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LIMIT THEORY FOR MOVING AVERAGES OF RANDOM VARIABLES WITH REGULARLY VARYING TAIL PROBABILITIES[1]

BY RICHARD DAVIS and SIDNEY RESNICK

Colorado State University

Let $\{Z_k, -\infty < k < \infty\}$ be iid where the Z_k's have regularly varying tail probabilities. Under mild conditions on a real sequence $\{c_j, j \geq 0\}$ the stationary process $\{X_n := \sum_{j=0}^{\infty} c_j Z_{n-j}, n \geq 1\}$ exists. A point process based on $\{X_n\}$ converges weakly and from this, a host of weak limit results for functionals of $\{X_n\}$ ensue. We study sums, extremes, excedences and first passages as well as behavior of sample covariance functions.

1. Introduction. Consider a sequence $(Z_k, -\infty < k < \infty)$ of real valued independent, identically distributed (iid) random variables. We assume

(1.1) $P(|Z_k| > x) = x^{-\alpha}L(x)$ where $L(x)$ is slowly varying at ∞ and $\alpha > 0$

and

$$\frac{P(Z_k > x)}{P(|Z_k| > x)} \to p \quad \text{and} \quad \frac{P(Z_k \leq -x)}{P(|Z_k| > x)} \to q \tag{1.2}$$

as $x \to \infty$, $0 \leq p \leq 1$ and $q = 1 - p$. Note that until Section 4 we do not require $0 < \alpha < 2$. Under mild conditions on a real sequence $\{c_j, j \geq 0\}$ (cf. Section 2 and Cline, 1983) the series

$$\sum_{j=0}^{\infty} c_j Z_{-j}$$

converges and we may define the stationary sequence of moving averages

$$X_n := \sum_{j=0}^{\infty} c_j Z_{n-j} \tag{1.3}$$

for $n \geq 1$. We study the weak limit behavior of various functionals of $\{X_n, n \geq 1\}$ such as extremes, sums and sample covariance functions.

Most of the work on extreme value theory for stationary sequences has focused primarily on the extension of the classical results to the stationary setting. In order to attain such an extension these processes are typically required to satisfy a mixing condition and a local dependence restriction such as D and D' formulated by Leadbetter (1974). Unfortunately many processes, such as $\{X_n\}$ above, rarely satisfy D'. However, the limit distribution of for example the maximum of $\{X_n\}$ can still be ascertained in some instances (cf. Rootzen, 1978; Finster, 1982; and Leadbetter, Lindgren, and Rootzen, 1983). In this paper we prove a

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point process convergence result which enables us to describe in some detail the weak limiting behavior of various functionals including the extremes of $\{X_n\}$. A survey of the techniques that will be used is given in Resnick (1984).

In Section 2, we prove our basic result which says that a sequence of point processes constructed from the sequence $\mathbf{Z}^{(k)} := (Z_k, Z_{k-1}, \cdots, Z_{k-m+1}) \in \mathbb{R}^m$, $k \geq 1$ converges weakly to a limit point process. An argument involving continuous mappings and Slutsky style proofs allows us to easily extend the basic result to show a sequence of point processes based on $\{X_n\}$ converges to another limit point process. The limiting point processes that we obtain are always derived from Poisson processes.

From these two limit theorems a variety of applications ensue in Section 3: (a) joint behavior of the upper extremes of $\{X_1, \cdots, X_n\}$ as $n \to \infty$; (b) joint behavior of the maximum and minimum of $\{X_1, \cdots, X_n\}$; (c) the first passage or inverse process to the maximum sequence of $\{X_n\}$ (cf. Finster, 1982); (d) the point process of excedences (cf. Rootzen, 1978).

For convenience, we state all of our results for one-sided moving averages. However, with routine modifications, these results will also be valid for two-sided moving averages with the constraint analogous to (2.6) on the coefficients. Hence, any stationary ARMA (p, q) process driven by a noise sequence with regularly varying tail probabilities will satisfy the hypotheses of our theorems.

Weak convergence notation and usage are as in Billingsley (1968) except that "$\Rightarrow$" is used to indicate weak convergence. For point processes we follow Neveu (1976); see also Kallenberg (1976). Let E be a state space which for our purposes is Euclidean. Let $\mathscr{E}$ be the σ-algebra generated by open sets. For $x \in E$, $F \in \mathscr{E}$, $\varepsilon_x(F) = 1$ if $x \in F$, 0 otherwise. A point measure m is defined to be a measure of the form $\sum_{i \in I} \varepsilon_{x_i}$ which is nonnegative integer valued and finite on relatively compact subsets of E. The class of such measures is $M_p(E)$ and $\mathscr{M}_p(E)$ is the smallest σ-algebra making the evaluation maps $m \to m(F)$ measurable where $m \in M_p(E)$ and $F \in \mathscr{E}$. A point process on E is a measurable map from a probability space $(\Omega, \mathscr{A}, P)$ to $(M_p(E), \mathscr{M}_p(E))$. Let $C_K^+(E)$ be the continuous functions $E \to R_+$ with compact support. A useful topology for $M_p(E)$ is the vague topology which renders $M_p(E)$ a complete separable metric space. If $\mu_n \in M_p(E)$, $n \geq 0$ then μ_n converges vaguely to μ_0 (written $\mu_n \to_v \mu_0$) if $\mu_n(f) \to \mu_0(f)$ for all $f \in C_K^+(E)$ where $\mu(f) = \int f \, d\mu$.

A Poisson process on $(E, \mathscr{E})$ with mean measure μ is a point process ξ satisfying for all $F \in \mathscr{E}$:

$$P[\xi(F) = k] = \begin{cases} e^{-\mu(A)}(\mu(A))^k/k! & \text{if } \mu(A) < \infty \\ 0 & \text{if } \mu(A) = \infty \end{cases}$$

and if $F_1, \cdots, F_n \in \mathscr{E}$ are mutually disjoint, then $\xi(F_1), \cdots, \xi(F_n)$ are independent. We assume μ is Radon. We will call ξ PRM (Poisson random measure) with mean measure μ on $(E, \mathscr{E})$, or PRM(μ) for short.

2. Basic convergences. Let $\{Z_k\}$ be an iid sequence having regularly varying tail probabilities as specified by (1.1) and (1.2). Further let $\{a_n\}$ be a sequence of positive constants such that

(2.1) $$nP(|Z_1| > a_n x) \to x^{-\alpha} \quad \text{for all} \quad x > 0.$$

In fact a_n may be defined as $\inf\{x: P(|Z_k| > x) \le n^{-1}\}$. On the space $(0, \infty) \times (\mathbb{R}\backslash\{0\})$ define the measure $\mu(dt, dx) = dt \times \lambda(dx)$ where $\lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)dx$. Then it may be shown (cf. Weissman, 1975a, b: Resnick, 1975, 1984; Mori and Oodaira, 1976) that $\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}Z_k)} \Rightarrow \sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k)}$ in $M_p((0, \infty) \times (\mathbb{R}\backslash\{0\}))$ where $\sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k)}$ is a PRM(μ) on $(0, \infty) \times (\mathbb{R}\backslash\{0\})$. Note the convention that if a point falls outside the state space it does not contribute to the sum. This result is useful for deriving asymptotic properties of various statistics which are functionals of the sequence $\{Z_k\}$. For our purposes, this result will be used to describe and interpret limits of other point processes.

