n^{-1/\alpha}(X_1 + \cdots + X_n) - n^{1-1/\alpha}E[X_1; X_1 \in B_{n^{1/\alpha}}]) \to_w c_1\text{Pois}\,\mu_{\alpha,\sigma}.$$

PROOF. Let $X_{nj} = n^{-1/\alpha}X_j$ for $j = 1, \cdots, n$ and $n \in N$. Corollary 5.8 shows that it is sufficient to prove that

$$\lim_{\delta\downarrow 0}\sup_n nE\,\|X_{n1\delta}\|^2 = 0 \quad \text{and} \quad \lim_k\sup_n nEd_k^2(X_{n1\beta}) = 0$$

for some $\beta > 0$. But as in Araujo and Giné [4, proof of Theorem 4.3] these conditions can be deduced from the relations

$$\sup_{t>0}t^\alpha P[\|X_1\| > t] < \infty \text{ and } \lim_k\sup_{t>0}t^\alpha P[d_k(X_1) > t] = 0$$

which are consequences of the hypothesis. □

5.10 COROLLARY. *Let* $\{X_j : j \in N\}$ *be a stationary,* ϕ*-mixing sequence of real random variables such that* $\phi(1) < 1$, $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$ *and* $\psi^* < \infty$. *Let* $\alpha \in (0, 2)$ *and suppose that*

(1) *there exist constants* $\ell_1 \geq 0$, $\ell_2 \geq 0$ *such that* $\ell_1 + \ell_2 > 0$ *and*

$$\lim_{x\to+\infty} \frac{P[X_1 < -x]}{P[X_1 > x]} = \frac{\ell_1}{\ell_2},$$

(2) *for every* $t > 0$,

$$\lim_{x\to+\infty} \frac{P[|X_1| > x]}{P[|X_1| > tx]} = t^\alpha.$$

Then there exist constants a_n, *with* $a_n \to \infty$, *such that*

$$\mathscr{L}(a_n^{-1}(X_1 + \cdots + X_n) - na_n^{-1}E[X_1; |X_1| \leq a_n]) \to_w c_1\,\text{Pois}\,\mu_{\alpha,\ell_1,\ell_2}$$

where $\mu_{\alpha,\ell_1,\ell_2}(dx) = \alpha\{I_{(-\infty,0)}(x)\ell_1|x|^{-1-\alpha} + I_{(0,+\infty)}(x)\ell_2x^{-1-\alpha}\}\,dx$.

PROOF. As in Gnedenko and Kolmogorov [10, pages 176–178] we can define a_n such that $a_n \to \infty$ and, for each $x > 0$,

$$\lim_n nP[X_1 < -xa_n] = \ell_1 x^{-\alpha}, \quad \lim_n nP[X_1 > xa_n] = \ell_2x^{-\alpha},$$

$$\lim_{\delta\downarrow 0}\lim\sup_n na_n^{-2}E[X_1^2; |X_1| \leq \delta a_n] = 0.$$

Now it suffices to define $X_{nj} = a_n^{-1}X_j$ *for* $j = 1, \cdots, n$, $n \in N$, and to apply Corollary 5.8. □

6. Infinitely divisible limits. In this section we consider ψ-mixing triangular arrays.

6.1 LEMMA. *Let $\{X_{nj}\}$ be a stationary, ψ-mixing triangular array such that $\phi(1) < 1$ and $\psi^* < \infty$. Assume $X_{n1} \to_P 0$, $\| X_{nj} \| \le M$ a.s. (for all n, j) for some M and that there exists $\delta > 0$ such that the triangular array $\{X_{nj\delta} - EX_{nj\delta}\}$ satisfies $(*)$ and $\{\mathscr{L}(S_{n,\delta} - ES_{n,\delta})\}$ is relatively compact. Then*

$$\lim_n E[f(S_{n,\delta} - ES_{n,\delta}) f(S_n^{(\delta)})] = 0$$

for every $f \in B'$.

PROOF. Let $Y_{nj} = X_{nj\delta} - EX_{nj\delta}$ for $j = 1, \cdots, j_n$. Since $X_{n1} \to_P 0$ we have $E\| X_{n1\delta} \| \to 0$, $E\| X_{n1}^\delta \| \to 0$ (write $E\| X_{n1}^\delta \| \le MP[\| X_{n1} \| > \delta]$) and $E \| Y_{n1} \| \to 0$ as $n \to \infty$. Theorem 3.4 applied to $\{Y_{nj}\}$ implies that $K = \sup_n j_n E \| X_{n1}^\delta \| < \infty$. Next we claim that

$$C = \sup_{n \in N} \max_{1 \le k \le j_n} E \| \textstyle\sum_{j=1}^k Y_{nj} \|$$

