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THE HYDRODYNAMICAL BEHAVIOR OF THE COUPLED BRANCHING PROCESS

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The coupled branching process (η_t^μ) is a Markov process on $(\mathbb{N})^S$ $(S = \mathbb{Z}^d)$ with initial distribution μ and the following time evolution: At rate $b\eta(x)$ a particle is born at site x, which moves instantaneously to a site y chosen with probability $q(x, y)$. All particles at a site die at rate pd, individual particles die independent from each other at rate $(1-p)d$. Furthermore, all particles perform independent continuous time random walks with kernel $p(x, y)$. We consider here the case $b = d$ and the symmetrized kernels $\hat{p}$, $\hat{q}$ are transient.

We show that the measures $\mathscr{L}(\eta_t^\mu(\cdot+[\alpha\sqrt{t}x]))$, $(\alpha \in \mathbb{R}^+, x \in \mathbb{R}^d)$ converge weakly for $t \to \infty$ to $\nu_{\tau(\alpha,x)}$. Here ν_ρ is the invariant measure of the process with: $E^{\nu_\rho}(\eta(x)) = \rho$ and which is also extremal in the set of all translationinvariant invariant measures of the process. The density profile $\tau(\alpha, x)$ is calculated explicitly; it is governed by the diffusion equation.

0. Introduction and main theorem. We are interested in transport phenomena for the coupled branching process. The "coupled branching process" is a Markov process on $(\mathbb{N})^S$ $(S = \mathbb{Z}^d)$ evolving as follows: (Denote by η an element of $(\mathbb{N})^S$). A new particle is born at site x at rate $b\eta(x)$; it moves instantaneously to the site y chosen with probability $q(x, y)$. Individual particles die at rate $(1-p)d$, while at rate pd all particles at a site are extinguished. Furthermore all particles perform independent from each other random walks with kernel $p(x, y)$ and rate m. The process with initial distribution μ is denoted by $(\eta_t^\mu)_{t\in\mathbb{R}^+}$.

In the case $b = d$, $p < p^*$ there exists a set $(\nu_\rho)_{\rho\in\mathbb{R}^+}$ of invariant measures with the properties ([2]):

(i) ν_ρ is translation invariant

(1) (ii) $\int \eta(x)\, d\nu_\rho = \rho, \quad \int \eta^2(x)\, d\nu_\rho < \infty$

(iii) $\lim_{t\to\infty} E^{\nu_\rho}(|\sum_{y\in S} r_t(x, y)\eta(y) - \rho|^2) = 0$ with

(2)
$$r_t(x, y) := \sum_{n=0}^\infty e^{-t}\frac{t^n}{n!} r^n(x, y)$$
$$r(x, y) := \frac{b}{m+b}\, q(x, y) + \frac{m}{m+b}\, p(x, y).$$

The property (iii) characterizes the measures which are extremal in the set of all translationinvariant invariant measures. (This is the most convenient character-

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ization in models where one has only knowledge about the first and second moments of $\{\eta(x)\}_{x\in S}$ under the invariant measure. Compare also [3].)

In the sequel we will always assume: $p < p^*$, $b = d$, the matrices $\hat{p}$ and $\hat{q}$ are transient. p^* can be calculated as follows: ([2])

$$p^* := \left(\frac{1}{2}\frac{b}{m+b}G_{\hat{r}}(0,0)\right)^{-1}, \quad G_{\hat{r}} = \textstyle\sum_0^\infty \hat{r}^n, \tag{3}$$

$$\hat{r}(x,y) = \frac{1}{2}(r(x,y) + r(y,x)). \tag{4}$$

A typical question of interest is: How does $\mathscr{L}(\eta_t^\mu)$ behave for $t \to \infty$ if μ is of the form $\delta_{\{\eta\equiv 1 \text{ on } H, \eta\equiv 0 \text{ on } \complement H\}}$ where $H := \{x \in \mathbb{Z}^d,\ x_1 < 0\}$. We will show that in the case where p, q are homogeneous and symmetric matrices such that: $\sum_{y\in S}(p(x,y) + q(x,y))\,\|y\|^2 < \infty$, then $\mathscr{L}(\eta_t^\mu)$ converges weakly to $\nu_{1/2}$ and an observer traveling through $\mathbb{Z}^d$ along the path $[\alpha\sqrt{t}x]$ $(\alpha \in \mathbb{R}^+, x \in \mathbb{R}^d)$ will observe the process converging weakly to $\eta_{\tau(\alpha,x)}$ where $\tau(\alpha, x)$ can be calculated explicitly (in terms of a normal distribution with mean 0 and covariance matrix (σ_{ij}) depending on r).