For a fixed positive integer $m > 1$, set

$$I_n = \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}\mathbf{Z}^{(k)})} \quad \text{and} \quad I = \sum_{k=1}^{\infty} \sum_{i=1}^{m} \varepsilon_{(t_k, j_k \mathbf{e}_i)}$$

where $\mathbf{Z}^{(k)} = (Z_k, Z_{k-1}, \cdots, Z_{k-m+1})$ and $\mathbf{e}_i \in \mathbb{R}^m$ is the basis element with ith component equal to one and the rest zero. If we neglect the t_k's temporarily, the process I may be described as follows: Take the j_k's and lay them on the axis determined by $\mathbf{e}_1$, then repeat this pattern on $\mathbf{e}_2$ and so on. The procedure is repeated deterministically on each axis.

The relevant state space for the processes $\{I_n\}$ and I is $E = (0, \infty) \times (\mathbb{R}^m\backslash\{(0, 0, \cdots, 0)\})$ where $\mathscr{S}$ is the usual product σ-algebra modified so that the compact sets of $\mathbb{R}^m\backslash\{(0, 0, \cdots, 0)\}$ are those compact sets in $\mathbb{R}^m$ which are bounded away from $(0, 0, \cdots, 0)$. We shall show $I_n \Rightarrow I$ but first we need to specify a convenient class of sets which generate $\mathscr{S}$. Let S be the collection of all sets B of the form

(2.2) $$B = (b_0, c_0] \times (b_1, c_1] \times \cdots \times (b_m, c_m]$$

where the m-dimensional rectangle $(b_1, c_1] \times \cdots \times (b_m, c_m]$ is bounded away from $(0, 0, \cdots, 0)$ and $b_i < c_i$, $b_i \neq 0$, $c_i \neq 0$ for $i = 1, \cdots, m$ and $b_0 \ge 0$. It is clear that S is a DC-semiring (cf. Kallenberg, 1976, page 3). Moreover, since $B \in S$ is bounded away from zero, either

(C1) $$(b_1, c_1] \times \cdots \times (b_m, c_m] \cap \{y\mathbf{e}_i : y \in \mathbb{R}\} = \phi \quad \text{for} \quad i = 1, \cdots, m$$

or

(C2) $$(b_1, c_1] \times \cdots \times (b_m, c_m] \cap \{y\mathbf{e}_i : y \in \mathbb{R}\} = \begin{cases} (b_{i'}, c_{i'}] & i = i' \\ \phi & i \neq i'. \end{cases}$$

That is, the rectangle $(b_1, c_1] \times \cdots \times (b_m, c_m]$ either has empty intersection with all of the coordinate axes or intersects exactly one in an interval. Note that in (C2), $b_i < 0 < c_i$ for $i \neq i'$ and $0 \notin (b_{i'}, c_{i'}]$. The following properties hold:

(2.3) $$P(I(\partial B) = 0) = 1 \quad \text{for all} \quad B \in S.$$

If B satisfies (C1) there is nothing to check since the points of I are located on the coordinate axes. However if B satisfies (C2), then $I(\partial B) \le \sum_{k=1}^{\infty} \varepsilon_{j_k}(\{b_{i'}, c_{i'}\}) = 0$ a.s. since the mean measure of $\sum_{k=1}^{\infty} \varepsilon_{j_k}$ is atomless.

(2.4) $P(I(B) = 0) = 1$ and $EI_n(B) \to 0$ if $B \in S$ satisfies (C1).

This follows easily since $(b_1, c_1] \times \cdots \times (b_m, c_m]$ has empty intersection with all of the coordinate axes and

$$EI_n(B) = \sum_{k/n \in (b_0, c_0]} P(a_n^{-1} Z_k \in (b_1, c_1], \cdots, a_n^{-1} Z_{k-m+1} \in (b_m, c_m])$$
$$\le \sum_{k/n \in (b_0, c_0]} \prod_{i=1}^{m} P(a_n^{-1} | Z_1 | > | b_i | \wedge | c_i |)$$

and from the definition of a_n it is clear that this sum goes to zero.

$$(2.5) \qquad P(I(B) = 0) = \exp(-\mu((b_0, c_0] \times (b_{i'}, c_{i'}])) \quad \text{and} \quad EI_n(B) \to \mu((b_0, c_0] \times (b_{i'}, c_{i'}])$$

if $B \in S$ satisfies (C2).

As above,

$$EI_n(B) = \sum_{k/n \in (b_0, c_0]} \prod_{i=1}^{m} P(a_n^{-1} Z_1 \in (b_i, c_i]).$$

Since $b_i < 0 < c_i$ for $i \ne i'$, $\prod_{i \ne i'} P(a_n^{-1} Z_1 \in (b_i, c_i]) \to 1$ and therefore

$$EI_n(B) \sim (c_0 - b_0) n P(a_n^{-1} Z_1 \in (b_{i'}, c_{i'}])$$
$$\to (c_0 - b_0) \lambda((b_{i'}, c_{i'}]) = \mu((b_0, c_0] \times (b_{i'}, c_{i'}]).$$

PROPOSITION 2.1. *Let* $\tilde{I}_n = \sum_{k=1}^{\infty} \sum_{i=1}^{m} \varepsilon_{(k/n, a_n^{-1} Z_k \cdot \mathbf{e}_i)}$. *Then* $I_n(B) - \tilde{I}_n(B) \to 0$ *in probability for all* $B \in S$.

PROOF. For simplicity we shall assume $B = (0, 1] \times (b_1, c_1] \times \cdots \times (b_m, c_m]$, the other cases being handled similarly. First consider the case when B satisfies (C1). Then by (2.4), $EI_n(B) \to 0$ and from the definition of $\tilde{I}_n$, $\tilde{I}_n(B) = 0$ which proves the result in this case.

Now suppose B satisfies (C2) in which case $0 \in (b_i, c_i]$, $i \ne i'$ and $0 \notin (b_{i'}, c_{i'}]$. By writing

$$I_n(B) = \sum_{k=1}^{i'-1} \varepsilon_{(k/n, a_n^{-1} \mathbf{Z}^{(k)})}(B) + \sum_{k=i'}^{\infty} \varepsilon_{(k/n, a_n^{-1} \mathbf{Z}^{(k)})}(B),$$

we see that the expectation of the first term is bounded above by $(i' - 1) \cdot P[a_n^{-1} Z_1 \in (b_{i'}, c_{i'}]] \to 0$ as $n \to \infty$ and thus this piece is $o_p(1)$. The second term can be bounded above by

$$\sum_{j=1}^{n-i'+1} \varepsilon_{((j+i'-1)/n, a_n^{-1} Z_j)}((0, 1] \times (b_{i'}, c_{i'}])$$
$$\le \sum_{j=1}^{n} \varepsilon_{(j/n, a_n^{-1} Z_j)}((0, 1] \times (b_{i'}, c_{i'}]) = \tilde{I}_n(B).$$

Clearly $E\tilde{I}_n(B) \to \mu((0, 1] \times (b_{i'}, c_{i'}])$ and $E(\sum_{k=i'}^{\infty} \varepsilon_{(k/n, a_n^{-1} \mathbf{Z}^{(k)})}(B)) \to \mu((0, 1] \times (b_{i'}, c_{i'}])$ by (2.5) which together with the inequality above gives $\tilde{I}_n(B) - \sum_{k=i'}^{\infty} \varepsilon_{(k/n, a_n^{-1} \mathbf{Z}^{(k)})}(B) \to 0$ in probability. This completes the proof. □

THEOREM 2.2. *Let* $\{Z_k\}$ *be iid satisfying* (1.1) *and* (1.2) *with* $\{a_n\}$ *satisfying*

(2.1). *For each fixed positive integer m,*

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}\mathbf{Z}^{(k)})} \Rightarrow \sum_{k=1}^{\infty} \sum_{i=1}^{m} \varepsilon_{(t_k, j_k \cdot \mathbf{e}_i)}$$

in $M_p((0, \infty) \times (\mathbb{R}^m \backslash \{(0, \cdots, 0)\}))$ *where* $\{(t_k, j_k): k \geq 1\}$ *are the points of a* $PRM(\mu)$ *on* $(0, \infty) \times (\mathbb{R}^m \backslash \{(0, \cdots, 0)\})$.

