The Annals of Probability
1984, Vol. 12, No. 3, 869–875

EXPONENTIAL APPROXIMATIONS FOR TWO CLASSES OF AGING DISTRIBUTIONS

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Inequalities are derived for the quality of exponential approximations to NBUE (new better than used) and NWUE (new worse than used) distributions.

1. Introduction. If a random variable is exponentially distributed with $\mu = EX$ and $\mu_2 = EX^2$, then $\mu_2 = 2\mu^2$. Defining $\rho = | \mu_2/2\mu^2 - 1 |$, it is tempting to conjecture that under mild restrictions a distribution with small ρ is approximately exponential. That restrictions are needed is seen by the example, $\Pr(X = 0) = \Pr(X = 1) = ½$, for which $\rho = 0$.

The scale invariant quantity, ρ, was suggested by Keilson [3]. It has an interesting interpretation. Define $\bar{F} = 1 - F$ and $G(x) = \mu^{-1} \int_0^x \bar{F}(s)\, ds$, the stationary renewal distribution corresponding to F. Then $\mu_G = \mu_2/2\mu$ and $\rho = | \mu_G/\mu - 1 |$. The parameter ρ is thus the scaled (by μ) distance between μ and μ_G. For F exponential, $F = G$ and thus $\mu = \mu_G$.

The problem of interest can be stated as follows: Given a class $\mathscr{S}$ of distributions, along with the first two moments μ and μ_2, find upper bounds for $\sup_{t,F\in\mathscr{S}} | \bar{F}(t) - e^{-t/\mu} |$ in terms of ρ.

The above problem for the class of completely monotone distributions (mixtures of exponential distributions) was studied by Keilson [8], Heyde [6], Heyde and Leslie [7], Hall [5], and Brown [2].

Brown [2] considered the class of IMRL (increasing mean residual life) distributions on $[0, \infty)$ deriving:

$$\sup_t | \bar{F}(t) - e^{-t/\mu} | \le \rho/(\rho + 1) \tag{1.1}$$

$$\sup_{B\in\beta} | F(B) - G(B) | \le \rho/(\rho + 1) \tag{1.2}$$

$$\sup_{B\in\beta} | G(B) - \int_B \mu^{-1} e^{-t/\mu} dt | \le \rho/(\rho + 1) \tag{1.3}$$

$$\sup_t | \bar{G}(t) - e^{-t/\mu_G} | \le \rho/(\rho + 1). \tag{1.4}$$

In (1.2) and (1.3) above, β is the collection of Borel subsets of $[0, \infty)$. The quantity $\rho/(\rho + 1)$ was shown to be the best upper bound for (1.1) and (1.2) even within the subclass of completely monotone distributions.

Brown [3] considered the class of IFR (increasing failure rate) distributions. It turns out that in this case (1.1) and (1.2) hold with $\rho/(\rho + 1)$ replaced by 2ρ,

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Received April 1983; revised January 1984.

[1] Research supported by Air Force Office of Scientific Research under AFOSR Grant No. 82-0024.

AMS 1980 *subject classifications.* Primary 60E15; secondary 60K10.

Key words and phrases. Exponential approximations, NBUE and NWUE distributions, inequalities, renewal theory, geometric sums.

and (1.3) and (1.4) hold with $\rho/(\rho+1)$ replaced by ρ. The bound 2ρ is the best bound for (1.1), among bounds of the form $c\rho^{\alpha}$.