is finite. To prove this, fix α such that $\phi(1) < \alpha < 1$ and put $\eta = 1 - \alpha$. Take $x_0 > 0$ such that

$$\sup_{0 \le t \le 1} \nu^t(B^c_{(x_0 - \eta)/2}) < \eta/2$$

and $n_0 \in N$ such that

$$\max_{1 \le k \le j_n} \rho(\mu_n^{(k)}, \nu^{k/j_n}) < \eta/2$$

if $n \ge n_0$, where $\mu_n^{(k)} = \mathscr{L}(\sum_{j=1}^k Y_{nj})$ (possible by Theorem 3.3). Hence if $n \ge n_0$ we have

$$\max_{1 \le k \le j_n} P[\| \textstyle\sum_{j=1}^k Y_{nj} \| > x/2] < 1 - \alpha$$

for $x \ge x_0$ and therefore, by Proposition 2.2,

$$\begin{aligned} \max_{1 \le k \le j_n} E \| \textstyle\sum_{j=1}^k Y_{nj} \| &\le x_0 + (\alpha - \phi(1))^{-1} \int_{x_0}^\infty P[\| \textstyle\sum_{j=1}^{j_n} Y_{nj} \| > x/2]\, dx \\ &\le x_0 + 2(\alpha - \phi(1))^{-1} E \| \textstyle\sum_{j=1}^{j_n} Y_{nj} \|. \end{aligned}$$

Applying Proposition 3.5 to $\{Y_{nj}\}$ we conclude that $C < \infty$.

Fix $f \in B'$. Let $h \in N$; if n is such that $j_n > 2h + 1$ and i satisfies $h + 1 \le i \le j_n - h$ write

$$U''_{ni} = \textstyle\sum_{j=1}^{i-h} f(Y_{nj}), \quad U'_{ni} = \sum_{j=i-h+1}^{i-1} f(Y_{nj}),$$
$$V'_{ni} = \textstyle\sum_{j=i+1}^{i+h-1} f(Y_{nj}), \quad V''_{ni} = \sum_{j=i+h}^{j_n} f(Y_{nj}).$$

For n sufficiently large we have

$$\begin{aligned}
&|E[f(S_n^{(\delta)})f(S_{n,\delta} - ES_{n,\delta})]| \\
&\quad\le \sum_{i=1}^{j_n} |E[f(X_{ni}^\delta) \sum_{j=1}^{j_n} f(Y_{nj})]| \\
&\quad\le j_n |E[f(X_{n1}^\delta)f(Y_{n1})]| \\
&\qquad + \sum_{i=1}^{h} |E[f(X_{ni}^\delta) \sum_{j\ne i} f(Y_{nj})]| \\
&\qquad + \sum_{i=h+1}^{j_n-h} |E[f(X_{ni}^\delta)\{U''_{ni} + U'_{ni} + V'_{ni} + V''_{ni}\}]| \\
&\qquad + \sum_{i=j_n-h+1}^{j_n} |E[f(X_{ni}^\delta) \sum_{j\ne i} f(Y_{nj})]| \\
&\quad= a_n + b_n + c_n + d_n \quad \text{(say)}.
\end{aligned}$$

Moreover

$$a_n = j_n |Ef(X_{n1}^\delta)| \; |f(EX_{n1\delta})| \le \|f\|^2 KE\|X_{n1\delta}\|,$$

$$b_n = \sum_{i=1}^{h} |E[f(X_{ni}^\delta)\{\sum_{j=1}^{i-1} f(Y_{nj}) + \sum_{j=i+1}^{j_n} f(Y_{nj})\}]| \le 2\, h\psi^* \|f\|^2 CE\|X_{n1}^\delta\|$$

by Proposition 2.7,

$$\begin{aligned}
c_n &\le \sum_{i=h+1}^{j_n-h} E|f(X_{ni}^\delta)| \\
&\quad \cdot \{\psi(h)E|U''_{ni}| + \psi^*E|U'_{ni}| + \psi^*E|V'_{ni}| + \psi(h)E|V''_{ni}|\} \\
&\le 2\|f\|^2 K\{\psi^* hE\|Y_{n1}\| + C\psi(h)\}
\end{aligned}$$

by Propositions 2.6 and 2.7 and d_n has the same bound that b_n. These inequalities and the remarks made above yield

$$\limsup_n |E[f(S_{n,\delta} - ES_{n,\delta})f(S_n^{(\delta)})]| \le 2\|f\|^2 KC\psi(h)$$

for every $h \in N$, then the ψ-mixing condition implies the desired result. □

6.2 THEOREM. *Let $\{X_{nj}\}$ be a stationary, ψ-mixing triangular array which satisfies $\phi(1) < 1$, $\psi^* < \infty$ and condition (*). Suppose that $\mathscr{L}(S_n) \to_w \nu$ with Lévy-Khintchine representation $\nu = \delta_{z_\tau} * \gamma * c_\tau \mathrm{Pois}\,\mu$ for $\tau \in C(\mu)$, where $z_\tau \in B$, γ is a centered Gaussian measure and μ is a Lévy measure. Then*