PROOF. By Theorem 4.2 in Kallenberg (1976), it suffices to show $(I_n(B_1), \cdots, I_n(B_j)) \Rightarrow (I(B_1), \cdots, I(B_j))$ for any $j \geq 1$ and sets $B_1, \cdots, B_j \in S$. However, in view of the above proposition, it is enough to prove $(\tilde{I}_n(B_1), \cdots, \tilde{I}_n(B_j)) \Rightarrow (I(B_1), \cdots, I(B_j))$ or equivalently $\tilde{I}_n \Rightarrow I$. But the composition of the two continuous mappings,

$$\sum_{k=1}^{\infty} \varepsilon_{(u_k, v_k)} \mapsto \left(\sum_{k=1}^{\infty} \varepsilon_{(u_k, v_k \cdot \mathbf{e}_1)}, \sum_{k=1}^{\infty} \varepsilon_{(u_k, v_k \cdot \mathbf{e}_2)}, \cdots, \sum_{k=1}^{\infty} \varepsilon_{(u_k, v_k \cdot \mathbf{e}_m)}\right)$$

$$\mapsto \sum_{k=1}^{\infty} \sum_{i=1}^{m} \varepsilon_{(u_k, v_k \cdot \mathbf{e}_i)}$$

is itself a continuous mapping from $M_p((0, \infty) \times (\mathbb{R} \backslash \{0\}))$ into $M_p(E)$. Thus by the continuous mapping theorem we obtain $\tilde{I}_n \Rightarrow I$ as desired. □

We now use these results to derive a point process result based on $\{X_n\}$ where recall $\{X_n\}$ is defined in (1.3) by $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ and as usual $\{Z_j\}$ satisfies (1.1) and (1.2). According to results in Cline (1983), the infinite series converges a.s. if

$$\sum_{j=0}^{\infty} |c_j|^{\delta} < \infty \quad \text{for some} \quad \delta < \alpha, \quad \delta \leq 1 \tag{2.6}$$

and in this case

$$\lim_{t \to \infty} \frac{P[|\sum_{j=0}^{\infty} c_j Z_{-j}| > t]}{P[|Z_1| > t]} = \sum_{j=0}^{\infty} |c_j|^{\alpha}. \tag{2.7}$$

We begin with a lemma which parallels Lemma 3.8.2(i) on page 74 of Leadbetter, Lindgren and Rootzen (1983).

LEMMA 2.3. *If* $\{c_j\}$ *satisfies* (2.6) *then for any* $\gamma > 0$

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\left[a_n^{-1} \bigvee_{k=1}^{n} \left|\sum_{j=0}^{m} c_j Z_{k-j} - X_k\right| > \gamma\right] = 0. \tag{2.8}$$

PROOF. We have

$$P\left[a_n^{-1} \bigvee_{k=1}^{n} \left|\sum_{j=0}^{m} c_j Z_{k-j} - X_k\right| > \gamma\right] = P\left[a_n^{-1} \bigvee_{k=1}^{n} \left|\sum_{j>m} c_j Z_{k-j}\right| > \gamma\right]$$

and since $\{\sum_{j>m} c_j Z_{k-j}, k = 1, \cdots, n\}$ is stationary the above is bounded by

$$nP\left[a_n^{-1} \left|\sum_{j>m} c_j Z_{k-j}\right| > \gamma\right] \sim \frac{P[|\sum_{j>m} c_j Z_{k-j}| > a_n \gamma]}{P[|Z_1| > a_n \gamma]} \cdot \frac{P[|Z_1| > a_n \gamma]}{P[|Z_1| > a_n]}$$

and by (2.1) and (2.7) as $n \to \infty$ this converges to

$$\sum_{j>m} |c_j|^{\alpha} \gamma^{-\alpha} \to 0$$

as $m \to \infty$. □

We may now state and prove a convergence result for point processes based on $\{X_k\}$.

THEOREM 2.4. *Suppose* $\{a_n\}$ *satisfies* (2.1), $\{c_j\}$ *satisfies* (2.6), $\{Z_k\}$ *satisfies* (1.1) *and* (1.2) *and* $\{X_k\}$ *is given by* (1.3). *Let* $\{(t_k, j_k)\}$ *be the points of* $PRM(\mu)$ *on* $(0, \infty) \times (\mathbb{R}\backslash\{0\})$.

(i) *In* $M_p((0, \infty) \times (R\backslash\{0\}))$ *as* $n \to \infty$

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, X_k/a_n)} \Rightarrow \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k c_i)}.$$

(ii) *For any positive integer* ℓ

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}(X_k, X_{k-1}, \cdots, X_{k-\ell}))} \Rightarrow \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \varepsilon_{(t_k, j_k(c_i, c_{i-1}, \cdots, c_{i-\ell}))}$$

in $M_p((0, \infty) \times (\mathbb{R}^{\ell+1}\backslash\{(0, 0, \cdots, 0)\}))$ *where the sum in the limit is taken over those points lying in the state space.*

PROOF. (i) From Theorem 2.2 we have for any positive integer m

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}\mathbf{Z}^{(k)})} \Rightarrow \sum_{i=1}^{m} \sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k \mathbf{e}_i)}.$$

The map

$$(z_k, z_{k-1}, \cdots, z_{k-m+1}) \to \sum_{i=0}^{m-1} c_i z_{k-i} \tag{2.9}$$

induces a continuous map from $M_p(E) \to M_p((0, \infty) \times (\mathbb{R}\backslash\{0\}))$ and so by the continuous mapping theorem

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}\sum_{i=0}^{m-1} c_i Z_{k-i})} \Rightarrow \sum_{i=0}^{m-1} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)} \quad \text{in} \quad M_p((0, \infty) \times (\mathbb{R}\backslash\{0\})).$$

As $m \to \infty$

$$\sum_{i=0}^{m-1} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)} \to \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)} \tag{2.10}$$

pointwise in the vague metric and so by Theorem 4.2, page 25 in Billingsley (1968) it suffices to show for any $\eta > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[\rho(\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}\sum_{i=0}^{m-1} c_i Z_{k-i})}, \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}X_k)}) > \eta] = 0$$

where ρ is the metric inducing the vague topology on $M_p(E)$. To accomplish this it is enough to prove (cf. Kallenberg, 1976, page 95) that for

$$f \in C_K^+((0, \infty) \times (\mathbb{R}\backslash\{0\}))$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[\,|\sum_{k=1}^{\infty} f(k/n, a_n^{-1} \sum_{i=0}^{m-1} c_i Z_{k-i}) - \sum_{k=1}^{\infty} f(k/n, a_n^{-1} X_k)\,| > \eta] = 0. \tag{2.11}$$

Suppose the support of f is contained in $[a, b] \times ([-K + \gamma_0, -K^{-1} - \gamma_0] \cup [K^{-1} + \gamma_0, K - \gamma_0])$ where $(K + K^{-1})/2 > \gamma_0 > 0$ and $0 < a < b$. Set $\omega(\gamma) = \sup\{\,|f(t, x) - f(t, y)|: x, y \in (0, \infty)$ or $x, y \in (-\infty, 0)$ and $|x - y| \le \gamma, t > 0\}$.