In this paper we consider the problem for F NBUE (new better than used in expectation) and for F NWUE (new worse than used in expectation). These are the weakest among the commonly studied classes of aging distributions, and it is often easy to demonstrate that a distribution belongs to one of these classes (NBUE and NWUE are defined in Section 2). The methods of Brown ([2], [3]) do not generalize to these cases because the partial ordering between F and G is too weak. Instead we use Fourier methods adopted from Feller [4]. Our main result is that for F NBUE or NWUE:

$$\sup_t |\bar{F}(t) - e^{-t/\mu}| \le A\rho^{1/2} \tag{1.5}$$

where $A = 4\sqrt{6}/\pi \approx 3.119$. For the NBUE case we show that the best bound of the form $c\rho^{\alpha}$ has $\alpha = ½$ and $1 \le c \le 4\sqrt{6}/\pi$. Thus the potential improvement in (1.5) for F NBUE is the lowering of the constant from 3.119 to 1. This remains true even within the subclass of IFRA distributions.

The standard reference for the classes of distributions discussed in this paper (NBUE, NWUE, IMRL, IFR, IFRA, NBU, completely monotone) is Barlow and Proschan [1].

2. Definitions and preliminary results. A distribution F on $[0, \infty)$ with $F(0) < 1$ and finite mean μ is defined to be NBUE if $E(X - t \mid X > t) \le \mu$ for all $t \ge 0$ with $\bar{F}(t) > 0$. Since $E(X - t \mid X > t) = \mu\bar{G}(t)/\bar{F}(t)$, it follows that F is NBUE if and only if F is stochastically larger than G, the stationary renewal distribution corresponding to F. Define h_G to be the failure rate function of G and note that $h_G(t) = [E(X - t \mid X > t)]^{-1}$, thus F is NBUE if and only if $h_G(t) \ge \mu^{-1}$ for all $t \ge 0$ with $\bar{F}(t) > 0$.

A distribution F on $[0, \infty)$ with $F(0) > 1$ and finite mean is defined to be NWUE if $E(X - t \mid X > t) \ge \mu$ for all $t \ge 0$ with $\bar{F}(t) > 0$. This is equivalent to F being stochastically smaller than G, and also to $h_G \le \mu^{-1}$.

LEMMA 2.1. *If F is* NBUE *then $\bar{G}(t) \le e^{-t/\mu}$ for all $t \ge 0$; for F* NWUE, *$\bar{G}(t) \ge e^{-t/\mu}$ for all $t \ge 0$.*

PROOF. For F NBUE let t_0 be the smallest number such that $\bar{F}(t_0) = 0$, with $t_0 = \infty$ if $\bar{F}(t) > 0$ for all t. Now $h_G(t) \ge \mu^{-1}$ for $0 \le t < t_0$, thus $\bar{G}(t) \le e^{-t/\mu}$ for $0 \le t < t_0$. If $t_0 < \infty$ then for $t > t_0$, $\bar{G}(t) = 0 \le e^{-t/\mu}$. If F is NWUE then $\bar{F}(t) > 0$ for all t, for if $\bar{F}(t_0) = 0$ for a finite t_0 then $\lim_{t\to t_0} E(X - t \mid X > t) = 0 < \mu$. Thus $h_G(t) \le \mu^{-1}$ for all $t \ge 0$ and $\bar{G}(t) \ge e^{-t/\mu}$ for all $t \ge 0$.

The following inequality (Lemma 2.2) is quite an important tool in deriving our subsequent results. It relies heavily on a smoothing result of Feller [4] (Lemma 1, page 510).

LEMMA 2.2. *Let F_1, F_2 be probability distributions on $[0, \infty)$ with finite means μ_1 and μ_2. Assume that F_1 is either stochastically larger or smaller than F_2, and* *that F_2 is differentiable with $F_2'(x) \le \mu_1^{-1}$ for all $x \ge 0$. Then:*

$$\sup_x |F_1(x) - F_2(x)| \le A[|\mu_1 - \mu_2|/\mu_1]^{1/2}$$

where $A = 4\sqrt{6}/\pi$.