(a) *for every $\tau \in C(\mu)$, $j_n \mathscr{L}(X_{n1}) | B_\tau^c \to_w \mu | B_\tau^c$,*

(b) *for every $f \in B'$,*

$$\lim_{\delta\downarrow 0} \begin{Bmatrix}\limsup_n \\ \liminf_n\end{Bmatrix} Ef^2(S_{n,\delta} - ES_{n,\delta}) = \lim_{\tau\downarrow 0, \tau\in C(\mu)} \lim_n Ef^2(S_{n,\tau} - ES_{n,\tau}) = \Phi_\gamma(f, f),$$

(c) *for every $\tau \in C(\mu)$,*

$$\mathscr{L}(S_n^{(\tau)}) \to_w \mathrm{Pois}(\mu | B_\tau^c), \quad \mathscr{L}(S_{n,\tau}) \to_w \delta_{z_\tau} * \gamma * c_\tau \mathrm{Pois}(\mu | B_\tau)$$

and

$$ES_{n,\tau} \to z_\tau \text{ in } B.$$

PROOF. Theorem 5.2 shows that (a) holds. Fix $\tau \in C(\mu)$. (a) and Theorem 5.6 applied to $\{X_{nj}^{\tau}\}$ imply the first assertion of (c) (note that $\mathscr{L}(X_{n1}^{\tau}) = \mathscr{L}(X_{n1})(B_\tau)\delta_0 + \mathscr{L}(X_{n1}) \mid B_\tau^c$). Hence $\{\mathscr{L}(S_{n,\tau})\}$ is relatively compact. Let λ be the Lévy measure of a limit point of this sequence; for every $\tau' < \tau$ such that $\tau' \in C(\lambda) \cap C(\mu)$ we have $j_n \mathscr{L}(X_{n1\tau}) \mid B_{\tau'}^c = j_n \mathscr{L}(X_{n1}) \mid B_\tau \cap B_{\tau'}^c$ and then $\lambda \mid B_{\tau'}^c = \mu \mid B_\tau \cap B_{\tau'}^c$ by (a). Therefore $\mu \mid B_\tau$ is the Lévy measure of every limit point of $\{\mathscr{L}(S_{n,\tau})\}$.

Now we will prove the following claim: (I) if a subsequence $\{\mathscr{L}(S_{n_k,\tau})\}$ converges to $\delta_z * \tilde{\gamma} * c_\tau \mathrm{Pois}(\mu \mid B_\tau)$ where $z \in B$ and $\tilde{\gamma}$ is a centered Gaussian measure then $\mathscr{L}(S_{n_k,\tau'}) \to_w \delta_z * \tilde{\gamma} * c_\tau \mathrm{Pois}(\mu \mid B_{\tau'})$ for every $\tau' > \tau$ such that $\tau' \in C(\mu)$.

In order to prove (I), fix such a τ' and observe that since $\{\mathscr{L}(S_{n_k,\tau'})\}$ is relatively compact it suffices to show that each one of its convergent subsequences has the desired limit. Let $\{\mathscr{L}(S_{n',\tau'})\}$ be such a subsequence with limit $\delta_{z'} * \gamma' * c_\tau \mathrm{Pois}(\mu \mid B_{\tau'}) = \delta_{z'+m} * \gamma' * c_{\tau'} \mathrm{Pois}(\mu \mid B_{\tau'})$ where $z' \in B$, γ' is a centered Gaussian measure and $m = \int_{B_\tau^c \cap B_{\tau'}} x\mu(dx)$ (we have used an elementary property of τ-centered Poisson measures; for this and other properties which we will use we refer to [3]). We have

$$\lim_{n'} ES_{n',\tau} = z + \int x c_\tau \mathrm{Pois}(\mu \mid B_\tau)(dx) = z, \ \lim_{n'} ES_{n',\tau'} = z' + m$$

by Proposition 3.5 (to prove that $\{X_{nj\tau}\}$ satisfies (*), write $X_{nj\tau} = X_{nj} - X_{nj}^{\tau}$ and use (a) and the property (*) of $\{X_{nj}\}$) and $m = \lim_{n'} (ES_{n',\tau'} - ES_{n',\tau})$ by (a); hence $z' = z$. On the other hand, we have for every $f \in B'$ (Proposition 3.5)