Since f has compact support it is also uniformly continuous and therefore $\omega(\gamma) \to 0$ if $\gamma \to 0$. If $\gamma < \gamma_0 \wedge K^{-1}$ then on the set $A_n = [a_n^{-1} \bigvee_{k=1}^n |\sum_{i=0}^{m-1} c_i Z_{k-i} - X_k| \le \gamma]$ we have $a_n^{-1} \sum_{i=0}^{m-1} c_i Z_{k-i} \in [-K, -K^{-1}] \cup [K^{-1}, K] =: B(K)$ implies

$$|f(k/n, a_n^{-1} \textstyle\sum_{i=0}^{m-1} c_i Z_{k-i}) - f(k/n, a_n^{-1} X_k)| \le \omega(\gamma)$$

and $a_n^{-1} \sum_{i=0}^{m-1} c_i Z_{k-i} \notin B(K)$ implies $f(k/n, a_n^{-1} \sum_{i=0}^{m-1} c_i Z_{k-i}) = 0 = f(k/n, a_n^{-1} X_k)$. Therefore (2.11) becomes

$$\begin{aligned}&\lim_{m\to\infty} \limsup_{n\to\infty} P\{[|\textstyle\sum_{k=1}^\infty f(k/n, a_n^{-1} \sum_{i=0}^{m-1} c_i Z_{k-i}) \\ &\qquad\qquad - \textstyle\sum_{k=1}^\infty f(k/n, a_n^{-1} X_n)| > \eta] \cap (A_n + A_n^c)\} \\ &\quad \le \lim_{m\to\infty} \limsup_{n\to\infty} P[\omega(\gamma) \textstyle\sum_{k=1}^\infty \varepsilon_{(k/n, \sum_{i=0}^{m-1} c_i Z_{k-i})}([a, b] \times B(K)) > \eta]\end{aligned}$$

(where the term involving A_n^c was killed via (2.8))

$$= \lim_{m\to\infty} P[\omega(\gamma) \textstyle\sum_{i=0}^{m-1} \sum_{k=1}^\infty \varepsilon_{(t_k, j_k c_i)}((0, 1] \times B(K)) > \eta]$$

from Theorem 2.2 and from (2.10) this is

$$= P[\omega(\gamma) \textstyle\sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon_{(t_k, j_k c_i)}([a, b] \times B(K)) > \eta].$$

Since

$$\textstyle\sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon_{(t_k, j_k c_i)}([a, b] \times B(K)) < \infty \quad \text{a.s.}$$

the desired result is obtained by letting $\gamma \to 0$.

(ii) The continuous map (2.9) is replaced by the map

$$(z_k, z_{k-1}, \cdots, z_{k-m+1}) \to (\textstyle\sum_{i=0}^{m-1} c_i z_{k-i}, \sum_{i=0}^{m-1} c_i z_{k-1-i}, \cdots, \sum_{i=0}^{m-1} c_i z_{k-\ell-i})$$

and (2.8) is changed (without difficulty) to

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(a_n^{-1} \textstyle\bigvee_{i=0}^{\ell} \bigvee_{k=1}^n |\sum_{j=0}^{m-1} c_j Z_{k-i-j} - Z_{k-i}| > \gamma) = 0.$$

The rest of the proof of (ii) is almost the same as that given in (i) and is omitted. □

3. Applications.

By applying functions and the continuous mapping theorem to Theorem 2.4, a variety of corollaries can be drawn. We now explore some of these.

We suppose throughout this section that the hypotheses of Theorem 2.4 are met. Set $c_+ = \max_j(c_j \vee 0)$, $c_- = \max_j(-c_j \vee 0)$, $M_n^r = r$th largest among $\{X_1, \cdots, X_n\}$, $M_n = M_n^1$.

(A) *Extremes.* We prove convergence of the sample extremal process to a limiting extremal process.

THEOREM 3.1. *Assume either $c_+ p > 0$ or $c_- q > 0$ and that* (1.1)–(1.3), (2.1), (2.6) *hold. Set*

$$Y_n(t) = \begin{cases} a_n^{-1} M_{[nt]} & \text{if } t \ge n^{-1} \\ a_n^{-1} X_1 & \text{if } 0 < t < n^{-1} \end{cases} \tag{3.1}$$

and suppose $(Y(t), t > 0)$ is an extremal process generated by the extreme value *distribution* $\exp(-(c_+^\alpha p + c_-^\alpha q)x^{-\alpha})$ *for* $x > 0$ (*cf. Dwass*, 1964, *Resnick* 1983, 1984). *Then* $Y_n \Rightarrow Y$ *in* $D(0, \infty)$.

PROOF. The functional T from $M_p((0, \infty) \times (\mathbb{R}\backslash\{0\})) \to D(0, \infty)$ defined by

$$(T \textstyle\sum_{k=1}^\infty \varepsilon_{(u_k, v_k)})(t) = \bigvee_{u_k \le t} v_k$$

is almost surely a continuous mapping (cf. Serfozo, 1982; Mori and Oodaira, 1976; Resnick, 1984). So by the continuous mapping theorem,

$$\begin{aligned} T(\textstyle\sum_{k=1}^\infty \varepsilon_{(k/n, a_n^{-1}X_k)}) &= Y_n(\cdot) = T(\textstyle\sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon_{(t_k, j_k c_i)}) \\ &= \bigvee_{t_k \le \cdot} (\bigvee_{i=0}^\infty j_k c_i) = \bigvee_{t_k \le \cdot} (c_+ j_k \vee (-c_- j_k)) =: Y(\cdot). \end{aligned}$$

Note that Y is an extremal process since $Y = T\xi$ where

$$\begin{aligned} \xi(\cdot) = &\textstyle\sum_{k=1}^\infty \varepsilon_{(t_k, c_+ j_k)}(\cdot \cap ((0, \infty) \times (0, \infty))) \\ &+ \textstyle\sum_{k=1}^\infty \varepsilon_{(t_k, -c_- j_k)}(\cdot \cap ((0, \infty) \times (0, \infty))); \end{aligned}$$

i.e. ξ is a PRM on $(0, \infty) \times (0, \infty)$ with mean measure of $(0, t) \times (x, \infty)$ equal to $t(c_+^\alpha p + c_-^\alpha q)x^{-\alpha}$ for $t > 0$, $x > 0$. The PRM ξ is obtained by taking points with negative ordinates and relecting about the horizontal axis up to the positive quadrant. For $t > 0$, $x > 0$

$$\begin{aligned} P[Y(t) \le x] &= P[\xi((0, t] \times (x, \infty)) = 0] \\ &= \exp\{-E\xi((0, t) \times (x, \infty))\} = \exp\{-t(c_+^\alpha p + c_-^\alpha q)x^{-\alpha}\} \end{aligned}$$

which completes the proof. □

Of course this method can be extended to get joint convergence of the k processes based on $(M_{[nt]}^i, i \le k)$. One merely needs to note that

$$[a_n^{-1} M_{[nt]}^r \le x] = [\textstyle\sum_{k=1}^\infty \varepsilon_{(k/n, X_k/a_n)}((0, t] \times (x, \infty)) \le r - 1].$$

The joint limiting distribution for any collection of upper extremes can thus be determined through the limiting point process. For example, letting

$$N(\cdot) = \textstyle\sum_{k=1}^\infty \sum_{i=0}^\infty \varepsilon_{(t_k, j_k c_i)}((0, 1] \times \cdot)$$

we have for $0 < y < x$,

$$P(a_n^{-1} M_n \le x, a_n^{-1} M_n^2 \le y) \to P(N((x, \infty)) = 0, N((y, x]) \le 1).$$

For convenience, suppose $c_- = 0$, define c_{+2} = second largest of $(c_j \vee 0)$ and for $y > 0$ set $G(y) = \exp\{-pc_+^\alpha y^{-\alpha}\}$. Then the above limit becomes

$$\begin{aligned} &P(\textstyle\sum_{k=1}^\infty \varepsilon_{j_k}(y/c_+, x/c_+ \wedge y/c_{+2}) \le 1, \sum_{k=1}^\infty \varepsilon_{j_k}((x/c_+ \wedge y/c_{+2}, \infty)) = 0) \\ &= G(x \wedge (c_+ y/c_{+2}))G(y)/G(x \wedge (c_+ y/c_{+2}))(1 - \log(G(y)/G(x \wedge (c_+ y/c_{+2})))) \\ &= G(y)(1 - \log(G(y)/G(x \wedge (c_+ y/c_{+2})))). \end{aligned}$$

By choosing $\rho(s) = 1 - s(1 \vee (s^{-1}(c_{+2}/c_+)^\alpha))$ the limit distribution of

$a_n^{-1}(M_n^1, M_n^2)$ may be rewritten as,

$$\begin{cases} G(x) & \text{if } x \le y \\ G(y)(1 - \rho(\log G(x)/\log G(y))\log G(y)) & \text{if } x > y \end{cases}$$

which is in agreement with the formula given on page 161 of Mori (1977) (cf. Welsch, 1972 and Mori, 1976).

