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RANDOMLY STARTED SIGNALS WITH WHITE NOISE[1]

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It is shown that if $B(t)$, $t \geq 0$, is a Wiener process, U is an independent random variable uniformly distributed on (0, 1), and ε is a constant, then the distribution of $B(t) + \varepsilon\sqrt{(t-U)^+}$, $0 \leq t \leq 1$, is absolutely continuous with respect to Wiener measure on $C[0, 1]$ if $0 < \varepsilon < 2$, and singular with respect to this measure if $\varepsilon > \sqrt{8}$.

1. Introduction. Let $C[0, \infty)$ be the space of continuous functions on $[0, \infty)$, let $\mathscr{F}$ be the Borel subsets of $C[0, \infty)$ for the topology of uniform convergence on compact sets, and let μ be Wiener measure on $\mathscr{F}$. For $t \geq 0$, define the random variable $B(t)$ on $(C[0, \infty), \mathscr{F}, \mu)$ by $B(t)(f) = f(t)$, so that $B(t)$, $t \geq 0$, is a standard Wiener process. Let U be a random variable independent of $B(t)$, $t \geq 0$, and uniformly distributed in (0, 1). (Formally, we must enlarge our probability space to permit such a U.) For a positive constant δ define $W_\delta(t)$, $t \geq 0$, by

$$W_\delta(t) = B(t) + \int_0^t \delta 2^{-1}(s-U)^{-1/2} I(U \leq s \leq U+1)\, ds,$$

where I denotes the indicator function, and let γ_δ be the distribution of W_δ. We prove

THEOREM 1. *If $0 < \delta < 2$, γ_δ is absolutely continuous with respect to μ. If $\delta > \sqrt{8}$, γ_δ is singular with respect to μ.*

We do not know what happens for $\delta \in [2, \sqrt{8}]$. We remark that Theorem 1 is essentially equivalent to the statement that the distribution of $B(t) + \delta\sqrt{(t-U)^+}$, $0 \leq t \leq 1$, is absolutely continuous with respect to Wiener measure on $C[0, 1]$ if $0 < \delta < 2$, and singular with respect to this measure if $\delta > \sqrt{8}$. Also, notice that it is easy to show that, for a fixed number a and any constant $\varepsilon > 0$, the distribution η of the process

$$\gamma(t) = B(t) + \int_0^t \varepsilon 2^{-1}(s-a)^{-1/2} I(a \leq s \leq a+1)\, ds$$

is singular with respect to μ. This can be done either using Girsanov's formula,

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which will be stated in Section 3, or by showing that if

$$F = \{f \in C[0, \infty): \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} (f(a + 2^{-k}) - f(a + 2^{-(k+1)}))2^{k/2} > 0\},$$

then $\mu(F) = 0$ while $\eta(F) = 1$, both statements holding by the strong law of large numbers for iid random variables.

The result ([1]) that, for constant ε, the probability

$$P_\varepsilon = P(\exists t: B(t + h) - B(t) > \varepsilon\sqrt{h} \text{ for all } h \in (0, 1))$$

equals zero for $\varepsilon > 1$, and equals one if $\varepsilon < 1$, has somewhat the same flavor as Theorem 1, although the proofs of these results are only related in that both the proof that $P_\varepsilon = 0$ if $\varepsilon > 1$, and the proof that γ_δ is singular with respect to μ if $\delta > \sqrt{8}$, have a common ancestor in Dvoretzky's argument in [2].

