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CONVERGENCE AND EXISTENCE OF RANDOM SET DISTRIBUTIONS

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We study the relation between distributions of random closed sets and their hitting functions T, defined by $T(B) = P\{\varphi \cap B \neq \emptyset\}$ for Borel sets B. In particular, a sequence of random sets converges in distribution iff the corresponding sequence of hitting functions converges on some sufficiently large class of bounded Borel sets. This class may be chosen to be countable.

1. Introduction. Let S be a locally compact second countable Hausdorff space and $\mathscr{S}$ its Borel σ-algebra. By a *random* (*closed*) *set* in S we mean a mapping φ of some probability space $(\Omega, \mathscr{R}, P)$ into the class $\mathscr{F}$ of closed subsets of S, satisfying

$$\{\varphi \cap B \neq \emptyset\} = \{\omega, \varphi(\omega) \cap B \neq \emptyset\} \in \mathscr{R}, \quad B \in \mathscr{S}.$$

Matheron (1975) gives the necessary background on random set theory, cf. also Kendall (1974) and Ripley (1976). The aim of this paper is to examine the relation between the distribution $P\varphi^{-1}$ and the hitting function T defined by

$$T(B) = P\{\varphi \cap B \neq \emptyset\}, \quad B \in \mathscr{S}.$$

Our main result is Theorem 2.1 which gives necessary and sufficient conditions for convergence in distribution w.r.t. the topology defined on page 3 in Matheron (1975). Loosely speaking, a sequence of random sets converges in distribution to some random set iff the corresponding sequence of hitting functions converges on some suitable class of bounded (i.e. relatively compact) Borel sets. This class of sets may even be chosen countable.

We give two existence criteria. The first one (Theorem 2.2, essentially due to Choquet, 1953) characterizes hitting functions, while the second one (Theorem 2.3) is in terms of a $\{0, 1\}$-valued random process.

Theorem 2.4 extends a characterization of the infinitely divisible distributions due to Matheron (1975). Finally, Theorem 2.5 gives necessary and sufficient conditions for convergence in distribution of the union of many uniformly small independent random sets.

We conclude this introduction with some remarks on terminology and notation. Let $\mathscr{K}$ and $\mathscr{G}$ denote respectively the classes of compact and open sets and put $\mathscr{B} = \{B \in \mathscr{S}, B^- \in \mathscr{K}\}$. Say that a class $\mathscr{A} \subset \mathscr{B}$ is *separating* if there exists, for all $K \in \mathscr{K}$ and $G \in \mathscr{G}$ with $K \subset G$, some $A \in \mathscr{A}$ such that $K \subset A \subset G$. In this case we may choose $A \in \mathscr{A}$ such that $K \subset A^0 \subset A^- \subset G$. Note that

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every separating class includes a countable separating subclass. Let $\mathscr{A}$ be separating and $\alpha: \mathscr{A} \to [0, \infty]$ *increasing*, i.e. $\alpha(A) \le \alpha(B)$ whenever $A \subset B$. We define the *outer limit* α^- and the *inner limit* α^0 of α by

$$\alpha^-(K) = \inf\{\alpha(A), A \in \mathscr{A}, K \subset A^0\}, \quad K \in \mathscr{K},$$

$$\alpha^0(G) = \sup\{\alpha(A), A \in \mathscr{A}, A^- \subset G\}, \quad G \in \mathscr{G}.$$

Note that α^- is defined on $\mathscr{K}$ while α^0 is defined on $\mathscr{G}$. If β is the restriction of α to some separating subclass then clearly $\beta^- = \alpha^-$ and $\beta^0 = \alpha^0$. Furthermore, $\alpha^{--} = \alpha^{0-} = \alpha^-$ and $\alpha^{00} = \alpha^{-0} = \alpha^0$. A set $B \in \mathscr{B}$ is said to be a *continuity set* of α if $\alpha^0(B^0) = \alpha^-(B^-)$. Let $\mathscr{B}_\alpha$ denote the class of such sets. Note that $B \in \mathscr{B}_\alpha$ iff $\alpha^0(B^0) \ge \alpha^-(B^-)$ (since $\alpha^0(B^0) \le \alpha^-(B^-)$, $B \in \mathscr{B}$) and that $\mathscr{B}_\alpha$ need not be included in $\mathscr{A}$.

