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ALMOST SURE EQUICONVERGENCE OF CONDITIONAL EXPECTATIONS

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If $(X, \mathscr{F}, P)$ is a probability space then a pseudo-metric δ can be defined on the sub-σ-fields of $\mathscr{F}$ by

$$\delta(\mathscr{A}, \mathscr{B}) = \sup_{A\in\mathscr{A}}\inf_{B\in\mathscr{B}}P(A \Delta B) \vee \sup_{B\in\mathscr{B}}\inf_{A\in\mathscr{A}}P(A \Delta B).$$

Boylan, Neveu, and Rogge, among others, have considered equiconvergence of conditional expectations of uniformly bounded measurable functions given sub-σ-fields $\{\mathscr{F}_n: 1 \le n \le \infty\}$ in probability and in L_p, $1 \le p < \infty$, as $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$. This paper proves the corresponding almost sure equiconvergence results when $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ or $\mathscr{F}_n \downarrow \mathscr{F}_\infty$. A sharp uniform bound for the rate of convergence is given. A consequence is that if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ or $\mathscr{F}_n \downarrow \mathscr{F}_\infty$ then the sequence of conditional expectations given $\mathscr{F}_n$ converges uniformly for all uniformly bounded measurable functions to the conditional expectation given $\mathscr{F}_\infty$ if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$.

1. Introduction. Let $(X, \mathscr{F}, P)$ be a probability space. If d is any pseudo-metric on $\mathscr{F}$ then a pseudo-metric can be defined on the sub-σ-fields of $\mathscr{F}$ by

$$\delta(\mathscr{A}, \mathscr{B}) = \sup_{A\in\mathscr{A}}\inf_{B\in\mathscr{B}}d(A, B) \vee \sup_{B\in\mathscr{B}}\inf_{A\in\mathscr{A}}d(A, B) \tag{1.1}$$

for sub-σ-fields $\mathscr{A}$ and $\mathscr{B}$. A different metric δ' (used by Boylan, 1971) may be defined by using "+" instead of "$\vee$" in (1.1). It is clear that δ and δ' are equivalent, $\delta \le \delta'$, and $\delta(\mathscr{A}, \mathscr{B}) = \delta'(\mathscr{A}, \mathscr{B})$ if $\mathscr{A} \subset \mathscr{B}$ or $\mathscr{B} \subset \mathscr{A}$. Using the "standard" pseudo-metric d given by $d(A, B) = P(A \Delta B)$, $A \in \mathscr{F}$, $B \in \mathscr{F}$, Boylan (1971) was able to show that for sub-σ-fields $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ or $\mathscr{F}_n \downarrow \mathscr{F}_\infty$, if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ as $n \to \infty$, then

$$\sup\{\| E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty] \|_1 : f \in \Phi\} \to 0 \quad \text{as} \quad n \to \infty, \tag{1.2}$$

where Φ is the collection of $\mathscr{F}$-measurable functions bounded by 0 and 1, and $\|\cdot\|_1$ denotes the usual L_1-norm. Rogge (1974) sharpened this result (see also Neveu, 1972) and showed that for any sub-σ-fields $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{F}$ with $\mathscr{A} \subset \mathscr{B}$

$$\sup\{\| E[f \mid \mathscr{A}] - E[f \mid \mathscr{B}] \|_1 : f \in \Phi\} \le 2\delta(\mathscr{A}, \mathscr{B})/(1 - \delta(\mathscr{A}, \mathscr{B})). \tag{1.3}$$

He also gave an example to show that this inequality is sharp. As an application, he was able to show that

$$\sup\{\| E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty] \|_1 : f \in \Phi\} \to 0 \quad \text{as} \quad n \to \infty \tag{1.4}$$

if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ as $n \to \infty$ for sub-σ-fields $\{\mathscr{F}_n: 1 \le n \le \infty\}$ without the assumption that $\{\mathscr{F}_n\}$ is nested. Thus δ appears to be just the right pseudo-

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metric for investigating the equiconvergence of conditional expectations in probability (and L_p) of uniformly bounded measurable functions. Extensions to sets of uniformly integrable functions have also been given by Neveu (1972) and Rogge (1974) as well as rates of convergence in L_p, $1 < p < \infty$.

There are natural almost sure analogs of all these results. Boylan (1971, page 558) gives a counterexample (due to Burgess Davis) to show that almost sure convergence of the conditional expectations of even a single indicator function fails in a case where $\delta'(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ as $n \to \infty$. However, in this example the σ-fields are not nested. The purpose of this paper is to derive a (sharp) upper bound for

$$\sup_{f\in\Phi} P\{\sup_{n\geq m} | E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty] | \geq \varepsilon\} \tag{1.5}$$

when $\{\mathscr{F}_n : 1 \leq n \leq \infty\}$ is a nested collection of sub-σ-fields of $\mathscr{F}$, and to use this to show that if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ or $\mathscr{F}_n \downarrow \mathscr{F}_\infty$ then (1.5) $\to 0$ as $m \to \infty$ for all $\varepsilon > 0$ if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$. Thus δ also appears to be just the right pseudo-metric for investigating almost sure equiconvergence of conditional expectations of uniformly bounded measurable functions when the σ-fields are nested. Extensions to uniformly integrable functions are also given. The methods also yield what appears to be new proofs of convergence of conditional expectations of a single integrable function when $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ or $\mathscr{F}_n \downarrow \mathscr{F}_\infty$ without any assumptions on $\delta(\mathscr{F}_n, \mathscr{F}_\infty)$.