PROOF. By Feller [4], Lemma 1, page 510,

$$\sup |F_1(x) - F_2(x)| \le 2 \sup_t |T_{\Delta(t)}| + 24/\pi\mu_1 T \tag{2.3}$$

where

$$\Delta(x) = F_1(x) - F_2(x), \quad T_{\Delta(t)} = \int_{-\infty}^{\infty} \Delta(t-x) V_T(x)\, dx,$$

$$V_T(x) = \frac{1 - \cos Tx}{\pi T x^2}.$$

Now, assume that F_1 is stochastically larger than F_2. Then:

$$\begin{aligned} |T_{\Delta(t)}| &= \left| \int_{-\infty}^{\infty} [F_1(t-x) - F_2(t-x)] \frac{1 - \cos Tx}{\pi T x^2}\, dx \right| \\ &= \int_{-\infty}^{t} [\bar{F}_1(t-x) - \bar{F}_2(t-x)] \frac{1 - \cos Tx}{\pi T x^2}\, dx \\ &\le \frac{T}{2\pi} \int_{-\infty}^{t} [\bar{F}_1(t-x) - \bar{F}_2(t-x)]\, dx = \frac{T}{2\pi}(\mu_1 - \mu_2). \end{aligned}$$

Thus from (2.3):

$$\sup |F_1(x) - F_2(x)| \le (1/\pi)[T(\mu_1 - \mu_2) + 24/\mu_1 T].$$

Define $L(T) = T(\mu_1 - \mu_2) + 24/\mu_1 T$; then a routine differentiation argument gives:

$$\min_{T>0} L(T) = L[(24/\mu_1(\mu_1 - \mu_2))^{1/2}] = 4\sqrt{6}\,[1 - (\mu_2/\mu_1)]^{1/2}$$

and the result is proved.

If F_2 is stochastically larger than F_1 the analogous result follows by similar argument.

3. NBUE results. Assume that F is NBUE. Recall that $\bar{G}(t) \le \bar{F}(t)$ and $\bar{G}(t) \le e^{-t/\mu}$ for all $t \ge 0$, where G is the stationary renewal distribution corresponding to F. Note that $G'(x) = \bar{F}(x)/\mu \le \mu^{-1}$ for all x. Applying Lemma 2.2 with $F_1 = F$, $F_2 = G$ we obtain:

$$\sup |\bar{F}(x) - \bar{G}(x)| \le (4\sqrt{6}/\pi)(1 - \mu_G/\mu)^{1/2} = (4\sqrt{6}/\pi)\rho^{1/2}. \tag{3.1}$$

By Brown [1], Remark 4.14, for F NBUE:

$$\sup |\bar{G}(t) - e^{-t/\mu}| \le \sup |G(B) - \int_B \mu^{-1} e^{-t/\mu}\, dt| \le \rho. \tag{3.2}$$

Since F NBUE implies $\bar{G}(x) \leq \min(\bar{F}(x), e^{-x/\mu})$ for all $x \geq 0$, it follows that:

$$\begin{aligned} &\sup |\bar{F}(x) - e^{-x/\mu}| \\ &\quad \leq \max(\sup |\bar{F}(x) - \bar{G}(x)|, \sup |\bar{G}(x) - e^{-x/\mu}|) \leq (4\sqrt{6}/\pi)\rho^{1/2}. \end{aligned} \tag{3.3}$$

Next, by simple computation:

$$\sup | e^{-t/\mu} - e^{-t/\mu_G}| \leq 1 - (\mu_G/\mu) \doteq \rho. \tag{3.4}$$

Moreover, $e^{-t/\mu} \geq \max(\bar{G}(t), e^{-t/\mu_G})$, thus:

$$\sup |\bar{G}(t) - e^{-t/\mu_G}| \leq \max(\sup |\bar{G}(t) - e^{-t/\mu}|, \sup | e^{-t/\mu} - e^{t/\mu_G}|) \leq \rho. \tag{3.5}$$

We summarize these results in Theorem 3.6.