If β is another increasing function defined on some separating class then $\mathscr{B}_\alpha \cap \mathscr{B}_\beta$ is separating. This may easily be seen by reduction to the corresponding result for increasing functions defined on [0, 1].

We shall write $N = \{1, 2, \cdots\}$.

2. Our results. The families $\{F, F \cap K = \varnothing\}$, $K \in \mathscr{K}$ and $\{F, F \cap G \ne \varnothing\}$, $G \in \mathscr{G}$ constitute an open subbase for a topology in $\mathscr{F}$. With this topology, $\mathscr{F}$ becomes a compact second countable Hausdorff space (see Matheron, 1975, page 3). Let $\mathscr{B}(\mathscr{F})$ denote the Borel-σ-algebra in $\mathscr{F}$ and write Σ for the σ-algebra in $\mathscr{F}$ generated by the family $\{F, F \cap B \ne \varnothing\}$, $B \in \mathscr{B}$.

Since $\mathscr{F}$ is second countable, $\mathscr{B}(\mathscr{F}) \subset \Sigma$. Conversely, every event in Σ is known to be universally measurable (see Matheron, 1975, page 30 or Rockafellar, 1976, page 164f).

Note that the hitting function T of a random set φ is increasing. Moreover, $T = T^-$ on $\mathscr{K}$ and $T = T^0$ on $\mathscr{G}$ (Matheron, 1975, page 28). Hence $B \in \mathscr{B}_T$ iff $T(B^0) = T(B^-)$, i.e. iff $P\{\varphi \cap B^0 = \varnothing, \varphi \cap B^- \ne \varnothing\} = 0$. Since $A^0 \cup B^0 \subset (A \cup B)^0$ and $(A \cup B)^- \subset A^- \cup B^-$, $\mathscr{B}_T$ is closed under finite unions.

Write $=_d$ for equality in distribution, i.e. $\varphi =_d \eta$ iff $P\varphi^{-1} = P\eta^{-1}$ on Σ, and $\to_d$ for convergence in distribution (see Billingsley, 1968).

Our first result gives necessary and sufficient conditions for convergence in distribution.

THEOREM 2.1. *Let $\varphi_1, \varphi_2, \cdots$ be random sets in S and $T_1, T_2, \cdots$ the corresponding hitting functions. If $\varphi_n \to_d$ some random set φ with hitting function T then*

$$\lim_n T_n(B) = T(B), \quad B \in \mathscr{B}_T. \tag{2.1}$$

Conversely, if there exists a separating class $\mathscr{A} \subset \mathscr{B}$ and an increasing set function $\alpha: \mathscr{A} \to [0, 1]$ such that

$$\alpha^0(A^0) \le \liminf_n T_n(A) \le \limsup_n T_n(A) \le \alpha^-(A^-), \quad A \in \mathscr{A} \tag{2.2}$$

then $\varphi_n \to_d$ some random set φ with hitting function T satisfying $T = \alpha^-$ on $\mathscr{K}$

and $T = \alpha^0$ *on* $\mathscr{G}$. *If further* $\mathscr{A} \subset \mathscr{B}_\alpha$ *then* $T = \alpha$ *on* $\mathscr{A}$. *Finally, if* $\mathscr{A} \subset \mathscr{K}$ *(or* $\mathscr{A} \subset \mathscr{G}$*) then* $\mathscr{A} \subset \mathscr{B}_\alpha$ *may be replaced by* $\alpha = \alpha^-$ *(or* $\alpha = \alpha^0$*) on* $\mathscr{A}$.

Note in particular that $\varphi_n \to_d \varphi$ follows from the condition $T_n(A) \to T(A)$ for A in an arbitrary separating class. Thus no continuity condition is needed in this direction.

PROOF. The necessity of (2.1) follows from Theorem 5.1 in Billingsley (1968), since the mapping

$$F \to 1_F(B) = \begin{cases} 1 & \text{if} \quad F \cap B \neq \emptyset \\ 0 & \text{else} \end{cases}$$

is, for fixed $B \in \mathscr{B}$, continuous at F if $F \cap B^0 \neq \emptyset$ or $F \cap B^- = \emptyset$.