When the σ-fields $\{\mathscr{F}_n\}$ are not nested, we prove almost sure equiconvergence as above if $\sum_n \delta(\mathscr{F}_n, \mathscr{F}_\infty) < \infty$. Examples are given to show that we may or may not have such equiconvergence when $\sum_n \delta(\mathscr{F}_n, \mathscr{F}_\infty) = \infty$.

2. Results. All upper case script letters, with or without subscripts, will indicate sub-σ-fields of $\mathscr{F}$. Φ will denote the collection of $\mathscr{F}$-measurable functions f such that $0 \leq f \leq 1$. All functions are real-valued.

To avoid technical problems involving measurability with respect to sub-σ-fields of $\mathscr{F}$, we will assume all sub-σ-fields are endowed with all the null-sets of $\mathscr{F}$ (note that this makes δ and δ' metrics). For the sake of brevity in the examples given below, we do not explicitly indicate the null sets so that the trivial σ-field, for example, is $\{\varnothing, X\} \cup \mathscr{N}$, where $\mathscr{N}$ = the null sets of $\mathscr{F}$.

Equalities (inequalities) among measurable functions are almost sure equalities (inequalities), and all convergences are almost sure convergences.

The indicator function of a set A is written both as I_A and $I(A)$. We write $A + B$ and $\sum_i A_i$ to indicate disjoint unions of sets. $AB = A \cap B$. We also write $A \subset B$ if $P(A - B) = 0$, so that $A = B$ and $A - B = \varnothing$ if $P(A \,\Delta\, B) = 0$. A' denotes the complement of A in X.

For future reference we note that if $\mathscr{A}$ and $\mathscr{B}$ are σ-fields and $B \in \mathscr{B}$, then

$$\begin{aligned} &\inf_{A\in\mathscr{A}} P(A\Delta B) \\ &\quad = \inf_{A\in\mathscr{A}}\{E[I_{A'}(E[I_B \mid \mathscr{A}] - I_A)] + E[I_A(I_A - E[I_B \mid \mathscr{A}])]\} \\ &\quad = \inf_{A\in\mathscr{A}} E[\,| E[I_B \mid \mathscr{A}] - I_A |\,] = E[E[I_B \mid \mathscr{A}] \wedge E[I_{B'} \mid \mathscr{A}]], \end{aligned} \tag{2.1}$$

and this infimum is a minimum actually achieved by any $\mathscr{A}$-measurable set S

such that $\{E[I_B|\mathscr{A}] > 1/2\} \subset S \subset \{E[I_B|\mathscr{A}] \geq 1/2\}$. This fact will be a key to all the proofs.

We first present a technical result before proceeding to the main results.

LEMMA. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, $\{B_1, B_2, \cdots, B_k\}$ is a pairwise disjoint collection of $\mathscr{B}$-measurable sets with $A = \sum_{1\leq i\leq k} B_i \in \mathscr{A}$, and $E[I_{B_i}|\mathscr{A}] \leq 1/2$, $1 \leq i \leq k$. Then there exists a $\mathscr{B}$-measurable set $B \subset A$ such that $\{E[I_B|\mathscr{A}] \geq 1/2\} = A$ and $P(B) \leq (2/3)P(A)$.*

PROOF. Let $k_i = \binom{k}{i}$, $1 \leq i \leq k$, and order the k_i distinct unordered i-tuples without repetition from $\{1, 2, \cdots, k\}$ in some arbitrary but fixed manner. Let I_{ij} be the jth i-tuple in this ordering, $S_{ij} = \sum_{n\in I_{ij}} E[I_{B_n}|\mathscr{A}]$, and $S_{i0} \equiv 0$. Now define $E_0 = A$, and for $1 \leq i \leq k$ and $1 \leq n \leq k_i$ define $E_i = A \cap \{S_{ij} < 1/3, 1 \leq j \leq k_i\}$, $F_{in} = E_{i-1} \cap \{S_{ij} < 1/3, 0 \leq j \leq n-1\}$, $G_{in} = F_{in} \cap \{1/3 \leq S_{in} \leq 1/2\}$, and $H_{in} = F_{in} \cap \{S_{in} > 1/2\}$. Since $H_{in} \subset E_{i-1} \subset E_1$ for $i \geq 2$ we have $1/2 < S_{in} < 2/3$ on H_{in}, $i \geq 2$; $H_{1n} \equiv \varnothing$ since $E[I_{B_j}|\mathscr{A}] \leq 1/2$ for all j.