$$\lim_{n'} Ef^2(S_{n',\tau'} - ES_{n',\tau}) = \int f^2 d[\delta_m * \gamma' * c_{\tau'} \mathrm{Pois}(\mu \mid B_{\tau'})]$$

$$= f^2(m) + \Phi_{\gamma'}(f, f) + \int f^2 d(\mu \mid B_{\tau'}),$$

$$\lim_{n'} Ef^2(S_{n',\tau} - ES_{n',\tau}) = \Phi_{\tilde{\gamma}}(f, f) + \int f^2 d(\mu \mid B_\tau),$$

$$\lim_{n'} Ef^2(S_{n',\tau'}^{(\tau)}) = f^2(m) + \int f^2 d(\mu \mid B_\tau^c \cap B_{\tau'})$$

(arguing as above we can obtain that $\mathscr{L}(S_{n,\tau'}^{(\tau)}) \to_w \mathrm{Pois}(\mu \mid B_\tau^c \cap B_{\tau'})$) and the equality

$$Ef^2(S_{n,\tau'} - ES_{n,\tau}) = Ef^2(S_{n,\tau} - ES_{n,\tau}) + Ef^2(S_{n,\tau'}^{(\tau)}) + 2E[f(S_{n,\tau} - ES_{n,\tau}) f(S_{n,\tau'}^{(\tau)})].$$

Therefore, Lemma 6.1 implies $\Phi_{\gamma'} = \Phi_\gamma$, that is, $\gamma' = \gamma$. Then (I) holds.

To complete the proof of (c) observe that its third assertion follows from the second (by Proposition 3.5). To prove it, let $\{n'\}$ be a subsequence of N such that $\mathscr{L}(S_{n',\tau}) \to_w \delta_z * \tilde{\gamma} * c_\tau \mathrm{Pois}(\mu \mid B_\tau)$ where $z \in B$ and $\tilde{\gamma}$ is a centered Gaussian measure; since $\{\mathscr{L}(S_{n,\tau})\}$ is relatively compact it is sufficient to show that $z = z_\tau$

and $\tilde{\gamma} = \gamma$. Take an increasing sequence $\{\tau_k\} \subset C(\mu)$ such that $\tau_1 > \tau$ and $\tau_k \uparrow \infty$; by (I) we have

$$\mathscr{L}(S_{n',\tau_k}) \to_w \delta_z * \tilde{\gamma} * c_\tau \text{Pois}(\mu \mid B_{\tau_k}) = \nu_k$$

(say) for every $k \in N$. Hence there exists a subsequence $\{n_k\}$ of $\{n'\}$ such that $\rho(\mathscr{L}(S_{n_k,\tau_k}), \nu_k) < 1/k$ for each $k \in N$. Note that $S_{n_k}^{(\tau_k)} \to_P 0$ (given $\varepsilon > 0$, by Theorem 3.4 we may choose $r > 0$ such that $\sup_n j_n \mathscr{L}(X_{n1})(B_r^c) \le \varepsilon$ which implies that $P[\| S_{n_k}^{(\tau_k)} \| > 0] \le j_{n_k} P[\| X_{n_k 1} \| > \tau_k] \le \varepsilon$ for sufficiently large k) and $c_\tau \text{Pois}(\mu \mid B_{\tau_k}) \to_w c_\tau \text{Pois}\,\mu$. Then

$$\nu = w - \lim_k \mathscr{L}(S_{n_k}) = w - \lim_k \mathscr{L}(S_{n_k,\tau_k}) = w - \lim_k \nu_k = \delta_z * \tilde{\gamma} * c_\tau \text{Pois}\mu$$

and the uniqueness of the Lévy-Khintchine representation implies $z = z_\tau$ and $\tilde{\gamma} = \gamma$. Thus (c) is proved.

Let $f \in B'$. (c) and Proposition 3.5 imply

$$\lim_n Ef^2(S_{n,\tau} - ES_{n,\tau}) = \Phi_\gamma(f, f) + \int f^2 d(c_\tau \text{Pois}(\mu \mid B_\tau))$$

for every $\tau \in C(\mu)$; arguing as in [3, proof of Theorem 2.10] we can deduce the second equality in (b). To obtain the first it is sufficient to show that $\liminf_n Ef^2(S_{n,\delta} - ES_{n,\delta})$ and $\limsup_n Ef^2(S_{n,\delta} - ES_{n,\delta})$ are increasing functions of δ. But this follows from Lemma 6.1 and the inequality

$$Ef^2(S_{n,\delta'} - ES_{n,\delta'}) \ge Ef^2(S_{n,\delta} - ES_{n,\delta}) + 2E[f(S_{n,\delta} - ES_{n,\delta}) f(S_{n,\delta'}^{(\delta)})]$$

$(0 < \delta < \delta')$. □

We use the notation $V_n(\delta, f)$ of Section 4.