Suppose conversely that (2.2) holds for some separating class $\mathscr{A} \subset \mathscr{B}$ and α: $\mathscr{A} \to [0, 1]$. Fix $K \in \mathscr{K}$ and choose $\{A_m\} \subset \mathscr{A}$ such that $K \subset A_m^0 \subset A_m^- \searrow K$. Then

$$\text{limsup}_n T_n(K) \leq \text{limsup}_n T_n(A_m) \leq \alpha^-(A_m^-) \to \alpha^-(K).$$

A similar argument yields

$$\text{liminf}_n T_n(K) \geq \text{liminf}_n T_n(K^0) \geq \alpha^0(K^0)$$

so

$$\text{lim}_n T_n(K) = \alpha^-(K), \quad K \in \mathscr{K} \cap \mathscr{B}_\alpha. \tag{2.3}$$

Since $\mathscr{F}$ is compact, the space of distributions on $\mathscr{F}$ is weakly compact. Hence if M is a subsequence of $\{\varphi_n\}$, there exists a further subsequence M_0 and a random set φ with hitting function T such that $\varphi_n \to_d \varphi$ through M_0. It follows from (2.3) and the direct part of the theorem that $T = \alpha^-$ on the separating class $\mathscr{K} \cap \mathscr{B}_\alpha \cap \mathscr{B}_T$. But $T = T^-$ and $\alpha^{--} = \alpha^-$. Hence $T = \alpha^-$ on $\mathscr{K}$. If φ_0 with hitting function T_0 is a limit point of $\{\varphi_n\}$ the same argument yields $T_0 = \alpha^-$ on $\mathscr{K}$. Hence $T = T_0$ on $\mathscr{K}$, and it follows that $P\varphi^{-1} = P\varphi_0^{-1}$ on $\mathscr{B}(\mathscr{F})$ (see Matheron, 1975, page 28). As noted above, each event in Σ is universally measurable so $\varphi =_d \varphi_0$. Now it follows by Theorem 2.3 in Billingsley (1968) that $\varphi_n \to_d \varphi$. Moreover, $T = \alpha^0$ on $\mathscr{G}$. Hence $\mathscr{B}_T = \mathscr{B}_\alpha$. The proofs of the last assertions are elementary. □

An interesting example of a random set is the support of a simple point process. Combining our theorem with Theorem 4.7 of Kallenberg (1975) leads to the conclusion that a sequence $\{\xi_n\}$ of point processes $\to_d$ a simple point process ξ iff the corresponding sequence of supports $\to_d$ the support of ξ and moreover

$$\text{lim}_{t\to\infty} \text{limsup}_n P\{\xi_n B > t\} = 0, \quad B \in \mathscr{B},$$

$$\text{limsup}_n P\{\xi_n I > 1\} \leq P\{\xi I > 1\}, \quad I \in \mathscr{I},$$

where $\mathscr{I} \subset \mathscr{B}$ is such that the class of finite unions of $\mathscr{I}$-sets is separating.

For increasing real-valued functions α defined on a separating class $\mathscr{A} \subset \mathscr{B}$, which is assumed to be closed under finite unions, we define recursively the successive differences of α w.r.t. $A_1, A_2, \cdots$ at A by the formulas

$$\Delta_{A_1}\alpha(A) = \alpha(A \cup A_1) - \alpha(A), \quad A, A_1 \in \mathscr{A} \tag{2.4}$$

$$\Delta_{A_{n+1}}\Delta_{A_n} \cdots \Delta_{A_1}\alpha(A) = \Delta_{A_n} \cdots \Delta_{A_1}\alpha(A \cup A_{n+1}) - \Delta_{A_n} \cdots \Delta_{A_1}\alpha(A), \quad n \in N, A, A_1, \cdots, A_{n+1} \in \mathscr{A}. \tag{2.5}$$

Recall that α is *alternating* (*of infinite order*) if

$$(-1)^{n+1}\Delta_{A_n} \cdots \Delta_{A_1}\alpha(A) \geq 0, \quad n \in N, \quad A, A_1, \cdots, A_n \in \mathscr{A}, \tag{2.6}$$

cf. Choquet (1953) page 170.

The following theorem is a slight extension of a theorem of Choquet (1953), cf. also Kendall (1974), Matheron (1975) and Ripley (1976).