Since $\sum_j B_j = A \in \mathscr{A}$ we have $\{E[I_{B_j}|\mathscr{A}] > 0\} \subset A$ for all j and $\sum_j E[I_{B_j}|\mathscr{A}] = I_A$. Now note that $A = E_0 = F_{11} \supset F_{12} \supset \cdots \supset F_{1k_1} \supset E_1 = F_{21} \supset \cdots \supset F_{kk_k} \supset E_k = \varnothing$. Indeed, since $\sum_j E[I_{B_j}|\mathscr{A}] = I_A$, $E_i = \varnothing$ for all $i \geq k - [k/2]$, where $[\cdot]$ is the largest integer function. From their definitions, $\{G_{in}, H_{in}\}_{i,n}$ is a collection of $\mathscr{A}$-measurable sets, $G_{in}H_{in} = \varnothing$ and $G_{in} + H_{in} = F_{in}F'_{i,n+1}$ (with the convention $F_{i,k_i+1} = E_i$) for all i and n, and $A = \sum_{i,n}(G_{in} + H_{in})$. Let $B = \sum_{i,n}[G_{in} \cap \sum_{j\notin I_{in}} B_j + H_{in} \cap \sum_{j\in I_{in}} B_j]$ which is a subset of A. Now for all i and n we have $1/2 \leq E[I(G_{in} \cap \sum_{j\notin I_{in}} B_j)|\mathscr{A}] = I_{G_{in}}(1 - S_{in}) \leq 2/3$ on G_{in} and 0 elsewhere and $1/2 < E[I(H_{in} \cap \sum_{j\in I_{in}} B_j)|\mathscr{A}] = I_{H_{in}}S_{in} < 2/3$ on H_{in} and 0 elsewhere (H_{in} and G_{in} may be null for some values of i and n). Thus $\{E[I_B|\mathscr{A}] \geq 1/2\} = \sum_{i,n}(G_{in} + H_{in}) = A$ and $P(B) = E[E[I_B|\mathscr{A}]] \leq (2/3)P(A)$. □

EXAMPLE 1. In the lemma if $k = 3$, $A = X$, $P(B_i) = 1/3$, $i = 1, 2, 3$, $\mathscr{A}$ is the trivial σ-field, and $\mathscr{B} = \sigma(B_1, B_2, B_3)$, then $\min\{P\{B \in \mathscr{B}: \{E[I_B|\mathscr{A}] \geq 1/2\} = A\}\} = 2/3 = (2/3)P(A)$. Thus the fraction $(2/3)$ cannot be reduced in general. □

PROPOSITION 2.1. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, $\{B_i: 1 \leq i \leq k\}$ is a partition of X by $\mathscr{B}$-measurable sets, $\{A_i: 1 \leq i \leq k\}$ is a pairwise disjoint collection of $\mathscr{A}$-measurable sets, and $\{0 \leq b_i \leq 1: 1 \leq i \leq k\}$ is a collection of real numbers. Then for every $\varepsilon > 0$*

$$P\{|E[\textstyle\sum_i b_i I_{B_i}|\mathscr{A}] - \sum_i b_i I_{B_i}| \geq \varepsilon\} \leq P(\sum_i A_i B_i')/\varepsilon + P(\sum_i A_i)',$$

where $\sum_i(\cdot) = \sum_{1\leq i\leq k}(\cdot)$.

PROOF. Fix $\varepsilon > 0$. For $1 \leq i \leq k$ let $G_i = \{E[I_{B_i}|\mathscr{A}] \leq 1 - \varepsilon\}$. Then on $A_j G_j'$ we have

$$\begin{aligned} b_j - \varepsilon < \textstyle\sum_i b_i E[I_{B_i}|\mathscr{A}] &\leq b_j + \textstyle\sum_{i\neq j} b_i E[I_{B_i}|\mathscr{A}] \\ &\leq b_j + E[I_{B_j'}|\mathscr{A}] < b_j + \varepsilon, \quad 1 \leq j \leq k. \end{aligned} \tag{2.2}$$

Thus

$$(2.3)\quad \{\,|E[\textstyle\sum_i b_i I_{B_i} \mid \mathscr{A}] - \sum_i b_i I_{B_i}| \geq \varepsilon\} \subset \sum_i A_i G_i B_i + \sum_i A_i B_i' + (\sum_i A_i)'.$$

Now for $1 \leq i \leq k$, $P(A_i G_i B_i) = E[I_{A_i G_i} E[I_{B_i} \mid \mathscr{A}]] \leq (1 - \varepsilon) P(A_i G_i)$ so that

$$(2.4)\quad \varepsilon P(A_i G_i B_i) \leq (1 - \varepsilon) P(A_i G_i B_i') \leq (1 - \varepsilon) P(A_i B_i').$$

Applying (2.4) to (2.3) completes the proof of the proposition. □

THEOREM 2.1. *Suppose $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, m is a positive integer, and $0 < \varepsilon < 1/3$. Then for all $f \in \Phi$*

$$(2.5)\quad P\{\sup_{n \geq m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \geq \varepsilon\} \leq \delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon.$$

PROOF. First suppose (2.5) is true for all $f \in \Phi$ such that $E[f \mid \mathscr{F}_\infty]$ is a simple function of the form $\sum_{1 \leq i \leq k} b_i I_{B_i}$, where $\{B_i\}$ is a partition of X by $\mathscr{F}_\infty$-measurable sets and $0 \leq b_i \leq 1$, $1 \leq i \leq k$. Now fix $f \in \Phi$. Since $E[f \mid \mathscr{F}_\infty] \in \Phi$, for every $0 \leq \eta < \varepsilon/2$ there exists a simple function of the form above such that $|E[f \mid \mathscr{F}_\infty] - \sum_i b_i I_{B_i}| \leq \eta$. Then $|E[f \mid \mathscr{F}_n] - E[\sum_i b_i I_{B_i} \mid \mathscr{F}_n]| = |E[E[f - \sum_i b_i I_{B_i} \mid \mathscr{F}_\infty] \mid \mathscr{F}_n]| \leq \eta$ also. Thus using the triangular inequality and our assumption, the l.h.s. of (2.5) is bounded by $\delta(\mathscr{F}_m, \mathscr{F}_\infty)/(\varepsilon - 2\eta)$. Since $0 \leq \eta < \varepsilon/2$ is arbitrary, this will complete the proof of the theorem. Thus it is sufficient to prove (2.5) when $E[f \mid \mathscr{F}_\infty]$ is a simple function of the form given above, and we do assume such is the case.