2. Singularity. Let $\varepsilon > \sqrt{8}$. The measure γ_ε will be shown to be singular with respect to μ by exhibiting a set $A_\varepsilon \in \mathscr{F}$ such that $\gamma_\varepsilon(A_\varepsilon) = 1$ and $\mu(A_\varepsilon) = 0$. Put $\varphi(s) = [2(s-1)\ln s]^{1/2}/(s^{1/2} - 1)$. Then $\varphi(s)$ decreases to $\sqrt{8}$ as s decreases to 1. Let $r(\varepsilon) = r > 1$ satisfy $\sqrt{8} < \varphi(r) < \varepsilon$, put $\beta = \varepsilon^2/\varphi^2(r) > 1$ and $\alpha = (\beta + 1)/2$. For integers $n \ge 1$ and $0 \le k \le [r^n]$, where [ ] is the greatest integer function, define the functions $Q_{k,n}$ on $C[0, \infty)$ by

$$Q_{k,n}(f) = n^{-1/2} \sum_{m=1}^{n} (r^{-m+1} - r^{-m})^{-1/2}(f(kr^{-n} + r^{-m+1}) - f(kr^{-n} + r^{-m})),$$

and put

$$S_n(f) = I(\max_{0\le k\le [r^n]} Q_{k,n}(f) \ge (2n\alpha \ln r)^{1/2}).$$

The set A_ε is defined by

$$A_\varepsilon = \{f: \limsup_{n\to\infty} S_n(f) = 1\}.$$

To show $\mu(A_\varepsilon) = 0$, we note that, considered as a random variable on $(C[0, \infty), \mathscr{F}, \mu)$, $Q_{k,n}$ is $n^{-1/2}$ times the sum of n independent standard normal random variables, so that $Q_{k,n}$ itself has a standard normal distribution. Thus if

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

$$\begin{aligned}\mu(S_n(f) = 1) &\le ([r^n] + 1)(1 - \Phi[(2n\alpha \ln r)^{1/2}])\\ &\le 2r^n \exp(-[(2n\alpha \ln r)^{1/2}]^2/2)\\ &= 2r^{n(1-2\alpha)}.\end{aligned}$$

Since $\alpha > 1$, $\sum_{n=1}^{\infty} \mu(S_n(f) = 1) < \infty$, so $\mu(A_\varepsilon) = 0$.

Now let $k(U, n) = k$ be that integer satisfying $kr^{-n} \le U < (k + 1)r^{-n}$. The conditional distribution of

$$(r^{-m+1} - r^{-m})^{-1/2}[W_\varepsilon(kr^{-n} + r^{-m+1}) - W_\varepsilon(kr^{-n} + r^{-m})]$$

given $U = u$ is normal with variance 1 and mean equal to

$$(r^{-m+1} - r^{-m})^{-1/2} \int_{kr^{-n}+r^{-m}}^{kr^{-n}+r^{-m+1}} \varepsilon 2^{-1}(s-u)^{-1/2}\, ds$$

$$\geq (r^{-m+1} - r^{-m})^{-1/2} \int_{kr^{-n}+r^{-m}}^{kr^{-n}+r^{-m+1}} \varepsilon 2^{-1}(s-kr^{-n})^{-1/2}\, ds$$

$$= \varepsilon(r-1)^{-1/2}(r^{1/2}-1)$$

$$= (2\beta \ln r)^{1/2},$$

so that conditioned on $U = u$

$$Y = n^{-1/2} \textstyle\sum_{m=1}^{n} (r^{-m+1} - r^{-m})^{-1/2}(W_\varepsilon(kr^{-n} + r^{-m+1}) - W_\varepsilon(kr^{-n} + r^{-m}))$$

is normal with variance 1 and mean exceeding $(2n\beta \ln r)^{1/2}$. In particular, $P(Y > (2n\alpha \ln r)^{1/2} \mid U = u) \geq \Phi[(2n\beta \ln r)^{1/2} - (2n\alpha \ln r)^{1/2}] = q_n$, so $\gamma_\varepsilon\{f \in C[0,\infty): S_n(f) = 1\} \geq q_n$. Since $q_n \to 1$ as $n \to \infty$ we get $\gamma_\varepsilon(A_\varepsilon) = 1$.