THEOREM 2.2. *If T: $\mathscr{B} \to [0, 1]$ is the hitting function of some random set then T is alternating and $T(\varnothing) = 0$. Conversely, let $\mathscr{A} \subset \mathscr{B}$ be a separating class, closed under formation of finite unions, and suppose that α: $\mathscr{A} \to [0, 1]$ is alternating with $\alpha(\varnothing) = 0$. Then there exists a random set φ with hitting function $T = \alpha^-$ on $\mathscr{K}$. If further $\mathscr{A} \subset \mathscr{B}_\alpha$ then $T = \alpha$ on $\mathscr{A}$. Finally, if $\mathscr{A} \subset \mathscr{K}$ (or $\mathscr{A} \subset \mathscr{G}$) then $\mathscr{A} \subset \mathscr{B}_\alpha$ may be replaced by $\alpha = \alpha^-$ (or $\alpha = \alpha^0$) on $\mathscr{A}$.*

PROOF. The direct part is immediate. The case $\mathscr{A} = \mathscr{K}$ of the converse part is proved in Matheron (1975) on page 31ff. The general case follows at once from the observation that α^- is alternating if α is so. □

This result follows also from Theorem 2.1. To see this, construct a sequence of random sets with finitely many points whose hitting functions converge to α on some suitable separating class. Proceed as in the proof of Theorem 5.6 in Kallenberg (1975).

We define the *hitting process* 1_φ of the random set φ in S by

$$1_\varphi(B) = \begin{cases} 1, & \text{if } \varphi \cap B \neq \varnothing \\ 0, & \text{else} \end{cases} \quad B \in \mathscr{B}.$$

Clearly 1_φ is increasing with $1_\varphi(\varnothing) = 0$. Moreover, if I is an arbitrary index set then

$$1_\varphi(\cup_i B_i) = \sup_i 1_\varphi(B_i), \quad B_i, \cup_i B_i \in \mathscr{B} \quad \text{for} \quad i \in I.$$

Let $\mathscr{A} \subset \mathscr{B}$ be a class of sets and let $\xi = (\xi(A), A \in \mathscr{A})$ and $\eta = (\eta(B), B \in \mathscr{B})$ be random processes. If $\xi(A) = \eta(A)$ a.s. whenever $A \in \mathscr{A}$ then we say that η is an *extension* of ξ. Our next result is similar to Theorem 9.2 of Vervaat (1982).

THEOREM 2.3. *Let $\mathscr{A} \subset \mathscr{B}$ be separating and closed under finite unions and $\xi = (\xi(A), A \in \mathscr{A})$ an increasing $\{0, 1\}$-valued random process satisfying $\xi(\varnothing) = 0$ and*

$$\xi(A \cup B) = \max(\xi(A), \xi(B)) \quad \text{a.s.}, \quad A, B \in \mathscr{A}. \tag{i}$$

Then there exists a random set φ such that with probability one $1_\varphi(K) = \xi^-(K)$,

$K \in \mathscr{K}$ *and* $1_\varphi(G) = \xi^0(G)$, $G \in \mathscr{G}$. *If further*

$$\text{(ii)} \qquad \xi^0(A^0) = \xi^-(A^-) \quad \text{a.s.,} \quad A \in \mathscr{A}$$

then 1_φ *is an extension of* ξ. *Finally, if* $\mathscr{A} \subset \mathscr{K}$ *then* (ii) *may be replaced by*

$$\text{(ii}'\text{)} \qquad \xi(A) = \xi^-(A) \quad \text{a.s.,} \quad A \in \mathscr{A},$$

and if $\mathscr{A} \subset \mathscr{G}$ *then* (i) *and* (ii) *may be replaced by*

$$\text{(i}'\text{)} \qquad \xi(\cup_i A_i) = \sup_i \xi(A_i) \quad \text{a.s.,} \quad A_1, A_2, \cdots, \cup_i A_i \in \mathscr{A}.$$

PROOF. A simple calculation yields $\xi^0(G_1 \cup G_2) = \max(\xi^0(G_1), \xi^0(G_2))$ a.s. whenever $G_1, G_2 \in \mathscr{G}$. Let $\mathscr{G}_0 \subset \mathscr{G}$ be countable, separating and closed under finite unions. Then, a.s.,

$$\xi^0(G_1 \cup G_2) = \max(\xi^0(G_1), \xi^0(G_2)), \quad G_1, G_2 \in \mathscr{G}_0.$$