(B) *Maxima and Minima.* It is just as easy to determine joint limiting behavior for any collection of upper and lower extremes using Theorem 2.4. We shall concentrate on the specific case of the maximum $M_n = \bigvee_{j=1}^n X_j$ and the minimum $W_n = \bigwedge_{j=1}^n X_j$.

THEOREM 3.2. *Suppose* (1.1)–(1.3), (2.1), (2.6) *all hold. Then we have*

$$P(a_n^{-1}M_n \le x,\ a_n^{-1}W_n \le y) \to G^p(x, \infty)G^q(\infty, x) - G^p(x, -y)G^q(-y, x)$$

where

$$G(x, y) = \begin{cases} \exp\{-c_+^\alpha x^{-\alpha}\} \wedge \exp\{-c_-^\alpha y^{-\alpha}\} & \textit{for } x > 0,\ y > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. We have for $x > 0$, $y < 0$

$$\begin{aligned} P(a_n^{-1}&M_n \le x,\ a_n^{-1}W_n > y) \\ &= P(\textstyle\sum_{k=1}^\infty \varepsilon_{(k/n, a_n^{-1}X_k)}((0, 1] \times ((-\infty, y) \cup (x, \infty))) = 0) \\ &\to P[\textstyle\sum_{k=1}^\infty \sum_{i=0}^\infty \varepsilon_{(t_k, j_k c_i)}((0, 1] \times ((-\infty, y) \cup (x, \infty))) = 0] \\ &= P[\textstyle\sum_{k=1}^\infty \varepsilon_{(t_k, j_k)}((0, 1] \times ((-\infty, -x/c_-) \cup (x/c_+, \infty) \cup (-\infty, y/c_+) \\ &\qquad\qquad \cup (-y/c_-, \infty))) = 0] \\ &= P[\textstyle\sum_{k=1}^\infty \varepsilon_{(t_k, j_k)}((0, 1] \times ((-\infty, (-x/c_-) \vee (y/c_+)) \\ &\qquad\qquad \cup ((x/c_+) \wedge (-y/c_-), \infty))) = 0] \\ &= \exp\{-[p(c_+^\alpha x^{-\alpha} \vee c_-^\alpha(-y)^{-\alpha}) + q(c_-^\alpha x^{-\alpha} \vee c_+^\alpha(-y)^{-\alpha})]\} \\ &= G^p(x, -y)G^q(-y, x). \end{aligned}$$

Thus,

$$P(a_n^{-1}M_n \le x,\ a_n^{-1}W_n \le y) = P(a_n^{-1}M_n \le x) - P(a_n^{-1}M_n \le x,\ a_n^{-1}W_n > y)$$

has the desired limit. □

Note that as specified by Theorem 4.1 in Davis (1982) the limit distribution of $a_n^{-1}(M_n, W_n)$ is of the form $H(x, \infty) - H(x, -y)$ where $H(x, y) = G^p(x, y)G^q(y, x)$ is a bivariate extreme value distribution. Moreover, it is easy to see that the maximum and minimum are asymptotically independent if and only if all the c_j's have the same sign. For further remarks on this point, see Davis (1983, 1984).

(C) *Inverses, overshoots and ranges.* We may trivially modify Theorem 2.4 to yield as $s \to \infty$.

$$\sum_{k=1}^{\infty} \varepsilon_{(k/s, X_k/a(s))} \Rightarrow \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)} \quad \text{in} \quad M_p((0, \infty) \times (\mathbb{R}\backslash\{0\})) .$$

where $a(s) = F^{\leftarrow}(1 - s^{-1})$ and $F^{\leftarrow}$ is the left continuous inverse of F. Let Y be the extremal process given in Theorem 3.1 and let $Y_s(t)$ be defined as in (3.1) with s replacing n. As in Resnick (1975) we have by an application of the continuous mapping theorem that $(Y_s, Y_s^{\leftarrow}) \Rightarrow (Y, Y^{\leftarrow})$ in $D(0, \infty) \times D(0, \infty)$ where for $x > 0$

$$Y_s^{\leftarrow}(x) = \inf(u: Y_s(u) > x) = \inf\{k: \vee_{i=1}^{k} X_i > a(s)x\}/s$$

and a similar definition holds for $Y^{\leftarrow}$. So setting $\tau(x) = \inf\{k: \vee_{i=1}^{k} X_i > x\}$ we have $\tau(a(s)\cdot)/s \Rightarrow Y^{\leftarrow}$ and changing variables we get

$$(1 - F(s))\tau(s\cdot) \Rightarrow Y^{\leftarrow}(\cdot) \tag{3.2}$$

as $s \to \infty$ in $D(0, \infty)$. Recall for $x > 0$, $t > 0$

$$P[Y(t) \leq x] = \exp(-t(pc_{+}^{\alpha} + qc_{-}^{\alpha})x^{-\alpha})$$

and therefore

$$P[Y^{\leftarrow}(x) \leq t] = 1 - \exp(-(pc_{+}^{\alpha} + qc_{-}^{\alpha})x^{-\alpha}t).$$

Now define $L(a(s), 1) = \inf\{k: X_k > a(s)\}$, $L(a(s), 2) = \inf\{k > L(a(s), 1): X_k > X_{L(a(s),1)}\}$ and so on. Then $\{X_{L(a(s),k)}/a(s), k \geq 1\}$ are those record values of $\{X_k/a(s)\}$ bigger than 1. As in Resnick (1975), Corollary 3 this sequence converges weakly in $\mathbb{R}^{\infty}$ to the range of Y above 1 which is a Poisson process with mean measure of $(a, b]$ equal to $\alpha \log b/a$ (Resnick, 1974, Theorem 2). In particular for $x > 0$

$$\lim_{s\to\infty} P[(X_{L(a(s),1)}/a(s)) - 1 \leq x] = 1 - (1 + x)^{-\alpha}. \tag{3.3}$$

(As before, we may change variables $t = a(s)$ to get the limit distribution for the overshoot past t.)