Let $\{A_i : 1 \leq i \leq k\}$ be an arbitrary pairwise disjoint collection of $\mathscr{F}_m$-measurable sets. Then $\{A_i\} \subset \mathscr{F}_n$ for all $n \geq m$. Now for $n \geq m$ and $1 \leq i \leq k$ define $G_{ni} = \{E[I_{B_i} \mid \mathscr{F}_n] \leq 1 - \varepsilon\}$ and $C_{ni} = \cup_{m \leq r \leq n} G_{ri}$. Note that G_{ni} and C_{ni} are $\mathscr{F}_r$-measurable for all $r \geq n$. From the proof of Proposition 2.1 it is clear that

$$(2.6)\quad \begin{aligned} &\{\sup_{n \geq m} |E[\textstyle\sum_i b_i I_{B_i} \mid \mathscr{F}_n] - \sum_i b_i I_{B_i}| \geq \varepsilon\} \\ &\quad \subset \textstyle\sum_i A_i B_i \cap (\cup_{n \geq m} G_{ni}) + \sum_i A_i B_i' + (\sum_i A_i)'. \end{aligned}$$

We now prove by induction

$$(2.7)\quad P(A_i C_{ri} B_i) \leq (1 - \varepsilon) P(A_i C_{ri} B_i')/\varepsilon \quad \text{for all} \quad r \geq m \quad \text{and} \quad 1 \leq i \leq k.$$

Fix $1 \leq i \leq k$. (2.7) is true for $r = m$ by (2.4). Now assume (2.7) is true for $m \leq r \leq n$. Then $P(A_i C_{ni}' G_{n+1,i} B_i) = E[I_{A_i C_{ni}' G_{n+1,i}} E[I_{B_i} \mid \mathscr{F}_{n+1}]] \leq (1 - \varepsilon) P(A_i C_{ni}' G_{n+1,i})$ so that

$$(2.8)\quad \varepsilon P(A_i C_{ni}' G_{n+1,i} B_i) \leq (1 - \varepsilon) P(A_i C_{ni}' G_{n+1,i} B_i').$$

From (2.7) (with $r = n$) and (2.8) we get $P(A_i C_{n+1,i} B_i) \leq (1 - \varepsilon) P(A_i C_{n+1,i} B_i')/\varepsilon \leq (1 - \varepsilon) P(A_i B_i')/\varepsilon$. This proves (2.7), and in conjunction with (2.6) we get an upper bound for the l.h.s. of (2.5) given by $P(\sum_i A_i B_i')/\varepsilon + P(\sum_i A_i)'$. Now for $1 \leq i \leq k$, let $A_i = \{E[I_{B_i} \mid \mathscr{F}_m] > 1/2\}$. Then $[A_i\}$ is a pairwise disjoint collection of $\mathscr{F}_m$-measurable sets. It is clear that $\{E[I_{A_i B_i} \mid \mathscr{F}_m] > 1/2\} = A_i$, $1 \leq i \leq k$. Now $(\sum_i A_i)' = \sum_j B_j \cap (\sum_i A_i)'$ and $E[I(B_j \cap (\sum_i A_i)') \mid \mathscr{F}_m] \leq E[I(B_j \cap A_j') \mid \mathscr{F}_m] = I_{A_j'} E[I_{B_j} \mid \mathscr{F}_m] \leq 1/2$, $1 \leq j \leq k$. Thus an application of the lemma to the sets

$\{B_j \cap (\sum_i A_i)'\}$ shows that there exists $B \in \mathscr{F}_\infty$ such that $B \subset (\sum_i A_i)'$, $P(B) \le (2/3)P(\sum_i A_i)'$, and $\{E[I_B \mid \mathscr{F}_m] \ge 1/2\} = (\sum_i A_i)'$. Thus $\{E[I_{\sum A_i B_i + B} \mid \mathscr{F}_m] \ge 1/2\} = X$. From the fact that $0 < \varepsilon < 1/3$ and (2.1), we finally get $P(\sum_i A_i B_i')/\varepsilon + P(\sum_i A_i)' \le P(\sum_i A_i B_i' + [(\sum_i A_i)' - B])/\varepsilon = \min_{A \in \mathscr{F}_m} P(A \,\Delta\, (\sum_i A_i B_i + B))/\varepsilon \le \delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon$, which completes the proof of the theorem. □

EXAMPLE 2. If in Theorem 2.1 $(X, \mathscr{F}, P) = ([0, 1), \mathscr{B}([0, 1)), P)$, where P is the uniform measure, $0 < \delta < \varepsilon < 1/3$, $A_1 = [0, \delta/2\varepsilon)$, $A_2 = [\delta/2\varepsilon, 1 - \delta/2\varepsilon)$, $A_3 = [1 - \delta/2\varepsilon, 1)$, $B = [(1 - \varepsilon)\delta/2\varepsilon, 1 - \delta/2)$, $\mathscr{F}_m = \sigma(A_1, A_2, A_3)$, and $\mathscr{F}_n = \mathscr{F}_{n+1} = \cdots = \mathscr{F}_\infty = \sigma(A_1, A_2, A_3, B)$, $n = m + 1$, then $E[I_B \mid \mathscr{F}_m] = \varepsilon$ on A_1, 1 on A_2, and $(1 - \varepsilon)$ on A_3, and $P\{\sup_{n \ge m} |E[I_B \mid \mathscr{F}_n] - I_B| \ge \varepsilon\} = P\{|E[I_B \mid \mathscr{F}_m] - I_B| \ge \varepsilon\} = P(A_1 + A_3) = \delta/\varepsilon = P(B \Delta (A_3 + A_2))/\varepsilon = \delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon$. Thus the bound on the r.h.s. of (2.5) is sharp. □

For $M > 0$ let Λ_M be the collection of $\mathscr{F}$-measurable functions f such that $|f| \le M$.