THEOREM 3.6. *Let F be* NBUE. *Then:*

$$\sup |\bar{F}(x) - e^{-x/\mu}| \leq A\rho^{1/2}, \quad \sup |\bar{F}(x) - \bar{G}(x)| \leq A\rho^{1/2}$$

$$\sup |\bar{G}(x) - e^{-x/\mu}| \leq \sup | G(B) - \int_B \mu^{-1}e^{-t/\mu}\, dt | \leq \rho$$

$$\sup |\bar{G}(x) - e^{-x/\mu_G}| \leq \rho$$

where $A = 4\sqrt{6}/\pi$ and $\rho = 1 - (\mu_2/2\mu^2)$.

Corollary (3.7) below presents a limit theorem for NBUE distributions.

COROLLARY 3.7. *Let $\{X_n,\ n \geq 1\}$ be a sequence of* NBUE *random variables with $\mu_n = EX_n$, $\mu_{2,n} = EX_n^2$ and $\rho_n = 1 - (\mu_{2,n}/2\mu_n^2)$. Then X_n/μ_n converges in distribution to an exponential distribution if and only if* $\lim \rho_n = 0$, *in which case the mean of the limiting exponential distribution equals* 1.

PROOF. The sufficiency of the condition $\lim \rho_n = 0$ follows from Theorem 3.6. To prove necessity assume that $\lim \Pr(X_n > t\mu_n) = e^{-ct}$ for all $t \geq 0$, and some $c > 0$. Let G_n denote the stationary renewal distribution corresponding to X_n, and H_n the stationary renewal distribution corresponding to X_n/μ_n. Then $\bar{H}_n(t) = \bar{G}_n(t\mu_n)$ and $\rho_n = 1 - \int_0^\infty \bar{H}_n(t)\, dt$.

Now:

$$\begin{aligned} \lim_{n\to\infty}\bar{H}_n(t) &= 1 - \lim_{n\to\infty}H_n(t) = 1 - \lim_{n\to\infty}\int_0^t \Pr(X_n > s\mu_n)\, ds \\ &= 1 - [(1 - e^{-ct})/c]. \end{aligned}$$

Since X_n is NBUE, so is X_n/μ_n, and it thus follows from Lemma 2.1 that:

$$\bar{H}_n(t) \leq e^{-t} \quad \text{for all} \quad n, t > 0.$$

Thus by the dominated convergence theorem:

$$\lim_{n\to\infty}\rho_n = 1 - \lim_{n\to\infty}\int_0^\infty \bar{H}_n(t)\,dt = 1 - \int_0^\infty [1 - \{(1-e^{-ct})/c\}]\,dt$$

$$= \begin{cases} \infty & \text{for} \quad c < 1 \\ 0 & \text{for} \quad c = 1 \\ -\infty & \text{for} \quad c > 1. \end{cases}$$

But NBUE distributions satisfy $0 \le \rho \le ½$ (since $\mu_2 \ge \mu^2$ by Chebychev's inequality). Thus c equals 1 and $\lim \rho_n = 0$.

4. Potential improvement of NBUE bound. In the following example we have a sequence of IFRA (and thus NBUE) distributions $\{F_n, n \ge 1\}$, with

$$\lim_{n\to\infty} \frac{\sup|\bar{F}_n(x) - e^{-x/\mu_n}|}{\rho_n^{1/2}} = 1.$$

It follows from this example and Theorem 1 that the best bound of the form $c\rho^\alpha$ has $\alpha = ½$ and $1 \le c \le 4\sqrt{6}/\pi$. Thus the maximum potential improvement in the bound $A\rho^{1/2}$ is the lowering of A to 1. This statement holds for the NBUE class as well as for the subclasses NBU and IFRA.