Consider jointly $(\{X_{L(a(s),k)}/a(s), k \geq 1\}, Y_s^{\leftarrow}(1))$ on $\mathbb{R}^{\infty} \times \mathbb{R}$. By the continuous mapping theorem this converges as $s \to \infty$ weakly to

$$(\{\text{points hit by } Y \text{ above } 1\}; Y^{\leftarrow}(1)) = (\{\text{times of jumps of } Y^{\leftarrow}(x), x > 1\}; Y^{\leftarrow}(1)).$$

Since $Y^{\leftarrow}$ has independent increments (Resnick, 1974; Shorrock, 1974; Dwass, 1974)

$$\{\text{times of jumps of } Y^{\leftarrow}(x), x > 1\} = \{\text{times of jumps of } Y^{\leftarrow}(x) - Y^{\leftarrow}(1), x > 1\}$$

is independent of $Y^{\leftarrow}(1)$. So for instance if we combine (3.2) and (3.3) jointly we get as $s \to \infty$

$$P[(1 - F(s))\tau(s) \leq x, (X_{L(s,1)} - s)/s > y] \to P[Y^{\leftarrow}(1) \leq x](1 + y)^{-\alpha}$$

for $x > 0$, $y > 0$. (Cf. Finster, 1982, Section 4.)

(D) *Excedences.* Rootzen (1978) and Leadbetter, Lindgren and Rootzen (1983) consider the indices when an observation X_k/a_n exceeds a given level $x > 0$. Suppose as a convenience for this subsection that $|c_j| \le 1$ for all j. The point process of points with ordinates bigger than $x > 0$ converges as $n \to \infty$; that is

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, X_k/a_n)}(\cdot \cap ((0, \infty) \times (x, \infty))) \Rightarrow \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)}(\cdot \cap ((0, \infty) \times (x, \infty)))$$

in $M_p((0, \infty) \times (x, \infty))$ from Theorem 2.4 and the fact that the map $m \to m(\cdot \cap ((0, \infty) \times (x, \infty)))$ from $M_p((0, \infty) \times (\mathbb{R}\backslash\{0\}))$ to $M_p((0, \infty) \times (x, \infty))$ is a.s. continuous. To evaluate the structure of the limit consider the following: Let $\{\Gamma_n, n \ge 1\}$ be the points of a homogeneous PRM on $(0, \infty)$ with rate $x^{-\alpha}$. Suppose $\{J_k, k \ge 1\}$ are iid on $(x, \infty) \cup (-\infty, -x)$ independent of $\{\Gamma_n\}$ and with common density

$$f(y) = (p\alpha y^{-\alpha-1} 1_{(x,\infty)}(y) + q\alpha(-y)^{-\alpha-1} 1_{(-\infty,-x)}(y))x^{\alpha}.$$

Then

$$\sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k)}(\cdot \cap (0, \infty) \times ((x, \infty) \cup (-\infty, -x))) =_d \sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k, J_k)}$$

on $M_p((0, \infty) \times ((x, \infty) \cup (-\infty, -x)))$. (This can be checked readily using Laplace functionals or from Cinlar, 1976.) Therefore the weak limit of the above point process is

$$\sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, c_i j_k)}(\cdot \cap ((0, \infty) \times (x, \infty)))$$
$$=_d \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k, c_i J_k)}(\cdot \cap ((0, \infty) \times (x, \infty))).$$

Finally define $\xi_k = \#\{c_i : c_i J_k > x\}$ so that $\{\xi_k, k \ge 1\}$ is iid. In the limit the point process of times of excedences is the compound Poisson point process $\sum_{k=1}^{\infty} \xi_k \varepsilon_{\Gamma_k}$ where $\{\xi_k\}$ and $\{\Gamma_k\}$ are independent (cf. Rootzen, 1978, page 858.)

4. Sums and sample covariances. In this section we determine the weak limiting behavior of the partial sums of the type decribed in Section 1. As before assume $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ where $\{Z_j\}$ satisfies (1.1) and (1.2) with $0 < \alpha < 2$. Further assume $\{c_j\}$ satisfies (2.6) which necessarily implies $\sum |c_j| < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$ and $b_n = EZ_1 1_{[|Z_1| \le a_n]}$ so that (cf. Feller, 1971)

$$a_n^{-1} \left(\sum_{i=1}^{n} (Z_i - b_n)\right) \Rightarrow S \tag{4.1}$$

in $\mathbb{R}$ where S is a stable random variable with index α.

THEOREM 4.1. *Suppose* (1.1) *and* (1.2) *are valid for* $0 < \alpha < 2$. *Also assume* (1.3), (2.1) *and* (2.6) *hold. Then*

$$a_n^{-1}\left(S_n - n \sum_{j=0}^{\infty} c_j b_n\right) \Rightarrow \left(\sum_{j=0}^{\infty} c_j\right) S$$

in $\mathbb{R}$, *where* S *has a stable distribution with index* α. (*If* $\sum_{j=0}^{\infty} c_j = 0$, *then* $a_n^{-1} S_n \to 0$ *in probability.*)

PROOF. We shall first establish the result for the partial sums of the truncated sequence $X_t^{(m)} = \sum_{j=0}^{m} c_j Z_{t-j}$ and then use an approximation argument to get the full result. Consider the sequence of $(m + 1)$-dimensional vectors

$$\mathbf{Y}_n = a_n^{-1}(\textstyle\sum_{t=1}^n (Z_t - b_n), \sum_{t=1}^n (Z_{t-1} - b_n), \cdots, \sum_{t=1}^n (Z_{t-m} - b_n)).$$

Relation (4.1) holds and furthermore for $1 \le k \le m$ and any $\eta > 0$

$$P(| a_n^{-1} \textstyle\sum_{t=1}^n (Z_t - b_n) - a_n^{-1} \sum_{t=1}^n (Z_{t-k} - b_n) | > \eta)$$
$$= P(a_n^{-1} | \textstyle\sum_{t=1-k}^0 Z_t + \sum_{t=n-k+1}^n Z_t | > \eta) = P(a_n^{-1} | \sum_{t=1-k}^k Z_t | > \eta) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, in $\mathbb{R}^{m+1}$, $\mathbf{Y}_n \Rightarrow (S, S, \cdots, S)$. By the continuous mapping theorem,

$$(c_0, c_1, \cdots, c_m) \cdot \mathbf{Y}_n = a_n^{-1}(\textstyle\sum_{j=1}^n X_j^{(m)} - (\sum_{j=0}^m c_j)nb_n) \Rightarrow (\sum_{j=0}^m c_j)S.$$

Now to prove the result in the general case it suffices to show by Theorem 4.2 in Billingsley (1968) that

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(a_n^{-1} | (S_n - \textstyle\sum_{j=0}^\infty c_j nb_n) - (\sum_{j=1}^n X_j^{(m)} - \sum_{j=0}^m c_j nb_n) | > \gamma) = 0 \tag{4.2}$$

and

$$(\textstyle\sum_{j=0}^m c_j)S \to (\sum_{j=0}^\infty c_j)S \quad \text{in probability as} \quad m \to \infty. \tag{4.3}$$

Clearly (4.3) is automatic (in fact we have a.s. convergence) since $\sum | c_j | < \infty$. As for (4.2) first note that

$$a_n^{-1}(\textstyle\sum_{t=1}^n X_t - (\sum_{j=0}^\infty c_j)nb_n) - a_n^{-1}(\sum_{t=1}^n X_t^{(m)} - \sum_{j=0}^m c_j nb_n)$$
$$= a_n^{-1} (\textstyle\sum_{t=1}^n \sum_{j>m} c_j Z_{t-j} - \sum_{j>m} c_j nb_n)$$

which may be written as

$$= a_n^{-1} \textstyle\sum_{k=-n+m+1}^\infty s_{k,n}(Z_{-k}1_{\{|Z_{-k}|\le a_n\}} - b_n) + a_n^{-1} \sum_{t=1}^n \sum_{j>m} c_j Z_{t-j}1_{\{|Z_{t-j}|>a_n\}}$$

where

$$s_{k,n} = \begin{cases} c_{m+1} + \cdots + c_{n+k} & \text{if} \quad -n + m + 1 \le k \le m \\ c_{k+1} + \cdots + c_{n+k} & \text{if} \quad k > m. \end{cases}$$