COROLLARY 2.1. *Suppose $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, m is a positive integer, $M > 0$, and $0 < \varepsilon < 2M/3$. Then*

$$\sup_{f \in \Lambda_M} P\{\sup_{n \ge m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \ge \varepsilon\} \le 2M\delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon. \tag{2.9}$$

OUTLINE OF PROOF. First assume $M = 1$. From the proof of Theorem 2.1 it is sufficient to prove (2.9) assuming $E[f \mid \mathscr{F}_\infty]$ is an $\mathscr{F}_\infty$-measurable simple function $\sum_{1 \le i \le k} b_i I_{B_i}$ as in the proof of Theorem 2.1 but with $-1 \le b_i \le 1$, $1 \le i \le k$. Let $A_i = \{E[I_{B_i} \mid \mathscr{F}_m] > 1/2\}$, $1 \le i \le k$, and $G_{ni} = \{E[I_{B_i} \mid \mathscr{F}_n] \le 1 - \varepsilon/2\}$, $n \ge m$, $1 \le i \le k$. Now note that for $n \ge m$ if $b_j \ge 0$, then on $A_j G_{nj}'$ we have $b_j - \varepsilon < b_j(1 - \varepsilon/2) - E[I_{B_j'} \mid \mathscr{F}_n] \le b_j(1 - \varepsilon/2) - \sum_{i \ne j} |b_i| E[I_{B_i} \mid \mathscr{F}_n] \le \sum_i b_i E[I_{B_i} \mid \mathscr{F}_n] \le b_j + \sum_{i \ne j} |b_i| E[I_{B_i} \mid \mathscr{F}_n] \le b_j + I_{B_j'} < b_j + \varepsilon/2$, and, similarly, if $b_j < 0$ then $b_j - \varepsilon/2 < \sum_i b_i E[I_{B_i} \mid \mathscr{F}_n] < b_j + \varepsilon$, $1 \le j \le k$. Thus (2.6) still holds. Now using the fact that $0 < \varepsilon < 2/3$ and a similar analysis as in Theorem 2.1, we get

$$\begin{aligned}
&P\{\sup_{n \ge m} |E[\textstyle\sum_i b_i I_{B_i} \mid \mathscr{F}_n] - \sum_i b_i I_{B_i}| \ge \varepsilon\} \\
&\le P(\textstyle\sum_i A_i B_i \cap (\bigcup_{n \ge m} G_{ni})) + P(\sum_i A_i B_i') + P(\sum_i A_i)' \\
&\le (2 - \varepsilon) P(\textstyle\sum_i A_i B_i')/\varepsilon + P(\sum_i A_i B_i') + P(\sum_i A_i)' \le 2P(\sum_i A_i B_i')/\varepsilon \\
&\quad + P(\textstyle\sum_i A_i)' \le 2P(\sum_i A_i B_i' + [(\sum_i A_i)' - B])/\varepsilon \le 2\delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon,
\end{aligned}$$

where the set $B \subset (\sum_i A_i)'$ is as defined in the proof of Theorem 2.1. The proof can now be completed using this result and the fact that $f \in \Lambda_M$ implies $-1 \le f/M \le 1$. □

Theorem 2.1 shows that if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ and $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ as $n \to \infty$ then $\sup_{f \in \Phi} P\{\sup_{n \ge m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \ge \varepsilon\} \to 0$ as $m \to \infty$ for all $\varepsilon > 0$. Since this implies (1.4) and Rogge's (1974) analysis shows that (1.4) implies

$\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ (even without the assumption that the $\{\mathscr{F}_n\}$ is nested) we have

THEOREM 2.2. *If $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, then*

$$\sup_{f\in\Phi} P\{\sup_{n\geq m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \geq \varepsilon\} \to 0$$

as $m \to \infty$ for all $\varepsilon > 0$ if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ as $n \to \infty$. □

Theorem 2.2 can be extended to any uniformly integrable collection of $\mathscr{F}$-measurable functions as can be seen from the following theorem.

THEOREM 2.3. *Suppose $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, Γ is a uniformly integrable collection of $\mathscr{F}$-measurable functions, m is a positive integer, $M > 0$, and $0 < \varepsilon < 2M/3$. Then $\sup_{f\in\Gamma} P\{\sup_{n\geq m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \geq \varepsilon\} \leq 4M\delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon + 4 \sup_{f\in\Gamma} E[|f| I(|f| > M)]/\varepsilon$.*

PROOF. Fix $f \in \Gamma$. Let $g = fI(|f| \leq M)$ and $h = fI(|f| > M)$. From Corollary 2.1 we have $P\{\sup_{n\geq m} |E[g \mid \mathscr{F}_n] - E[g \mid \mathscr{F}_\infty]| \geq \varepsilon/2\} \leq 4M\delta(\mathscr{F}_m, \mathscr{F}_\infty)/\varepsilon$. The proof is completed by noting that $P\{\sup_{n\geq m} |E[h \mid \mathscr{F}_n]| \geq \varepsilon/2\}$ and $P\{|E[h \mid \mathscr{F}_\infty]| \geq \varepsilon/2\}$ are both bounded above by $2E[|h|]/\varepsilon$ by the maximal inequality for martingales. □

The following example shows that Theorem 2.2 does not hold for all tight collections of functions even with uniformly bounded expectations.