6.3 Corollary. *Let $\{X_{nj}\}$ be as in Theorem 6.2. Assume that either* (i) $\sum_{j=1}^\infty \phi^{1/2}(j) < \infty$ *and for every $f \in B'$ there exists $\delta > 0$ such that*

$$C_{\delta,f} = \sup_n j_n Ef^2(X_{n1\delta} - EX_{n1\delta}) < \infty,$$

or (ii) $\sum_{j=1}^\infty \psi(j) < \infty$ *and for every $f \in B'$ there exists $\delta > 0$ such that*

$$M_{\delta,f} = \sup_n j_n^{1/2} E \mid f(X_{n1\delta} - EX_{n1\delta}) \mid < \infty.$$

Then for every $f \in B'$

(b′) $$\lim_{\delta \downarrow 0} \begin{Bmatrix} \limsup_n \\ \liminf_n \end{Bmatrix} V_n(\delta, f) = \lim_{\tau \downarrow 0, \tau \in C(\mu)} \lim_n V_n(\tau, f) = \Phi_\gamma(f, f).$$

Proof. Let $f \in B'$. First let us observe that conclusion (a) of Theorem 6.2 implies that we may suppose that $C_{\delta,f} < \infty$ for every $\delta > 0$ in (i) and $M_{\delta,f} < \infty$ for every $\delta > 0$ in (ii).

As in the proof of Corollary 4.3 we have for each $\delta > 0$

$$\mid Ef^2(S_{n,\delta} - ES_{n,\delta}) - V_n(\delta, f) \mid \le 4(j_n^{-1} \textstyle\sum_{j=1}^{j_n-1} j\phi^{1/2}(j)) C_{\delta,f}$$

and analogously (but using Proposition 2.6) we obtain the bound

$$2(j_n^{-1} \sum_{j=1}^{j_n-1} j\psi(j))M_{\delta,f}^2.$$

Now we can deduce (b′) from (b) of Theorem 6.2 and (i) or (ii). □

REMARK. Let $\{X_{nj}\}$ be as in Theorem 6.2. If $\sum_{j=1}^{\infty} \phi^{1/2}(j) < 1/4$ or $\sum_{j=1}^{\infty} \psi(j) < 1/2$ then (b′) holds for every $f \in B'$ (argue as in the remark following Corollary 4.3).

6.4 THEOREM. *Let $\{X_{nj}\}$ be a stationary, ψ-mixing triangular array such that $\phi(1) < 1$, $\psi^* < \infty$. Assume*

(1) *for some $\alpha > 0$, the triangular array $\{X_{nj\alpha} - EX_{nj\alpha}\}$ satisfies (*),*

(2) *there exists a σ-finite measure μ such that for every $\tau \in C(\mu)$*

$$j_n \mathscr{L}(X_{n1}) \mid B_\tau^c \rightarrow_w \mu \mid B_\tau^c,$$

(3) *there exist a sequentially w^*-dense subset W of B' and a sequence $\delta_k \downarrow 0$ such that*

$$\Phi(f) = \lim_k \begin{Bmatrix} \limsup_n \\ \liminf_n \end{Bmatrix} Ef^2(S_{n,\delta_k} - ES_{n,\delta_k})$$

exists for every $f \in W$,

(4) *there exist $\beta > 0, p > 0$ and a sequence $\{F_k\}$ of finite-dimensional subspaces of B such that*

$$\lim_k \sup_n Ed_{F_k}^p(S_{n,\beta} - ES_{n,\beta}) = 0.$$

Then (a) μ is a Lévy measure, (b) *there exists a centered Gaussian measure γ such that $\Phi_\gamma(f, f) = \Phi(f)$ for every $f \in W$,* (c) *$\mathscr{L}(S_n - ES_{n,\tau}) \rightarrow_w \gamma * c_\tau \mathrm{Pois}\,\mu$ for every $\tau \in C(\mu)$.*

PROOF. By an application of [1, Theorem 2.3] and using Lemma 5.5 and Proposition 3.5 we can deduce from the hypotheses that $\{\mathscr{L}(S_n - ES_{n,\tau})\}$ is relatively compact for every $\tau \in C(\mu)$ (see the proof of Theorem 5.7).

Fix $\tau \in C(\mu)$ and write $Y_{nj} = X_{nj} - EX_{nj\tau}$, $T_n = \sum_{j=1}^{j_n} Y_{nj}$. Note that $EX_{n1\tau} \rightarrow 0$ in B by (2) and that $\{Y_{nj}\}$ satisfies (*) (write $Y_{nj} = (X_{nj\alpha} - EX_{nj\alpha}) + X_{nj}^\alpha + EX_{n1\alpha} - EX_{n1\tau}$ and use (1) and (2)).