So the probability in (4.2) is bounded above by

$$P(a_n^{-1} \textstyle\sum_{k=-n+m+1}^\infty | s_{k,n} | | Z_{-k}1_{\{|Z_{-k}|\le a_n\}} - b_n | > \gamma/2) + P(a_n^{-1} \sum_{t=1}^n \sum_{j>m} | c_j | | Z_{t-j} | 1_{\{|Z_{t-j}|>a_n\}} > \gamma/2). \tag{4.4}$$

Observe that

$$(1/n) \textstyle\sum_{k=-n+m+1}^m s_{k,n}^2 = (1/n) \sum_{j=1}^n (c_{m+1} + \cdots + c_{j+m})^2 \to (\sum_{j=m+1}^\infty c_j)^2 \quad \text{as} \quad n \to \infty$$

and

$$(1/n) \sum_{k=m+1}^{\infty} s_{k,n}^2$$

$$= (1/n) \sum_{k=m+1}^{\infty} (c_{k+1} + \cdots + c_{k+n})^2 \le (1/n) \sum_{k=m+1}^{\infty} (|c_{k+1}| + \cdots + |c_{k+m}|)^2$$

$$\le (1/n)(\sum_{j=1}^{\infty} |c_j|) \sum_{k=m+1}^{\infty} (|c_{k+1}| + \cdots + |c_{k+n}|) \le (\sum_{j=0}^{\infty} |c_j|)(\sum_{k=m+1}^{\infty} |c_k|)$$

from which we conclude that $\lim_{m\to\infty}\limsup_{n\to\infty}(1/n) \sum_{k=-n+m+1}^{\infty} s_{k,n}^2 = 0$. Now using Chebyshev's inequality, the first term in (4.3) is bounded by

$$\frac{4}{\gamma^2}\left[\frac{1}{n} \sum_{k=-n+m+1}^{\infty} s_{k,n}^2 \cdot \frac{n}{a_n^2} \operatorname{Var}(Z_1 1_{\{|Z_1|\le a_n\}})\right]$$

$$\le \frac{4}{\gamma^2}\left[\frac{1}{n} \sum_{k=-n+m+1}^{\infty} s_{k,n}^2\right]\left[\frac{n}{a_n^2} EZ_1^2 1_{[Z_1 \le a_n]}\right].$$

By Karamata's Theorem (cf. page 283, Feller, 1971) $\limsup_{n\to\infty} na_n^{-2}EZ_1^2 1_{[|Z_1|\le a_n]} < \infty$ so that the $\lim_{m\to\infty}\limsup_{n\to\infty}$ of the first piece in (4.4) is zero.

For $\alpha > 1$, the second term in (4.4) is bounded by

$$\frac{2n}{\gamma} E(a_n^{-1} \sum_{j>m} |c_j| |Z_j| 1_{[|Z_j|>a_n]}) = \frac{2}{\gamma} \sum_{j>m} |c_j| \frac{n}{a_n} E|Z_1| 1_{[|Z_1|>a_n]}$$

$$\to \frac{2}{\gamma} \sum_{j>m} |c_j| \frac{\alpha}{\alpha - 1}$$

as $n \to \infty$ by Karamata's Theorem. Therefore, $\lim_{m\to\infty}\limsup_{n\to\infty}$ of this piece is also zero when $\alpha > 1$. For the case $0 < \alpha \le 1$, let $\delta < \alpha$ be as specified in (2.6) and note that

$$(a_n^{-1} \sum_{t=1}^{n} \sum_{j>m} |c_j| |Z_{t-j}| 1_{[|Z_{t-j}|>a_n]})^{\delta} \le a_n^{-\delta} \sum_{t=1}^{n} \sum_{j>m} |c_j|^{\delta} |Z_{t-j}|^{\delta} 1_{[|Z_{t-j}|^{\delta}>a_n^{\delta}]}.$$

Since $|Z_t|^{\delta}$ has index $\alpha/\delta > 1$, we can apply the above argument to get

$$\lim_{m\to\infty}\limsup_{n\to\infty} P[a_n^{-1} \sum_{t=1}^{n} \sum_{j>m} |c_j| |Z_{t-j}| 1_{[|Z_{t-j}|>a_n]} > \gamma/2]$$

$$\le \lim_{m\to\infty}\limsup_{n\to\infty} P[a_n^{-\delta} \sum_{t=1}^{n} \sum_{j>m} |c_j|^{\delta} |Z_{t-j}|^{\delta} 1_{[|Z_{t-j}|^{\delta}>a_n^{\delta}]} > (\gamma/2)^{\delta}] = 0$$

which establishes Theorem 4.1 as claimed. □

We now consider the asymptotic properties of the sample covariance function and correlation functions which will follow quite easily from Theorem 2.4. For h a nonnegative integer let

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=h+1}^{n} X_t X_{t-h}, \quad \hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \frac{\sum_{t=h+1}^{n} X_t X_{t-h}}{\sum_{t=1}^{n} X_t^2}$$

denote, respectively, the sample covariance and sample correlation function at lag h. If $\mathrm{Var}(Z_t) = \sigma^2 < \infty$, then

$$\gamma(h) = \mathrm{cov}(X_t, X_{t+h}) = (\textstyle\sum_{j=0}^{\infty} c_j c_{j+h})\sigma^2 \quad \text{and} \quad \rho(h) = \dfrac{\sum_{j=0}^{\infty} c_j c_{j+h}}{\sum_{j=0}^{\infty} c_j^2}$$

which does not depend on the nuisance parameter σ^2. In the following theorem, we show that $na_n^{-2}\,\hat{\gamma}(h)$ has a stable limit and $\hat{\rho}(h) \to \rho(h)$ in probability.

THEOREM 4.2. *Let* $\sum_{k=1}^{\infty} \varepsilon_{j_k}$ *be a PRM*(λ) *on* $\mathbb{R}\backslash\{0\}$ *with*

$$\lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)dx, \quad 0 < \alpha < 2.$$

Suppose (1.1)–(1.3), (2.1), (2.6) *hold with* $0 < \alpha < 2$. *Then for every nonnegative integer* ℓ, *as* $n \to \infty$:

(i)
$$(n/a_n^2)(\hat{\gamma}(0), \hat{\gamma}(1), \cdots, \hat{\gamma}(\ell)) \Rightarrow \textstyle\sum_{i=1}^{\infty} j_i^2 \cdot (\sum_{j=0}^{\infty} c_j^2, \sum_{j=0}^{\infty} c_j c_{j+1}, \cdots, \sum_{j=0}^{\infty} c_j c_{j+\ell})$$

and

(ii)
$$\hat{\rho}(\ell) \to \rho(\ell) = \frac{\sum_{j=0}^{\infty} c_j c_{j+\ell}}{\sum_{j=0}^{\infty} c_j^2} \quad \textit{in probability.}$$

REMARK. $\sum_{k=1}^{\infty} j_k^2$ is a stable random variable of index $\alpha/2$.