Following H. Rost [4] we call $\tau(\cdot,\cdot)$ *the density profile*. By $[x]$ we denote $([x_1],\cdots,[x_d]) \in \mathbb{Z}^d$. The methods we use to prove the results outlined above can be adapted to other models as for example the processes introduced in [3] (coupled random walk, potlatch, etc.). For other systems like simple exclusion, voter model, the hydrodynamical limit has been studied by different methods. [4], [5].

In order to formulate our main theorem we have to introduce some notions: consider a translation invariant measure ν on $(\mathbb{N})^S$ which is shift ergodic and has the properties: $\int \eta^2(x)\,d\nu < \infty$, $\int \eta(x)\,d\nu = \rho$ and $\lim_{|u-v|\to\infty} E^\nu(\eta(u)\eta(v)) = E(\eta(u))\cdot E(\eta(v)) = \rho^2$. Now let $H \subseteq \mathbb{R}^d$ be a halfspace.

We define a measure μ through:

$$\mu(\{\eta \in A\}) = \nu(\{\eta\, 1_H \in A\}) \quad \forall\text{: measurable } A \subseteq (\mathbb{N})^S (S = \mathbb{Z}^d). \tag{5}$$

Our method of analysis works also for more general initial distributions. To avoid complicated notation, we focus here on the case described above. In order to make our coupling arguments work we need some homogenity ($\int \eta(x)\,d\nu$ does not vary too much in x) and some (very weak) form of mixing: $E(\eta(u)\eta(v))$ converges to $E(\eta(u))E(\eta(v))$ for $(u - v) \to \infty$. For more details, see the Appendix. $\mathscr{N}_r$ denotes the normal measure on $\mathbb{R}^d$ with mean 0 and covariance matrix $\sigma_{i,j}$ given by:

$$\sigma_{i,j} = \frac{\partial^2}{\partial\theta_i\partial\theta_j}\left(\textstyle\sum_{x\in S} r(x,0)e^{i\theta\cdot x}\right)\big|_{\theta=0} \quad \theta \in [0, 2\pi]^d. \tag{6}$$

THEOREM 1. *Assume that p, q are homogeneous and symmetric matrices and have the property*:

$$\textstyle\sum_{x\in S} r(0,x)\,\|x\|^2 < \infty \qquad \sum_{n=0} r^n(0,x) > 0 \quad \forall{:}x \in \mathbb{Z}^d. \tag{7}$$

Then we have:

(8) a) $\mathscr{L}(\eta_t^\mu) \to_{t\to\infty} \nu_\tau \quad \tau = \rho\, \mathscr{N}_r(H)$

(9) b) $\mathscr{L}(\eta_t^\mu(\cdot+[\alpha\sqrt{t}x])) \to_{t\to\infty} \nu_{\tau(\alpha,x)} \quad \forall\colon \alpha \in \mathbb{R}^+, \quad x \in \mathbb{R}^d$

(10) $E(\eta_t^\mu(\cdot+[\alpha\sqrt{t}x])) \to_{t\to\infty} \tau(\alpha, x)$

(11) c) $\tau(\alpha, x) = \rho\, \mathscr{N}_r(H - \alpha x)$.

This theorem tells us how the population starting in one half-space spreads out through the whole space. Define $\hat{\tau}(\alpha, x) := \tau(\alpha^{-1}, x)$. Then: $\hat{\tau}$ is governed by the diffusion equation, $\hat{\tau}(0, \cdot) = \rho 1_H(\cdot)$.

The density profile is the same as for a system of particles performing independent random walks. But in the latter case the local equilibria are Poisson systems (propagation of chaos), while in our model the local equilibria have slowly decreasing positive correlations [2].