EXAMPLE 3. Let $(X, \mathscr{A}, P) = ((0, 1), \mathscr{B}((0, 1)), P)$ where P is the uniform measure. For $2 \leq n < \infty$ let $\mathscr{F}_n = \sigma((0, 1/n) \cup (½, 1), \mathscr{B}([1/n, ½]))$ and $\mathscr{F}_\infty = \mathscr{A}$. Let $f_k(x) = kI_{(0,1/k)}(x)$, $x \in X$, $k \geq 2$. Then $\{f_k\}$ is tight and $\mathscr{F}_\infty$-measurable, $E[f_k] \equiv 1$, $f_k = E[f_k \mid \mathscr{F}_\infty] = 0$ on $(½, 1)$ and $E[f_k \mid \mathscr{F}_n] = 2n/(n + 2)$ on $(½, 1)$ for all $k \geq n \geq 2$. □

The methods above yield what appears to be a new proof for a special type of martingale convergence.

THEOREM 2.4. *Suppose f is an $\mathscr{F}$-measurable and integrable function and $\mathscr{F}_n \uparrow \mathscr{F}_\infty$. Then $E[f \mid \mathscr{F}_n] \to E[f \mid \mathscr{F}_\infty]$.*

PROOF. The maximal inequality for martingales allows us to consider by truncation only the case $|f| \leq M$ for some $M > 0$. From the proof of Theorem 2.1 it is sufficient to consider only the case when $E[f \mid \mathscr{F}_\infty]$ is a simple function of the form $\sum_{1\leq i\leq k} b_i I_{B_i}$, with $|b_i| \leq M$ and $\{B_i\} \subset \mathscr{F}_\infty$ a partition of X.

Let $0 < \varepsilon < 2M/3$ be arbitrary. Since $\mathscr{F}_\infty = \sigma(\cup_n \mathscr{F}_n)$, from a standard result in measure theory there exists a positive integer m and $\{C_i{:}1 \leq i \leq k\} \subset \mathscr{F}_m$ such that $P(C_i \,\Delta\, B_i) \leq \varepsilon^2/2Mk$, $1 \leq i \leq k$. Now define $A_i = \{E[I_{B_i} \mid \mathscr{F}_m] > ½\}$, $1 \leq i \leq k$. From (2.1), $P(A_i \,\Delta\, B_i) \leq P(C_i \,\Delta\, B_i) \leq \varepsilon^2/2Mk$, $1 \leq i \leq k$. From the analysis of Corollary 2.1 it is clear that

$$P\{\sup_{n\ge m} | E[\textstyle\sum_i b_i I_{B_i} | \mathscr{F}_n] - \sum_i b_i I_{B_i} | \ge \varepsilon\} \tag{2.10}$$
$$\le 2MP(\textstyle\sum_i A_i B_i')/\varepsilon + MP(\sum_i A_i)'.$$

Now $(\sum_i A_i)' \subset \bigcup_i A_i' B_i$ and $P(\sum_i A_i)' \le \sum_i P(A_i' B_i)$. Thus the r.h.s. of (2.10) is bounded by $2M[P(\sum_i A_i B_i') + \sum_i P(A_i' B_i)]/\varepsilon = 2M \sum_i P(A_i \Delta B_i)/\varepsilon \le \varepsilon$. Since $0 < \varepsilon < 2M/3$ is arbitrary, the proof is complete. □

Theorems 2.1–2.4 continue to hold if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ is replaced by $\mathscr{F}_n \downarrow \mathscr{F}_\infty$, the proofs needing only slight modifications for Theorems 2.1–2.3.

THEOREM 2.5. *Theorem 2.1–2.3 continue to hold if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ is replaced by $\mathscr{F}_n \downarrow \mathscr{F}_\infty$.*

INDICATION OF PROOF. Here we indicate only the modifications necessary to prove Theorem 2.1 when $\mathscr{F}_n \downarrow \mathscr{F}_\infty$.

Fix $f \in \Phi$ and $0 < \varepsilon < 1/3$. It is sufficient to prove only the case when $E[f | \mathscr{F}_m]$ is an $\mathscr{F}_m$-measurable simple function in Φ of the form $\sum_{1\le i\le k} b_i I_{B_i}$. For $m \le n \le \infty$ let $G_{ni} = \{E[I_{B_i} | \mathscr{F}_n] \le 1 - \varepsilon\}$ and $A_i = \{E[I_{B_i} | \mathscr{F}_\infty] > 1/2\}$, $1 \le i \le k$. On $A_i B_i \cap (\bigcap_{m\le n\le\infty} G_{ni}')$ we have $b_i - \varepsilon < E[\sum_i b_i I_{B_i} | \mathscr{F}_n] < b_i + \varepsilon$, $1 \le i \le k$, and thus $\bigcup_{m\le n\le\infty}\{| E[\sum_i b_i I_{B_i} | \mathscr{F}_n] - E[\sum_i b_i I_{B_i} | \mathscr{F}_\infty]| \ge \varepsilon\} \subset \sum_i A_i B_i \cap ((\bigcup_{m\le n<\infty} G_{ni}) \cup G_{\infty i}) + \sum_i A_i B_i' + (\sum_i A_i)'$. The analog of the key equation (2.7), $\varepsilon P\{A_i B_i \cap ((\bigcup_{m\le r\le n} G_{ri}) \cup G_{\infty i})\} \le (1 - \varepsilon)P\{A_i B_i' \cap ((\bigcup_{m\le r\le n} G_{ri}) \cup G_{\infty i})\} \le (1 - \varepsilon)P(A_i B_i')$ for all $n \ge m$ and $1 \le i \le k$ can be shown by (backward) induction in a manner similar to the proof of (2.7). The rest of the proof is the same as that of Theorem 2.1. □