PROOF. (i) By restricting attention in Theorem 2.4 to points in $(0, 1] \times \mathbb{R}^{\ell+1}\backslash\{(0, \cdots, 0)\}$, we have

$$\textstyle\sum_{k=1}^{n} \varepsilon_{a_n^{-1}(X_k, X_{k-1}, \cdots, X_{k-\ell})} \Rightarrow \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \varepsilon_{j_k(c_i, c_{i-1}, \cdots, c_{i-\ell})}. \tag{4.5}$$

Let $\gamma > 0$, and for each integer h, $0 \le h \le \ell$ define the mapping $\phi_{h,\gamma}$ from $M_p(\mathbb{R}^{\ell+1}\backslash\{(0, \cdots, 0)\}$ into $\mathbb{R}$ by

$$\phi_{h,\gamma}(\textstyle\sum_{k=1}^{\infty} \varepsilon_{(u_{k0}, u_{k1}, \cdots, u_{k\ell})}) = \sum_{k=1}^{\infty} u_{k0} u_{kh} 1_{[|u_{k0}| > \gamma \text{ or } |u_{kh}| > \gamma]}.$$

Using an argument similar to that given in Section 3 of Resnick (1984) it may be shown that $\phi_{h,\gamma}$ is a.s. continuous relative to the limit point process in (4.5). Thus by the continuous mapping theorem, the $\ell + 1$ vector of random variables with components

$$\phi_{h,\gamma}(\textstyle\sum_{k=1}^{n} \varepsilon_{a_n^{-1}(X_k, X_{k-1}, \cdots, X_{k-\ell})}) = a_n^{-2} \sum_{k=1}^{n} X_k X_{k-h} 1_{\{|X_k| > a_n\gamma \text{ or } |X_{k-h}| > a_n\gamma\}},$$

$h = 0, \cdots, \ell$, converges in distribution in $\mathbb{R}^{\ell+1}$ to the random vector whose corresponding components are

$$\phi_{h,\gamma}(\textstyle\sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \varepsilon_{j_k(c_i, c_{i-1}, \cdots, c_{i-\ell})}) = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} j_k^2 c_i c_{i-h} 1_{[|j_k| > \gamma(c_i^{-1} \wedge c_{i-h}^{-1})]}. \tag{4.6}$$

It is easy to check that $(j_k^2;\ k \ge 1)$ are the points of a PRM($\tilde{\lambda}$) on $(0, \infty)$ with $\tilde{\lambda}(dx) = x^{-\alpha/2-1}(\alpha/2)dx$. An alternative representation for $\sum_{k=1}^{\infty} \varepsilon_{j_k^2}$ is $\sum_{k=1}^{\infty} \varepsilon_{\Gamma_k^{-2/\alpha}}$ where $\Gamma_k = E_1 + \cdots + E_k$ is the sum of k iid unit exponentials. By the strong law of large numbers, $\Gamma_k^{-2/\alpha} \sim k^{-2/\alpha}$ a.s. $k \to \infty$, which ensures that $\sum_{k=1}^{\infty} j_k^2 = \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} < \infty$. Consequently, the limit random variable in (4.5) approaches $\sum_{k=1}^{\infty} j_k^2 \sum_{i=0}^{\infty} c_i c_{i-h}$ as $\gamma \to 0$.

Next we show that for any $\eta > 0$,

(4.7) $\lim_{\gamma\to 0}\limsup_{n\to\infty} P(a_n^{-2}\,|\sum_{k=1}^n X_k X_{k-h} 1_{[|X_k|\le a_n\gamma, |X_{k-h}|\le a_n\gamma]}| > \eta) = 0.$

This probability can be bounded above by

$$\frac{n}{a_n^2\gamma} E(|X_k|\,1_{[|X_k|\le\gamma a_n]}\,|X_{k-h}|\,1_{[|X_{k-h}|\le\gamma a_n]})$$

which by Cauchy–Schwartz has an upper bound $(n/a_n^2\gamma)EX_1^2 1_{[|X_1|\le a_n\gamma]}$. According to (2.7), the distribution of X_k has regularly varying tails with index $0 < \alpha < 2$ and so

$$\lim_{\gamma\to 0}\limsup_{n\to\infty}(n/a_n^2)EX_1^2 1_{[|X_1|\le a_n\gamma]} = 0$$

by Karamata's Theorem (Feller, 1971, page 283) which verifies (4.7). Invoking Theorem 4.2 in Billingsley (1968), we have

$$a_n^{-2}(\textstyle\sum_{t=1}^n X_t^2,\ \sum_{t=1}^n X_tX_{t-1},\ \cdots,\ \sum_{t=1}^n X_tX_{t-\ell})$$
$$\Rightarrow \textstyle\sum_{k=1}^\infty j_k^2(\sum c_i^2,\ \sum c_ic_{i+1},\ \cdots,\ \sum c_ic_{i+\ell}).$$

Finally it is easy to see that the limit is unaltered if we commence the summing of $\sum_{t=1}^n X_tX_{t-h}$ at $t = h + 1$, for $h = 0, \cdots, \ell$ and since (ii) follows trivially from (i) we are done. □

COROLLARY. *The same limit laws are attained in Theorem* 4.2 *if* $\hat\gamma(h)$ *and* $\hat\rho(h)$ *are replaced by their mean corrected versions,*

$$\tilde\gamma(h) = (1/n)\textstyle\sum_{t=h+1}^n (X_t - \bar X)(X_{t-h} - \bar X) \quad \text{and} \quad \tilde\rho(h) = \tilde\gamma(h)/\tilde\gamma(0),$$

respectively, where $\bar X = (1/n)\sum_{i=1}^n X_i$.

PROOF. It suffices to show $R_n := na_n^{-2}(\hat\gamma(h) - \tilde\gamma(h)) \to_p 0$. R_n also has the representation

(4.8) $$R_n = a_n^{-2}((n-h)\bar X^2 - \bar X\textstyle\sum_{t=h+1}^n X_t - \bar X\sum_{t=h+1}^n X_{t-h}).$$

For the first term observe

$$\frac{\sqrt{n}\bar X}{a_n} = \frac{\sum_{i=1}^n X_i}{\sqrt{n}a_n} = \frac{1}{\sqrt{n}}\left(\frac{\sum_{i=1}^n X_i - nb_n\sum_{i=0}^\infty c_i}{a_n}\right) + \frac{nb_n\sum_{i=0}^\infty c_i}{\sqrt{n}a_n}$$

and from Theorem 4.1 the first term converges to zero in probability. If $\alpha < 1$ no centering is necessary (Feller, 1971) by Karamata's Theorem and $nb_n/a_n \to \alpha(1-\alpha)$. If $1 < \alpha < 2$ then $b_n \to EZ_1 < \infty$ and since a_n is regularly varying with index $1/\alpha \in (½, 1)$ we have $\sqrt{n}/a_n \to 0$. If $\alpha = 1$ then

$$|\sqrt{n}b_n/a_n| \le \sqrt{n}a_n^{-1}E|Z_1|1_{[|Z_1|\le a_n]}.$$

Since $E|Z_1|1_{[|Z_1|\le a_n]}$ is slowly varying (Feller, 1971, page 315 and Karamata's Theorem) and a_n is regularly varying with index 1 we have

$$\sqrt{n}b_n/a_n \to 0.$$

In all three cases $\sqrt{n} b_n a_n^{-1} \sum_{i=0}^{\infty} c_i \to 0$. The remaining terms of (4.8) converge in probability to zero by similar arguments.

REMARKS.

(1) The weak consistency of the estimate $\hat{\rho}(h)$ was established by Kanter and Steiger (1974) for AR(p) processes under stronger assumptions on the errors $\{Z_n\}$.

(2) If $\{X_t\}$ is the AR(p) process

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

where $\phi(z) = 1 - \phi_1 z \cdots - \phi_p z^p \neq 0$ for $|z| \leq 1$,then using the above theorem it can be shown that the least squares estimates of $(\phi_1, \cdots, \phi_p)$ are weakly consistent. This result was also obtained by Kanter and Steiger while Hannan and Kanter (1977) considered strong consistency properties of the least squares estimator.

(3) From the above theorem, we also have weak consistency of the estimates of the parameters of a causal invertible ARMA(p, q) process based on the method of moments.

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DEPARTMENT OF STATISTICS
COLORADO STATE UNIVERSITY
FORT COLLINS, COLORADO 80523