Discard the exceptional null-set. By compactness we may now prove that

$$\xi^-(K) = \sup_{s \in K} \xi^-(\{s\}), \quad K \in \mathscr{K}.$$

Define for each $\omega \in \Omega$ a closed set $\varphi = \{s \in S, \xi^-(\{s\}) = 1\}$. Clearly $\xi^-(K) = 1$ iff $\varphi \cap K \neq \varnothing$, i.e. iff $1_\varphi(K) = 1$. Hence $1_\varphi = \xi^-$ on $\mathscr{K}$. This implies that $1_\varphi = \xi^0$ on $\mathscr{G}$. Fix $A \in \mathscr{A}$. If $\xi^0(A^0) = \xi^-(A^-)$, then $1_\varphi(A^0) = 1_\varphi(A^-)$ and therefore $\xi(A) = 1_\varphi(A)$. Hence 1_φ extends ξ if (ii) holds. The next assertion is obvious. To prove the last assertion, note first that (i) is obvious if $\mathscr{A} \subset \mathscr{G}$ and (i′) is true. Moreover, it follows that $\xi(A) = \xi^0(A^0)$ a.s. whenever $A \in \mathscr{A}$. □

Recall that a random set φ with hitting function T is *infinitely divisible*, if for each $n \in N$ there exists some independent and identically distributed random sets $\varphi_1, \cdots, \varphi_n$ with $\varphi =_d \cup_{i=1}^n \varphi_i$, cf. Matheron (1975) page 54. This holds iff the function $1-(1-T)^{1/n}$ is a hitting function whenever $n \in N$, and by Theorem 2.2 this is true iff $1-(1-T)^{1/n}$ is alternating.

Before characterizing the infinitely divisible distributions, we must extend the definition of an alternating function to cover the case where the function may take the value $+\infty$. Let $\mathscr{A} \subset \mathscr{B}$ be separating and closed under finite unions and $\alpha: \mathscr{A} \to [0, \infty]$ increasing and *subadditive*, i.e. $\alpha(A \cup B) \le \alpha(A) + \alpha(B)$, $A, B \in \mathscr{A}$. Thus $\alpha(A \cup B)$ is finite iff both $\alpha(A)$ and $\alpha(B)$ are so. In the present case, the successive differences of α are only defined when $\alpha(A), \alpha(A_1), \alpha(A_2), \cdots < \infty$, and we shall still say that α is alternating if (2.6) holds.

Our next result is a slight extension of a theorem by Matheron (1975).

THEOREM 2.4. *If the random set* φ *with hitting function* T *is infinitely divisible then* $\psi = -\log(1-T)$ *is alternating with* $\psi(\varnothing) = 0$. *Conversely, let* $\mathscr{A} \subset \mathscr{B}$ *be a separating class, closed under finite unions and suppose that* $\psi: \mathscr{A} \to [0, \infty]$ *is alternating with* $\psi(\varnothing) = 0$. *Then there exists an infinitely divisible random set* φ *with hitting function* $T = 1 - \exp(-\psi^-)$ *on* $\mathscr{K}$. *If further* $\mathscr{A} \subset \mathscr{B}_\psi$ *then* $T = 1 - \exp(-\psi)$ *on* $\mathscr{A}$. *Finally, if* $\mathscr{A} \subset \mathscr{K}$ (*or* $\mathscr{A} \subset \mathscr{G}$) *then* $\mathscr{A} \subset \mathscr{B}_\psi$ *may be replaced by* $\psi = \psi^-$ (*or* $\psi = \psi^0$) *on* $\mathscr{A}$.

Note in particular that a random set φ in S with hitting function T is infinitely divisible iff $\psi = -\log(1 - T)$ is alternating.

PROOF. If φ is infinitely divisible then ψ is alternating, since $\psi = \lim_n n(1 - (1 - T)^{1/n})$. The converse part is proved for the case $\psi(A) < \infty$, $A \in \mathscr{A} = \mathscr{K}$ in Matheron (1975) on page 56f. This proof may be extended to cover the case where ψ may take the value $+\infty$. Note that it is necessary to assume ψ to be subadditive. The truth for general $\mathscr{A}$ follows from the fact that ψ^- is alternating if ψ is so. □

Let $\{\varphi_{nj}, n \in N, j \in J_n\}$ be a family of random sets in S and $\{T_{nj}\}$ the corresponding family of hitting functions. Say that $\{\varphi_{nj}\}$ is a *null-array* if for each $n \in N$ the random sets $\{\varphi_{nj}, j \in J_n\}$ are independent and, moreover, $\sup_j T_{nj}(K) \to 0$ for all $K \in \mathscr{K}$. Note that J_n may be infinite, provided that $\cup_j \varphi_{nj}$ is a random set.