3. Absolute continuity. If $f(s)$, $s \geq 0$, is a measurable function such that $\int_0^\infty f^2(s)\, ds < \infty$, Girsanov's formula (see [3]) gives that if ρ is the distribution of the process $B(t) + \int_0^t f(s)\, ds$, $t \geq 0$, then the Radon Nikodym derivative of ρ with respect to μ is

$$\frac{d\rho}{d\mu} = \exp\left(\int_0^\infty f(s)\, dB(s) - \frac{1}{2}\int_0^\infty f^2(s)\, ds\right).$$

We let EX stand for $\int_{C[0,\infty)} X d\mu$. Of course, $E(d\rho/d\mu) = 1$.

For an integer $n > 1$ and a constant $\delta > 0$ put $\alpha_n(v, t, \delta) = \alpha_n(v, t) = \delta 2^{-1}(v-t)^{-1/2} I(t + n^{-1} \leq v \leq t+1)$. Let

$$W_\delta^n(t) = B(t) + \int_0^t \alpha_n(s, U)\, ds,$$

and let γ_δ^n be the distribution of W_δ^n. We will show that, for $0 < \delta < 2$,

$$E(d\gamma_\delta^n/d\mu)^2 \leq M_\delta < \infty,$$

which gives that the random variables $d\gamma_\delta^n/d\mu$ are uniformly integrable with respect to μ. Since $| W_\delta^n(t) - W_\delta(t) | \leq \delta/\sqrt{n} \to 0$ as $n \to \infty$, this implies that γ_δ is absolutely continuous with respect to μ if $0 < \delta < 2$.

We have

$$E\left(\frac{d\gamma_\delta^n}{d\mu}\right)^2 = E\left[\left(\int_0^1 \exp\left(\int_0^\infty \alpha_n(v,t)\, dB(v) - \frac{1}{2}\int_0^\infty \alpha_n^2(v,t)\, dv\right) dt\right)^2\right]$$

$$= E \int_0^1 \int_0^1 \exp\left(\int_0^\infty (\alpha_n(v,t) + \alpha_n(v,s))\, dB(v) - \frac{1}{2}\int_0^\infty (\alpha_n^2(v,t) + \alpha_n^2(v,s))\, dv\right) ds\, dt$$

$$= \int_0^1 \int_0^1 E \exp\left(\int_0^\infty (\alpha_n(v, t) + \alpha_n(v, s))\, dB(v) - \frac{1}{2}\int_0^\infty (\alpha_n^2(v, t) + \alpha_n^2(v, s))\, dv\right) ds\, dt$$

$$= \int_0^1 \int_0^1 \exp \int_0^\infty \alpha_n(v, t)\alpha_n(v, s)\, dv \cdot E \exp\left(\int_0^\infty (\alpha_n(v, t) + \alpha_n(v, s))\, dB(v) - \frac{1}{2}\int_0^\infty (\alpha_n(v, t) + \alpha_n(v, s))^2\, dv\right) ds\, dt$$

$$= \int_0^1 \int_0^1 \exp\left(\int_0^\infty \alpha_n(v, t)\alpha_n(v, s)\, dv\right) ds\, dt$$

$$= 2 \int_0^1 \int_s^1 \exp\left(\frac{\delta^2}{4} \int_{t+n^{-1}}^{s+1} [(v - t)(v - s)]^{-1/2}\, dv\right) ds\, dt.$$

Now if $s < t < s + 1$,

$$\int_{t+n^{-1}}^{s+1} [(v - t)(v - s)]^{-1/2}\, dv \le \int_t^{s+1} [(v - t)(v - s)]^{-1/2}\, dv$$
$$= \ln[(2 - (t - s) + 2\sqrt{1 - (t - s)})/(t - s)]$$
$$\le \ln[4/(t - s)],$$

so that $E(d\gamma_\delta^n/d\mu)^2 \le 2 \int_0^1 \int_s^1 (4/(t - s))^{\delta^2/4}\, dt\, ds < \infty$ if $0 < \delta < 2$.

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