Now we prove that

$$j_n \mathscr{L}(Y_{n1}) \mid B_\delta^c \rightarrow_w \mu \mid B_\delta^c \tag{6.1}$$

for every $\delta \in C(\mu)$. Fix such a δ. If $0 < \varepsilon < \delta$ we have for sufficiently large n

$$j_n P[X_{n1} \in B_{\delta+\varepsilon}^c] \le j_n P[Y_{n1} \in B_\delta^c] \le j_n P[X_{n1} \in B_{\delta-\varepsilon}^c]$$

since $EX_{n1\tau} \rightarrow 0$; then (2) implies $\lim_n j_n \mathscr{L}(Y_{n1})(B_\delta^c) = \mu(B_\delta^c)$ because $\delta \in C(\mu)$. Similarly, if F is a closed set and $\varepsilon > 0$

$$\limsup_n j_n P[Y_{n1} \in B_\delta^c \cap F] \le \mu\,(((\mathring{B}_\delta)^c \cap F) + B_\varepsilon),$$

which shows that $\limsup_n (j_n \mathscr{L}(Y_{n1}) \mid B_\delta^c)(F) \le (\mu \mid B_\delta^c)(F)$. Hence (6.1) holds.

As in the proof of Theorem 6.2 we can prove, using Lemma 6.1, that $\liminf_n Ef^2(S_{n,\delta} - ES_{n,\delta})$ and $\limsup_n Ef^2(S_{n,\delta} - ES_{n,\delta})$ are increasing functions of δ for each $f \in B'$. Then (3) implies that

$$\Phi(f) = \lim_{\delta \downarrow 0} \begin{Bmatrix} \limsup_n \\ \liminf_n \end{Bmatrix} Ef^2(S_{n,\delta} - ES_{n,\delta}) \tag{6.2}$$

for every $f \in W$.

Next, we show that for each $f \in W$ we have

$$\lim_{\delta \downarrow 0, \delta \in C(\mu)} \begin{Bmatrix} \limsup_n \\ \liminf_n \end{Bmatrix} Ef^2(T_{n,\delta} - ET_{n,\delta}) = \Phi(f). \tag{6.3}$$

In order to prove this, fix $f \in W$ and observe that by (6.2) it is sufficient to show that

$$\lim_n (Ef^2(S_{n,\delta} - ES_{n,\delta}) - Ef^2(T_{n,\delta} - ET_{n,\delta})) = 0 \tag{6.4}$$

for each $\delta \in C(\mu)$. Let $\delta \in C(\mu)$ and write $\tilde{S}_n = S_{n,\delta} - ES_{n,\delta}$, $\tilde{T}_n = T_{n,\delta} - ET_{n,\delta}$. Using the Cauchy-Schwarz inequality we obtain

$$|Ef^2(\tilde{S}_n) - Ef^2(\tilde{T}_n)| \le \{(Ef^2(\tilde{S}_n))^{1/2} + (Ef^2(T_{n,\delta}))^{1/2}\}(Ef^2(\tilde{S}_n - \tilde{T}_n))^{1/2}$$

and moreover we have $\sup_n Ef^2(\tilde{S}_n) < \infty$ and $\sup_n Ef^2(T_{n,\delta}) < \infty$ (use Lemma 5.5 and Proposition 3.5). By Proposition 3.5 ($\{(X_{nj\delta} - EX_{nj\delta}) - (Y_{nj\delta} - EY_{nj\delta})\}$ satisfies (*)), (6.4) will follow if we prove that $\tilde{S}_n - \tilde{T}_n \to_P 0$. One has

$$\begin{aligned} E\|\tilde{S}_n - \tilde{T}_n\| &\le j_n E\|(X_{n1\delta} - EX_{n1\delta}) - (Y_{n1\delta} - EY_{n1\delta})\| \\ &\le j_n E[\|EX_{n1\tau} - EX_{n1\delta} + EY_{n1\delta}\|; \|X_{n1}\| \le \delta, \|Y_{n1}\| \le \delta] \\ &\quad + j_n E[\|X_{n1} - EX_{n1\delta} + EY_{n1\delta}\|; \|X_{n1}\| \le \delta, \|Y_{n1}\| > \delta] \\ &\quad + j_n E[\| - Y_{n1} - EX_{n1\delta} + EY_{n1\delta}\|; \|X_{n1}\| > \delta, \|Y_{n1}\| \le \delta] \\ &\quad + j_n E[\|EY_{n1\delta} - EX_{n1\delta}\|; \|X_{n1}\| > \delta, \|Y_{n1}\| > \delta] \\ &= a_n + b_n + c_n + d_n \quad \text{(say).} \end{aligned}$$