The distributions F_n is defined by:

$$\bar{F}_n(t) = \begin{cases} 1 & t < 1/n \\ e^{-t} & t \ge 1/n. \end{cases}$$

Then:

$$\mu_n = n^{-1} + e^{-n^{-1}}, \quad \mu_{2,n} = n^{-2} + 2(n^{-1} + 1)e^{-n^{-1}}$$

$$\rho_n = 1 - [(1 + 2n(n+1)e^{-n^{-1}})/2(1 + ne^{-n^{-1}})^2]$$

$$D_n = \sup|\bar{F}_n(x) - e^{-x/\mu_n}| = 1 - \exp[-1/(1 + ne^{-n^{-1}})].$$

It follows that:

$$D_n = n^{-1} + o(n^{-1})$$

and

$$\rho_n = n^{-2} + o(n^{-2}).$$

Thus:

$$\lim_{n\to\infty}[D_n/\rho_n^{1/2}] = 1.$$

5. NWUE results. Assume that F is NWUE. Applying Lemma 2.2 with $F_1 = F$ and $F_2 = G$ we obtain:

$$\sup|\bar{F}(x) - \bar{G}(x)| \le A\rho^{1/2}. \tag{5.1}$$

We do not know of an analogue of (3.2) for NWUE distributions, but applying

Lemma 2.2 with $\bar{F}_1(x) = e^{-x/\mu}$ and $F_2 = G$ we obtain:

$$\sup | \bar{G}(x) - e^{-x/\mu} | \leq A\rho^{1/2}. \tag{5.2}$$

Since $\bar{G}(x) \geq \max(\bar{F}(x), e^{-x/\mu})$ it follows from (5.1) and (5.2) that:

$$\sup | \bar{F}(x) - e^{-x/\mu} | \leq A\rho^{1/2}. \tag{5.3}$$

Finally since $e^{-x/\mu} \leq \min(\bar{G}(t), e^{-t/\mu_G})$ and $\sup | e^{-t/\mu_G} - e^{-t/\mu} | \leq 1 - (\mu/\mu_G) = \rho/(\rho + 1)$, we obtain:

$$\sup | \bar{G}(x) - e^{-x/\mu_G} | \leq A\rho^{1/2}. \tag{5.4}$$

Corollary (3.7) does not hold for NWUE distributions. While (5.3) insures that $\lim \rho_n = 0$ is sufficient for convergence to an exponential distribution, $\lim \rho_n = 0$ is not a necessary condition. To see this consider the distribution F with failure rate:

$$h(x) = \begin{cases} 2 & 0 \leq x \leq 1 \\ 2x^{-1} & x > 1. \end{cases}$$

Clearly F is DFR, with finite mean, and infinite second moment. Now, for $n = 1, 2, \cdots$ define:

$$F_n(x) = n^{-1}F(x) + (1 - n^{-1})(1 - e^{-x}).$$

Since F_n is a mixture of DFR distributions, F_n is DFR and thus NWUE. Clearly, F_n converges to an exponential distribution with mean 1. However since F has infinite second moment, so does F_n, and thus $\rho_n = \infty$ for all n.

6. Geometric sums. Define Y to be a geometric sum of X with parameter p if Y can be represented as $\sum_1^N X_i$ with $\{X_i, i \geq 1\}$ i.i.d. as X, N geometrically distributed with parameter p, and N and $\{X_i\}$ independent. One naturally occurring example of a geometric sum is the last renewal epoch in a defective renewal process (Feller [4], Chapter XI, Sections 6 and 7).

THEOREM 6.1. *Let X be either* NBUE *or* NWUE *with finite second moment. Suppose that Y is a geometric sum of X with parameter p. Then*:

$$\sup | \Pr(Y > t) - \exp(-tp\mu^{-1}) | \leq A(p\rho)^{1/2}$$

where $A = 4\sqrt{6}/\pi$ and $\rho = | (EX^2/2(EX)^2) - 1 |$.

PROOF. The result follows from Theorem 3.6 and the following easily verified results:

$$\rho_Y = p\rho_X \quad \text{where} \quad \rho_Y = | (EY^2/2(EY)^2) - 1 |. \tag{6.2}$$

(6.3) If Y is a geometric sum of X and X is NBUE (NWUE) then Y is NBUE (NWUE).

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