The proof of our theorem proceeds by means of two propositions, formulated and proved in the next two sections.

1. The density profile. The first step in the proof of Theorem 1 is to show the existence of a density profile and to identify it. (Assume $m + b = 1$.)

PROPOSITION 1. *Make the same assumptions as in Theorem* 1. *Let t_n be a sequence with $t_n \uparrow \infty$. Then for all $\alpha \in \mathbb{R}^+$, $x \in \mathbb{R}^d$: $\{\mathscr{L}(\eta^\mu(\cdot+[\alpha\sqrt{t_n}x]))\}_{n\in\mathbb{N}}$ is weakly relative compact and a limit-point λ has the property*:

(12) $$E^\lambda(\eta(z)) = \rho\mathscr{N}_r(H - \alpha x) \quad \forall\colon z \in \mathbb{Z}^d.$$

PROOF. One observes first that our process has the property (denote by $S(t)$ the semigroup of the process):

(13) $$\lambda_1 \geq \lambda_2 \Rightarrow \lambda_1 S(t) \geq \lambda_2 S(t) \quad \forall\colon t \in \mathbb{R}^+.$$

Here $\lambda_1 \geq \lambda_2$ is defined as follows:

$\lambda_1 \geq \lambda_2 \Leftrightarrow \langle \lambda_1, f\rangle \geq \langle \lambda_2, f\rangle$ for all positive increasing functions f on $(\mathbb{N})^S$.

The property (13) implies (using the construction of μ:(5)):

(14) $$E^\mu(\eta_t^2(z + [\alpha\sqrt{t}x])) \leq E^\nu(\eta_t^2(z + [\alpha\sqrt{t}x])) \leq C < \infty \quad \forall\colon z \in \mathbb{Z}^d, \quad t \in \mathbb{R}^+.$$

The last inequality follows from Theorem 1(b) in [2]. So we obtained so far that the sequence $\{\mathscr{L}(\eta_{t_n}^\mu(\cdot + [\alpha\sqrt{t_n}x]))\}_{n\in\mathbb{N}}$ is weakly relative compact and the random variables $\{\eta(z + [\alpha\sqrt{t_n}x])\}_{n\in\mathbb{N}}$ are uniformly integrable under the measures $\mu S(t_n)$.

Now we finish the proof by showing:

(15) $$\lim_{t\to\infty} E(\eta_t^\mu(z + [\alpha\sqrt{t}x])) = \rho\mathscr{N}_r(H - \alpha x) \quad \forall\colon z \in \mathbb{Z}^d.$$

We have the following closed system of differential equations for

(16) $$h_t(x) := E^{\mu}(\eta_t(x)) \quad \text{(see [2])}$$

(17) $$\frac{d}{dt} h_t(x) = \sum_{y \in S} r(y, x) h_t(y) - h_t(x)$$

or in other words:

(18) $$\frac{d}{dt} h_t(x) = (Gh_t)_{(x)} \quad \text{with} \quad G(x, y) := r(y, x) - \delta(x, y).$$

As generator, G determines uniquely a semigroup $(U_t)_{t \in \mathbb{R}^+}$ on $L_\infty(S)$. We can give the following alternative description of this semigroup: Denote by $(X_t^{(z)})_{t \in \mathbb{R}^+}$ the continuous-time random walk on $\mathbb{Z}^d$ with kernel $r(y, x)$ jump rate 1 and initial position z. Then:

(19) $$U_t(f)_{(z)} = E(f(X_t^{(z)})) \quad \forall: z \in \mathbb{Z}^d, \quad f \in L^\infty(S).$$

For our special case we obtain therefore:

(20) $$h_t(x) = \rho \operatorname{Prob}(X_t^{(x)} \in H).$$

Then we calculate as follows:

(21) $$\begin{aligned} E(\eta_t^{\mu}(z + [\alpha\sqrt{t}x])) &= \rho \operatorname{Prob}(X_t^{\{[\alpha\sqrt{t}x]+z\}} \in H) \\ &= \rho \operatorname{Prob}(X_t^0 \in H - ([\alpha\sqrt{t}x] + z)) \\ &= \rho \operatorname{Prob}\left(\left(\frac{X_t^{\{0\}}}{\sqrt{t}} \in \frac{H}{\sqrt{t}} - \left(\frac{[\alpha\sqrt{t}x]}{\sqrt{t}} + \frac{z}{\sqrt{t}}\right)\right)\right). \end{aligned}$$

Now the central limit theorem gives us immediately our assertion (15). □

2. The coupling. We start with the following observation: Denote by ν_ρ the invariant measure of the process with property (1)–(2). Then we have:

(22) $$\mathscr{L}(\eta_t^{\nu_\rho}(\cdot + [\alpha\sqrt{t}x])) = \nu_\rho \quad \forall: t \in \mathbb{R}^+,$$

so we will try to construct a coupling between the processes:

(23) $$(\eta_t^{\nu_\beta}(\cdot + [\alpha\sqrt{t}x]))_{t \in \mathbb{R}^+}, \quad (\eta_t^{\mu}(\cdot + [\alpha\sqrt{t}x]))_{t \in \mathbb{R}^+}$$

with $\beta = \tau(\alpha, x)$. This means we construct a process $(\tilde{\eta}_t)_{t \in \mathbb{R}^+}$ with state space $(\mathbb{N})^S \times (\mathbb{N})^S$, initial distribution $\nu_\beta \otimes \mu$ and the property that the marginals of $(\tilde{\eta}_t) = (\eta_t^1, \eta_t^2)$ are versions of $(\eta_t^{\nu_\beta})$, (η_t^{μ}).

Such a coupling was introduced in [2]. One makes a birth at x for both components appear at rate $\sum_y r(y, x)\eta^1(y) \wedge \eta^2(y)$, a death for both components appears at rate $d(1 - p)\eta^1(x) \wedge \eta^2(x)$. Death for all particles at a site for both components occurs at rate pd. Similar is the procedure for the random motion. Then one introduces transitions of the single components so that the marginals become versions of the original process.

We introduce the following abbreviations:

(24) $\xi_t^1(\alpha, x)_{(\cdot)} := \eta_t^{\mu}(\cdot + [\alpha\sqrt{t}x]), \quad \xi_t^2(\alpha, x)_{(\cdot)} := \eta_t^{\nu_\beta}(\cdot + [\alpha\sqrt{t}x])$

(25) $\tilde{\xi}_t(\alpha, x)_{(\cdot)} := \tilde{\eta}_t(\cdot + [\alpha\sqrt{t}x]).$

PROPOSITION 2. *Make the same assumption as in Theorem* 1. *Then*:

(26)
$$\lim_{t\to\infty}\text{Prob}(\xi_t^1(\alpha, x)_{(z)} = \xi_t^2(\alpha, x)_{(z)}\ \forall: z \in A) = 1 \qquad \forall: A \subseteq \mathbb{Z}^d \ \textit{with} \ |A| < \infty.$$

We summarize the main steps of the proof of this proposition in the following two lemmata:

LEMMA 1.

(27)
$$\lim_{t\to\infty}\text{Prob}(\xi_t^1(\alpha, x)_{(z)} > \xi_t^2(\alpha, x)_{(z)}, \xi_t^1(\alpha, x)_{(\bar{z})} < \xi_t^2(\alpha, x)_{(\bar{z})}) = 0 \qquad \forall: z, \bar{z} \in \mathbb{Z}^d.$$

Knowing (27) it remains, in order to prove (26), to show that the coupled process does not put in the limit $t \to \infty$ any mass on the set $\{\xi^1 > \xi^2 \text{ or } \xi^1 < \xi^2\} \subseteq \mathbb{N}^S \times \mathbb{N}^S$. This is excluded by:

LEMMA 2. *Let t_n be a sequence with $t_n \uparrow \infty$. Suppose λ is a weak limit point of* $\{\mathscr{L}(\eta_{t_n}^{\mu}(\cdot + [\alpha\sqrt{t_n}x]))\}_{n\in\mathbb{N}}$. *Then we have*:

(28)
(i) λ *is translation invariant*

(ii) $\lim_{t\to\infty}E^{\lambda}(|\sum_{y\in S} r_t(x, y)\eta(y) - \tau(\alpha, x)|^2) = 0.$

These two lemmata together with Proposition 1 prove of course immediately Proposition 2.