Take ε such that $0 < \varepsilon < \delta$; for sufficiently large n we have $\|EX_{n1\tau}\| \le \varepsilon$ and then

$$\begin{aligned} a_n &\le j_n \|EX_{n1\tau} - EX_{n1\delta} + E[X_{n1} - EX_{n1\tau}; \|Y_{n1}\| \le \delta]\| \\ &= j_n \|(EX_{n1\tau})P[\|Y_{n1}\| > \delta] - E[X_{n1}; \|X_{n1}\| \le \delta, \|Y_{n1}\| > \delta] \\ &\quad + E[X_{n1}; \|Y_{n1}\| \le \delta, \|X_{n1}\| > \delta]\| \\ &\le \|EX_{n1\tau}\| j_n P[\|Y_{n1}\| > \delta] + 3\delta j_n P[\delta - \varepsilon \le \|X_{n1}\| \le \delta + \varepsilon], \\ b_n &\le 3\delta j_n P[\delta - \varepsilon \le \|X_{n1}\| \le \delta], \\ c_n &\le 3\delta j_n P[\delta \le \|X_{n1}\| \le \delta + \varepsilon], \\ d_n &\le \{\|EY_{n1\delta}\| + \|EX_{n1\delta}\|\} j_n P[\|X_{n1}\| > \delta]. \end{aligned}$$

Applying (2) we obtain that

$$\limsup_n E\|\tilde{S}_n - \tilde{T}_n\| \le 9\delta\mu(\{x: \delta - \varepsilon \le \|x\| \le \delta + \varepsilon\})$$

for every $\varepsilon \in (0, \delta)$ and therefore $\lim_n E\|\tilde{S}_n - \tilde{T}_n\| = 0$ since $\delta \in C(\mu)$. As remarked above this implies (6.4) and thus (6.3) is proved.

On the other hand, we claim that

$$ET_{n,\tau} \to 0 \text{ in } B. \tag{6.5}$$

The equalities

$$\begin{aligned} ET_{n,\tau} &= j_n E[X_{n1} - EX_{n1\tau};\ \|Y_{n1}\| \le \tau] \\ &= j_n E[X_{n1};\ \|Y_{n1}\| \le \tau,\ \|X_{n1}\| > \tau] - j_n E[X_{n1};\ \|X_{n1}\| \le \tau,\ \|Y_{n1}\| > \tau] \\ &\quad + (EX_{n1\tau}) j_n P[\|Y_{n1}\| > \tau] \end{aligned}$$

and the fact that $EX_{n1\tau} \to 0$ imply that

$$\limsup_n \|ET_{n,\tau}\| \le 2(\tau + \varepsilon)\mu(\{x: \tau - \varepsilon \le \|x\| \le \tau + \varepsilon\})$$

for every $\varepsilon \in (0, \tau)$ and this implies (6.5) because $\tau \in C(\mu)$.

By the relative compactness of $\{\mathscr{L}(S_n - ES_{n,\tau})\} = \{\mathscr{L}(T_n)\}$, claims (6.1), (6.3) and (6.5) and Theorem 6.2 applied to $\{Y_{nj}\}$ we may conclude the proof through a standard argument. □

6.5 COROLLARY. *Suppose that B is a Hilbert space. Let $\{X_{nj}\}$ be a stationary, ψ-mixing triangular array such that $\phi(1) < 1$, $\psi^* < \infty$. Assume*

(1) *there exists a σ-finite measure μ such that for every $\tau \in C(\mu)$*

$$j_n \mathscr{L}(X_{n1}) \,|\, B_\tau^c \to_w \mu \,|\, B_\tau^c,$$

(2) *one of the conditions* (i) *or* (ii) *of Corollary* 6.3 *holds and there exists a sequence $\delta_k \downarrow 0$ such that*

$$\Phi(f) = \lim_k \begin{Bmatrix} \limsup_n \\ \liminf_n \end{Bmatrix} V_n(\delta_k, f)$$

exists for every $f \in B'$,

(3) *there exists $\beta > 0$ such that*

$$\lim_k \sup_n j_n E d_k^2(X_{n1\beta} - EX_{n1\beta}) = 0.$$

Then (a) *μ is a Lévy measure,* (b) *there exists a centered Gaussian measure γ with covariance $\Phi_\gamma(f, f) = \Phi(f)(f \in B')$,* (c) *$\mathscr{L}(S_n - ES_{n,\tau}) \to_w \gamma * c_\tau \mathrm{Pois}\,\mu$ for every $\tau \in C(\mu)$.*

PROOF. We will show that the hypotheses of the previous result are satisfied.