In proving Theorem 2.4 where $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, since $\mathscr{F}_\infty$ is the σ-field generated by the field $\bigcup_{n\ge1}\mathscr{F}_n$, we were able to approximate a finite number of fixed sets in $\mathscr{F}_\infty$ arbitrarily closely (in terms of probabilities of symmetric differences) by sets from $\mathscr{F}_n$ for all n sufficiently large. This technique cannot be used when $\mathscr{F}_n \downarrow \mathscr{F}_\infty$. However, the following proposition, interesting in its own merit, offers a different method for proving Theorem 2.4 when $\mathscr{F}_n \downarrow \mathscr{F}_\infty$.

PROPOSITION 2.2. *Suppose $\mathscr{F}_n \downarrow \mathscr{F}_\infty$ and $B \in \mathscr{F}$. Then for $t \in (0, 1)$*

$$\{E[I_B | \mathscr{F}_\infty] < t\} \subset \liminf_n\{E[I_B | \mathscr{F}_n] \le t\} \subset \{E[I_B | \mathscr{F}_\infty] \le t\}$$

and

$$\{E[I_B | \mathscr{F}_\infty] > t\} \subset \liminf_n\{E[I_B | \mathscr{F}_n] \ge t\} \subset \{E[I_B | \mathscr{F}_\infty] \ge t\}.$$

PROOF. Fix $t \in (0, 1)$. Let $F_\infty = \{E[I_B | \mathscr{F}_\infty] < t\}$, $G_n = \{E[I_B | \mathscr{F}_n] \le t\}$ for $1 \le n \le \infty$, $C_n = \bigcap_{k\ge n} G_k$ for $n \ge 1$, and $C = \liminf_n\{E[I_B | \mathscr{F}_n] \le t\} = \lim_n C_n$. If n is a positive integer then $C_n \subset G_n$ and $P(C_n G_\infty' B) = E[I_{C_n G_\infty'} E[I_B | \mathscr{F}_n]] \le tP(C_n G_\infty') \le tP(CG_\infty')$. Since $C_n G_\infty' B \uparrow CG_\infty' B$, we have $P(CG_\infty' B) \le tP(CG_\infty')$. But

C and G'_∞ are $\mathscr{F}_\infty$-measurable. Thus $P(CG'_\infty B) = E[I_{CG'_\infty} E[I_B \mid \mathscr{F}_\infty]] > tP(CG'_\infty)$ if $P(CG'_\infty) > 0$. Thus $P(CG'_\infty) > 0$ leads to a contradiction. Hence, $C \subset G_\infty$.

Now for $1 \le m \le n$ let $C'_{mn} = \bigcup_{m \le k \le n} G'_k$. Then $C'_{mn} \uparrow C'_m$ as $n \to \infty$ and $C'_m \downarrow C' = \limsup_n G'_n$. By induction $P(C'_{mn} F_\infty B) > tP(C'_{mn} F_\infty)$ for all $m \le n$. $C'_{mn} F_\infty \uparrow C'_m F_\infty$ as $n \to \infty$. Thus $P(C'_m F_\infty B) \ge tP(C'_m F_\infty) \ge tP(C' F_\infty)$, $m \ge 1$. Since $C'_m F_\infty B \downarrow C' F_\infty B$, we have $P(C' F_\infty B) \ge tP(C' F_\infty)$. However, C' and F_∞ are $\mathscr{F}_\infty$-measurable and thus $P(C' F_\infty B) < tP(C' F_\infty)$ if $P(C' F_\infty) > 0$. Thus $P(C' F_\infty) = 0$ and $F_\infty \subset C$.

The other half of the proposition is proven in a similar manner. □

It is natural to conjecture that $\liminf_n G_n = \limsup_n G_n = G_\infty$, or at least $\liminf_n G_n = \limsup_n G_n$, where G_n is defined as in the proof of Proposition 2.2. The following example shows that both are false.