PROOF OF LEMMA 1. For this proof we use a "Liapunov function". We need the following definitions:

(29) $\tilde{a}_t(u) := E^{\mu\otimes\nu_\beta}(\eta_t^1(u) - \eta_t^2(u))^+$

(30) $\tilde{b}_t(x, y) := E^{\mu\otimes\nu_\beta}(\eta_t^1(x) - \eta_t^2(x))^+ 1_{\{(\eta_t^1(x)-\eta_t^2(x))(\eta_t^1(y)-\eta_t^2(y))<0\}}$

(31) $\tilde{A}_t(\alpha, x)_{(y)} := \tilde{a}_t(y + [\alpha\sqrt{t}x])$

(32) $\tilde{B}_t(\alpha, x)_{(y,z)} := \tilde{b}_t(y + [\alpha\sqrt{t}x], z + [\alpha\sqrt{t}x]).$

$\tilde{A}_t(\alpha, x)_{(y)}$ will be our "Liapunov function". Denote by (t_n) the points on $\mathbb{R}^+$ where $[\alpha\sqrt{t}x]$ changes it's value. (We consider α, x to be fixed for the moment.) Now we decompose $\tilde{A}_t(\alpha, x)_{(y)}$ as follows:

(33) $\tilde{A}_t(\alpha, x)_{(y)} = \tilde{A}_t^d(\alpha, x)_{(y)} + \tilde{A}_t^s(\alpha, x)_{(y)}$

(34) $\tilde{A}_t^s(\alpha, x)_{(y)} := \sum_{t_n \le t} (\lim_{s \to t_n^+}(\tilde{A}_s(\alpha, x)_{(y)}) - \lim_{s \to t_n^-}(\tilde{A}_s(\alpha, x)_{(y)}))$.

Now we calculate the derivative of $\tilde{A}_t^d$ (on $\mathbb{R}\backslash\{t_n\}$) by applying the generator $\tilde{G}$ of the coupled process to the function $(\eta^1(u) - \eta^2(u))^+$ $((\mathbb{N}^S) \times (\mathbb{N})^S \to \mathbb{R}^+)$. We find: (Assume $m + b = 1$ for convenience.)

$$\frac{d}{dt} \tilde{A}_t^d(\alpha, x)_{(u)} = (\textstyle\sum_{y \in S} r(y, u + [\alpha\sqrt{t}x])\tilde{A}_t^d(\alpha, x)_{(y)} - \tilde{A}_t^d(\alpha, x)_{(u)}) \tag{35}$$

$$- \textstyle\sum_{y \in S} r(y, u + [\alpha\sqrt{t}x])\tilde{B}_t(\alpha, x)_{(y,u)}. \tag{36}$$

Altogether we obtain the following expression for $\tilde{A}_t(\alpha, x)_{(u)}$:

$$\begin{aligned} \tilde{A}_t(\alpha, x)_{(u)} = \tilde{A}_0(\alpha, x)_{(u)} &- \textstyle\sum_{y \in S} \int_0^t r(y, u)\tilde{B}_s(\alpha, x)_{(y,u)}\, ds \\ &+ \int_0^t (\textstyle\sum_{y \in S} r(y, u)\tilde{A}_s^d(\alpha, x)_{(y)} - \tilde{A}_s^d(\alpha, x)_{(u)})\, ds \\ &+ \tilde{A}_t^s(\alpha, x)_{(u)}. \end{aligned} \tag{37}$$

We will show later that:

$$\begin{aligned} &\tilde{A}_t^s(\alpha, x)_{(0)} \quad \text{is decreasing in } t \text{ for} \quad x \in \mathscr{C}H \\ &|\textstyle\sum_{y \in s} r(y, 0)\tilde{A}_t(\alpha, x)_{(y)} - \tilde{A}(\alpha, x)_{(0)}| = O(t^{-1/2}). \end{aligned} \tag{38}$$