Write $Y_{nj} = X_{nj\beta} - EX_{nj\beta}$. We prove that $\{Y_{nj}\}$ satisfies (*). Let $\{r_n\} \subset N$ such that $r_n \le j_n$ and $r_n/j_n \to 0$. We have, as in the proof of Corollary 4.5,

$$Ed_k^2\left(\sum_{j=1}^{r_n} Y_{nj}\right) \le \left(1 + 4\sum_{j=1}^{\infty} \phi^{1/2}(j)\right) r_n E d_k^2(Y_{n1})$$

and

$$Ef^2(\sum_{j=1}^{r_n} Y_{nj}) \leq (1 + 4\sum_{j=1}^{\infty} \phi^{1/2}(j))(r_n/j_n)C_{\beta,f}$$

for each $f \in B'$. If (i) of Corollary 6.3 holds we have by hypothesis (1) that $C_{\beta,f} < \infty$ for each $f \in B'$ (see the proof of Corollary 6.3) and therefore, using (3) and [1, Theorem 2.3], $\sum_{j=1}^{r_n} Y_{nj} \to_P 0$. In case (ii) holds note that

$$Ed_k^2(\sum_{j=1}^{r_n} Y_{nj}) \leq (1 + 2\sum_{j=1}^{\infty} \psi(j))r_n Ed_k^2(Y_{n1}),$$

$$Ef^2(\sum_{j=1}^{r_n} Y_{nj}) \leq (1 + 2\sum_{j=1}^{\infty} \psi(j))(r_n/j_n)M_{\beta,f}^2$$

for each $f \in B'$ (use Proposition 2.6) and argue as above. Then (1) of Theorem 6.4 holds.

On the other hand, hypothesis (2) implies (3) of the previous result (see the proof of Corollary 6.3).

Finally, the inequalities

$$Ed_k^2(S_{n,\beta} - ES_{n,\beta}) \leq (1 + 4\sum_{j=1}^{\infty} \phi^{1/2}(j))j_n Ed_k^2(Y_{n1}),$$

$$Ed_k^2(S_{n,\beta} - ES_{n,\beta}) \leq (1 + 2\sum_{j=1}^{\infty} \psi(j))j_n Ed_k^2(Y_{n1})$$

together with hypothesis (3) and condition (i) or (ii) imply (4) of Theorem 6.4. □

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REFERENCES

[1] DE ACOSTA, A. (1970). Existence and convergence of probability measures in Banach spaces. *Trans. Amer. Math. Soc.* **152** 273–298.

[2] DE ACOSTA, A. (1982). Invariance principles in probability for triangular arrays of B-valued random vectors and some applications. *Ann. Probab.* **10** 346–373.

[3] DE ACOSTA, A., ARAUJO, A. and GINÉ, E. (1978). On Poisson measures, Gaussian measures and the central limit theorem in Banach spaces. In Probability in Banach spaces. *Advances in Probability, IV* (J. Kuelbs, ed.) 1–68. Dekker, New York.

[4] ARAUJO, A. and GINÉ, E. (1979). On tails and domains of attraction of stable measures in Banach spaces. *Trans. Amer. Math. Soc.* **248** 105–119.

[5] ARAUJO, A. and GINÉ, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables.* Wiley, New York.

[6] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[7] BILLINGSLEY, P. (1971). Weak convergence of measures: applications in probability. *Regional Conference Series in Applied Mathematics* **5**. SIAM, Philadelphia.

[8] EBERLEIN, E. (1979). An invariance principle for lattices of dependent random variables. *Z. Wahrsch. verw. Gebiete* **50** 119–133.

[9] GIHMAN, I. I. and SKOROHOD, A. V. (1974). *The Theory of Stochastic Processes I.* Springer-Verlag, New York.

[10] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables.* Addison Wesley, Reading, Mass.

[11] HOFFMANN-JORGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

[12] IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.

[13] IOSIFESCU, M. and THEODORESCU, R. (1969). *Random Processes and Learning*. Springer-Verlag, Berlin.

[14] KUELBS, J. (1973). The invariance principle for Banach space valued random variables. *J. Multivariate Anal.* **3** 161–172.

[15] PHILIPP, W. (1969). The central limit problem for mixing sequences of random variables. *Z. Wahrsch. verw. Gebiete* **12** 155–171.

[16] PHILIPP, W. (1980). Weak and L^p-invariance principles for sums of B-valued random variables. *Ann. Probab.* **8** 68–82.

[17] STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Statist.* **36** 423–439.

DEPARTAMENTO DE MATEMÁTICA
FACULTAD DE CIENCIAS EXACTAS, UNLP
CASILLA DE CORREO 172
1900 LA PLATA
ARGENTINA