EXAMPLE 4. Let $(X, \mathscr{F}, P) = ((0, 1), \mathscr{B}((0, 1)), P)$, where P is the uniform measure. For $n \ge 1$ let

$$\mathscr{F}_n = \sigma(\mathscr{B}((0, a_n)), [a_n, b_n), \mathscr{B}([b_n, 1/2)), [1/2, 1 - 1/2^n), \mathscr{B}([1 - 1/2^n, 1))),$$

where $a_n = 1/2^{n-[n/2]+1}$ and $b_n = 1/2 - 1/2^{[n/2]+2}$. Let $\mathscr{F}_\infty = \bigcap_{n \ge 1} \mathscr{F}_n = \sigma((0, 1/2), [1/2, 1))$ and $B = (1/4, 3/4)$. Then $E[I_B \mid \mathscr{F}_n] = 0$ on $(0, a_n)$, more than $1/2$ on $[a_n, b_n)$ if n is even but equal to $1/2$ if n is odd, 1 on $[b_n, 1/2)$, more than $1/2$ on $[1/2, 1 - 1/2^n)$, and 0 on $[1 - 1/2^n, 1)$. Thus $\{E_\infty I_B = 1/2\} = X$, $\limsup_n \{E[I_B \mid \mathscr{F}_n] \le 1/2\} = (0, 1/2)$, and $\liminf_n \{E[I_B \mid \mathscr{F}_n] \le 1/2\} = \varnothing$. □

THEOREM 2.6. *Suppose $\mathscr{F}_n \downarrow \mathscr{F}_\infty$ and f is an $\mathscr{F}$-measurable and integrable function. Then $E[f \mid \mathscr{F}_n] \to E[f \mid \mathscr{F}_\infty]$ as $n \to \infty$.*

PROOF. First suppose $f = I_B$ for some $\mathscr{F}$-measurable set B. Let $\varepsilon > 0$ be arbitrary. Choose $0 < t_1 < t_2 < \cdots < t_k < 1$ such that $X = \bigcup_{1 \le i \le k} C_i$ for some positive integer k, where $C_i = \{t_i - \varepsilon/2 < E[I_B \mid \mathscr{F}_\infty] < t_i + \varepsilon/2\}$, $1 \le i \le k$. From Proposition 2.2 we have

$$C_i \subset \liminf_n \{E[I_B \mid \mathscr{F}_n] \le t_i + \varepsilon/2\} \cap \liminf_n \{E[I_B \mid \mathscr{F}_n] \ge t_i - \varepsilon/2\}, \quad 1 \le i \le k.$$

Thus $x \in C_i$ implies $|E[I_B \mid \mathscr{F}_\infty](x) - E[I_B \mid \mathscr{F}_n](x)| < \varepsilon$ eventually as $n \to \infty$, $1 \le i \le k$. Since $X = \bigcup_{1 \le i \le k} C_i$ and $\varepsilon > 0$ is arbitrary $E[I_B \mid \mathscr{F}_n] \to E[I_B \mid \mathscr{F}_\infty]$ as $n \to \infty$. A standard argument of truncating f and approximating the truncated f by simple functions now completes the proof of the theorem. □

The last theorem concerns uniform convergence of conditional expectations when the σ-fields are not nested.

THEOREM 2.7. *Suppose $\{\mathscr{F}_n : 1 \le n \le \infty\}$ is an arbitrary collection of σ-fields with $\mathscr{F}_n \subset \mathscr{F}_\infty$, $n \ge 1$. Then $\sum_{n \ge 1} \delta(\mathscr{F}_n, \mathscr{F}_\infty) < \infty$ implies $\sup_{f \in \Phi} P\{\sup_{n \ge m} |E[f \mid \mathscr{F}_n] - E[f \mid \mathscr{F}_\infty]| \ge \varepsilon\} \to 0$ as $m \to \infty$ for all $\varepsilon > 0$.*

PROOF. Let $\varepsilon > 0$ be arbitrary. From the proof of Theorem 2.1, if $f \in \Phi$, then

$$P\{\,|E[f\,|\,\mathscr{F}_n] - E[f\,|\,\mathscr{F}_\infty] \geq \varepsilon\} \leq \delta(\mathscr{F}_n, \mathscr{F}_\infty)/\varepsilon, \quad n \geq 1.$$

Thus $\sup_{f\in\Phi} P\{\sup_{n\geq m}|E[f\,|\,\mathscr{F}_n] - E[f\,|\,\mathscr{F}_\infty]\,| \geq \varepsilon\} \leq \sum_{n\geq m} \delta(\mathscr{F}_n, \mathscr{F}_\infty)/\varepsilon \to 0$ as $m \to \infty$ since $\sum_{n\geq 1} \delta(\mathscr{F}_n, \mathscr{F}_\infty) < \infty$. □

The previously mentioned example given by Boylan (1971) shows that conditional expectations of even a single indicator function may fail to converge when $\sum_n \delta(\mathscr{F}_n, \mathscr{F}_\infty) = \infty$ if the σ-fields are not nested. The following trivial example shows that almost sure uniform convergence is possible even when $\sum_n \delta(\mathscr{F}_n, \mathscr{F}_\infty) = \infty$ and the σ-fields are not nested.

EXAMPLE 5. Let $(X, \mathscr{F}, P) = ((0, 1), \mathscr{B}((0, 1)), P)$, where P is the uniform measure. For $n \geq 1$ let

$$\mathscr{F}_n = \sigma(\mathscr{B}((0, \tfrac{1}{2}^n)), [\tfrac{1}{2}^n, 1/n), \mathscr{B}([1/n, 1)))$$

and let $\mathscr{F}_\infty = \mathscr{F}$. Then $\{\mathscr{F}_n\}$ is not nested, $\sum_n \delta(\mathscr{F}_n, \mathscr{F}_\infty) = \sum_n \tfrac{1}{4}^n = \infty$, and for $0 < \varepsilon \leq \tfrac{1}{2}$, $\sup_{f\in\Phi} P\{\sup_{n\geq m}|E[f\,|\,\mathscr{F}_n] - E[f\,|\,\mathscr{F}_\infty]\,| \geq \varepsilon\} = 1/m \to 0$ as $m \to \infty$. □

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