Suppose first that $\tilde{A}_t(\alpha, x)_{(0)}$ is decreasing in t for $t \ge t_0$. (Assume without loss of generality that $x \in \mathscr{C}H$ from now on). Then we have,

$$\lim_{t \to \infty} \tilde{B}_t(\alpha, x)_{(y,u)} = 0 \quad \forall: u, y \quad \text{with} \quad r(y, u) > 0. \tag{39}$$

We can rewrite this as:

$$\begin{aligned} &\lim_{t \to \infty} \text{Prob}(\xi_t^1(\alpha, x)_{(u)} > \xi_t^2(\alpha, x)_{(u)}, \xi_t^1(\alpha, x)_{(y)} < \xi_t^2(\alpha, x)_{(y)}) = 0 \\ &\qquad \forall: u, y \quad \text{with} \quad r(y, u) + r(u, y) > 0. \end{aligned} \tag{40}$$

(The symmetrization of the restriction on the pair u, y is obtained by interchanging u and y in (37).)

With the techniques developed in [2] we extend (40) to (details are left to the reader):

$$\begin{aligned} &\lim_{t \to \infty} \text{Prob}(\xi_t^1(\alpha, x)_{(z)} > \xi_t^2(\alpha, x)_{(z)}, \quad \xi_t^1(\alpha, x)_{(\bar{z})} < \xi_t^2(\alpha, x)_{(\bar{z})}) = 0 \\ &\qquad \forall: z, \bar{z} \in \mathbb{Z}^d. \end{aligned} \tag{41}$$

If $\tilde{A}_t(\alpha, x)_{(0)}$ is not eventually decreasing, it is either increasing (which is obviously excluded by (38) together with arguments as above) or an unbounded set of local maxima exist. Pick a sequence (t_n) of those local maxima with $t_n \nearrow \infty$. Using Proposition 1 and Lemma 2 it is easy to conclude with similar arguments as above that a weak limit point of $\{\mathscr{L}(\tilde{\xi}_{t_n}(\alpha, x))\}_{n \in \mathbb{N}}$ is concentrated on $\{\xi^1 = \xi^2\}$

and therefore $\tilde{A}_{t_n}(\alpha, x)_{(0)}$ converges to 0 and (41) holds. Now let us prove the statements in (38).

The following fact is crucial: consider a measure $\hat{\mu}$ constructed from ν by (5) but using $H + (z - u)$ instead of H. Then we have:

$$\mathscr{L}((\eta_t^1(u), \eta_t^2(u))) = \mathscr{L}(\hat{\eta}_t^1(z), \hat{\eta}_t^2(z)) \tag{42}$$

where we construct $(\hat{\eta}_t^1, \hat{\eta}_t^2)$ as the coupled process for the initial distribution $\hat{\mu} \otimes \nu_\beta$. We can now think of the process $(\hat{\eta}_t^1)$ as follows: consider a process with initial distribution λ constructed from ν by (5) but using $H \,\Delta\, (H + (z - u))$ instead of H. Then

$$\hat{\eta}_t^1 \mathrel{\hat{=}} \eta_t^1 \pm \xi_t. \tag{43}$$

The above relation allows us immediately to conclude that for $x \in \mathscr{C}H$ we have:

$$\tilde{a}_t(y + x) \leq \tilde{a}_t(y) \quad \text{for every} \quad y \in \mathbb{Z}^d. \tag{44}$$

Furthermore, we obtain:

$$\begin{aligned} |\tilde{a}_t(u) - \tilde{a}_t(z)| &\leq 2\rho \operatorname{Prob}(X_t^{\{0\}} \in ((H - u)\Delta(H - u - (z - u)))) \\ &= 2\rho \operatorname{Prob}(X_t^{\{0\}} \in (H\Delta(H - (z - u)) - u)). \end{aligned} \tag{45}$$