THEOREM 2.5. *Let $\{\varphi_{nj}\}$ be a null-array of random sets and $\{T_{nj}\}$ the corresponding hitting functions. Put $\varphi_n = \cup_j \varphi_{nj}$ and let T_n be the hitting function of φ_n. If $\varphi_n \to_d$ some random set φ with hitting function T then φ is infinitely divisible and*

$$\lim_n \textstyle\sum_j T_{nj}(B) = -\log(1 - T(B)), \quad B \in \mathscr{B}_T. \tag{2.7}$$

Conversely, if there exists a separating class $\mathscr{A} \subset \mathscr{B}$ and an increasing function $\psi: \mathscr{A} \to [0, \infty]$ such that

$$\psi^0(A^0) \le \liminf_n \textstyle\sum_j T_{nj}(A) \le \limsup_n \sum_j T_{nj}(A) \le \psi^-(A^-), \quad A \in \mathscr{A} \tag{2.8}$$

then $\varphi_n \to_d$ some infinitely divisible random set φ with hitting function $T = 1 - \exp(-\psi^-)$ on $\mathscr{K}$. If further $\mathscr{A} \subset \mathscr{B}_\psi$ then $T = 1 - \exp(-\psi)$ on $\mathscr{A}$. Finally, if $\mathscr{A} \subset \mathscr{K}$ (or $\mathscr{A} \subset \mathscr{G}$) then $\mathscr{A} \subset \mathscr{B}_\psi$ may be replaced by $\alpha = \alpha^-$ (or $\alpha = \alpha^0$) on $\mathscr{A}$.

Note in particular that $\varphi_n = \cup_j \varphi_{nj} \to_d \varphi$ follows from the condition $\sum_j T_{nj}(A) \to -\log(1 - T(A))$ for A in an arbitrary separating class. Note also that the convergence in (2.7) and (2.8) is in $[0, \infty]$.

Our proof depends on the following lemma.

LEMMA 2.6. *Let $B \in \mathscr{B}$ and $\tau \in [0, \infty]$. Then $\sum_j T_{nj}(B) \to \tau$ iff $T_n(B) \to 1 - \exp(-\tau)$.*

PROOF. Let $x \in [0, d]$ for some $d \in (0, 1)$, and put $D = 1/(1 - d)$. Then

$$x \le -\log(1 - x) \le Dx \tag{2.9}$$

and

$$0 \le -x - \log(1 - x) \le \frac{D}{2} x^2. \tag{2.10}$$

Since $\sup_j T_{nj}(B) \to 0$, we may assume that $T_{nj}(B) \le d$. By (2.9),

$$\textstyle\sum_j T_{nj}(B) \le -\log(1 - T_n(B)) \le D \sum_j T_{nj}(B),$$

and from (2.10) we get

$$\begin{aligned} 0 &\le -\textstyle\sum_j T_{nj}(B) - \log(1 - T_n(B)) \\ &\le \frac{D}{2} \textstyle\sum_j T_{nj}(B)^2 \le \frac{D}{2} \sum_j T_{nj}(B) \sup_j T_{nj}(B). \end{aligned}$$

Now the stated equivalence follows. □

PROOF OF THEOREM 2.5. If $\varphi_n \to_d \varphi$, then (2.7) follows immediately from Theorem 2.1 and Lemma 2.6. Morever, $-\log(1 - T)$ must be alternating, since $\sum_j T_{nj}$ is so. Hence φ is infinitely divisible.

Suppose conversely that (2.8) is true for some separating class $\mathscr{A} \subset \mathscr{B}$ and function $\psi: \mathscr{A} \to [0, \infty]$. It follows, as in the proof of (2.3), that $\sum_j T_{nj}(K) \to \psi^-(K)$, $K \in \mathscr{K} \cap \mathscr{B}_\psi$. Now all assertions follow from Lemma 2.6 and Theorem 2.1. □

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