In the case $u = [\alpha\sqrt{t}x]$, $z = [\alpha\sqrt{t}x] + y$ we can rewrite this as:

$$\begin{aligned} &|\tilde{A}_t(\alpha, x)_{(y)} - \tilde{A}_t(\alpha, x)_{(0)}| \\ &\qquad \leq 2\rho \operatorname{Prob}\left(\frac{X_t^{\{0\}}}{\sqrt{t}} \in \frac{1}{\sqrt{t}}\left((H\Delta(H - y)) - [\alpha\sqrt{t}x]\right)\right). \end{aligned} \tag{46}$$

This allows us to conclude with the local central limit theorem that:

$$|\tilde{A}_t(\alpha, x)_{(y)} - \tilde{A}_t(\alpha, x)_{(0)}| = O(t^{-1/2}). \tag{47}$$

The last relation allows us immediately to conclude (38).

PROOF OF LEMMA 2. Let λ be a weak-limit point of

$$\{\mathscr{L}(\eta_{t_n}^{\mu}(\cdot + [\alpha\sqrt{t_n}x]))\}_{n\in\mathbb{N}} \quad (t_n \uparrow \infty).$$

The translation invariance is easily established using (42) and arguments based on (43). We know already that $E^\lambda(\eta(u)) = \tau(\alpha, x)$ $\forall$: $u \in \mathbb{Z}^d$. Therefore a straightforward calculation shows that in order to prove (28) it suffices to show:

$$\limsup_{|x-y|\to\infty} E^\lambda(\eta(x)\eta(y)) \leq (\tau(\alpha, x))^2. \tag{48}$$

With the results in Section 1 of [2] we obtain:

$$\begin{aligned} &\limsup_{|y-z|\to\infty} E^\lambda(\eta(y)\eta(z)) \\ &\qquad \leq \limsup_{|y-z|\to\infty} \limsup_{t\to\infty} \left(\textstyle\sum_{u,v} r(y_t, u) r(z_t, v) f(u, v)\right) \\ &\qquad\qquad y_t := y + [\alpha\sqrt{t}x], \quad z_t := z + [\alpha\sqrt{t}x] \\ &\qquad f(u, v) := E^\mu(\eta(u)\eta(v) - \delta(u, v)\eta(u)). \end{aligned} \tag{49}$$

The way in which μ is constructed from ν (see (5)) allows us to conclude:

(50) $$\lim_{\varepsilon\to\infty} f(u + \varepsilon x, v) = \begin{cases} 0 & \text{for } x \in \mathscr{C}H \\ \rho^2 & \text{for } x \in H \end{cases}$$

and with the central limit theorem we obtain (48). (Note that $r_t(x, y)$ describes a continuous time symmetric random walk.)

APPENDIX

A thorough examination of our proofs shows that we can apply our methods to every initial distribution μ having the properties ($f(u, v)$ as in (49), $\rho(u) := \int \eta(u)\, d\mu$):

(51) $$\lim_{t\to\infty}\left(\sum_{y\in S} r_t(z, y)\rho(\cdot + [\alpha\sqrt{t}x])\right) = \tau(\alpha, x)$$
$$\lim_{|u-v|\to\infty}(f(u, v) - \rho(u)\rho(v)) = 0, \quad \sup_u f(u, u) < \infty$$

(52) $$\int_0^t \sum_{u\in S} r_t(y_t, u)\,|\,\rho(u + z) - \rho(u)\,|\, dt < \infty$$
$$\text{for all } z \text{ with } T_z(\mu) \neq \mu.$$

Here T_z denotes translation by z, $y_t := y + [\alpha\sqrt{t}x]$.

For all $z \in \mathbb{Z}^d$ there exists a distribution $\lambda \in (\mathbb{N})^S \times (\mathbb{N})^S$ with marginals $T_z(\mu)$, μ such that:

(53) $$\int |\,\eta^1(u) - \eta^2(u)\,|\, d\lambda \le |\,\rho(u + z) - \rho(u)\,| \quad \text{for all } u \in \mathbb{Z}^d.$$

For given $\rho(u)$ the condition (52) can be checked using large deviation theory for the random walk with kernel $r(x, y)$.

A way to generate measures fulfilling (53) is to pick a measure ν which is translation invariant, shift ergodic and has property (51). Then selected independently at each site a random number of particles. The induced new measure has then the property (53).

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