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SELF-SIMILAR PROCESSES WITH STATIONARY INCREMENTS GENERATED BY POINT PROCESSES[1]

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A real-valued process $X = (X(t))_{t\in\mathbb{R}}$ is self-similar with exponent H(H-ss), if $X(a\cdot) =_d a^H X$ for all $a > 0$. The present paper studies H-ss processes X_H with stationary increments that can be represented for $t > 0$ by $X_H(t) := \int |x|^H \operatorname{sgn} x \, \Pi((0, t], dx) =: \int x \, \Pi^H((0, t], dx)$, where Π is a point process in $\mathbb{R}^2$ that is Poincaré, i.e., invariant in distribution under the transformations $(t, x) \mapsto (at + b, ax)$ of $\mathbb{R}^2$. In particular, X_H allows such a representation if it is a jump process, Π^H being the graph of its jumps. Several examples of Poincaré processes Π are presented. These lead in many cases to new examples of H-ss processes X_H with stationary increments. Furthermore, it is investigated for which H the integral expression for X_H converges, conditionally or absolutely. If Π has finite intensity $\mathbb{E}\Pi$, then $(1, \infty)$ is wp1 the set of H for which X_H converges absolutely. If $\mathbb{E}\Pi$ is not finite, then the situation is more complicated, as is the case for conditional convergence. Several examples illustrate this. In the final section the integrator Π in the expression for X_H is replaced by $\Pi - \mathbb{E}\Pi$, which gives conditional convergence for more H in $(0, 1)$.

1. Introduction and basic results. In the present paper, stochastic processes are random $\mathbb{R}$-valued functions $X = (X(t))_{t\in\mathbb{R}}$ on $\mathbb{R}$. Two stochastic processes X and Y have the same distribution or "are versions of each other" (notation $X =_d Y$) if they have the same finite-dimensional distributions. A process X is self-similar with exponent $H > 0$ (H-ss) if

$$X(a\cdot) =_d a^H X(\cdot) \quad \text{for all} \quad a > 0, \tag{1.1}$$

and has stationary increments (is si) if

$$X(\cdot + b) - X(b) =_d X(\cdot) - X(0) \quad \text{for all} \quad b \in \mathbb{R}. \tag{1.2}$$

The reader is referred to Vervaat (1985) for a more elaborate introduction to H-ss si processes. In particular it is shown there that only trivial processes are admitted by extension of the definition to all $H \in \mathbb{R}$, that (in the present case $H > 0$) $X(0) = 0$ with probability one (wp1) and that X is continuous in probability. Thus we can and do always take a separable, measurable version for X.

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We wish to investigate the special class of H-ss si processes for which $X(b) - X(a)$ can be represented as the sum of the heights of all points above $(a, b]$ of a point process in the plane. This is a natural approach in case X is a jump process, but the above class is actually somewhat wider than that.

In order to clarify our approach, let us first consider any *jump process* X, i.e. a right-continuous process whose left-hand limits exist everywhere, such that

$$X(b) - X(a) = \sum_{a<u\le b} (X(u) - X(u-)) \tag{1.3}$$

for all $a, b \in \mathbb{R}$, $a < b$. Note that the series on the right-hand side of (1.3) has only countably many nonzero terms. To simplify things further, assume for now that the series converges absolutely wp1 for all $a, b \in \mathbb{R}$, $a < b$, so that we need not worry about the order of summation. If $X(0) = 0$ wp1, which is the case if X is H-ss si, then (1.3) can be rewritten as

$$X(t) = (\operatorname{sgn} t) \sum_{u\in I_t} (X(u) - X(u-)), \tag{1.4}$$

where

$$I_t = \begin{cases} (0, t] & \text{if } t > 0, \\ (t, 0] & \text{if } t < 0, \\ \varnothing & \text{if } t = 0. \end{cases}$$

We are going to describe the sample paths of X in the same way as has been a tradition since Itô (1942) for processes with stationary independent increments without normal component. Set

$$\Pi := \{(t, x) \in \mathbb{R} \times (\mathbb{R}\setminus\{0\}): x = X(t) - X(t-)\}.$$

Then $\Pi = \Pi_X$ is a random countable subset of $E := \mathbb{R} \times (\mathbb{R}\setminus\{0\})$. We call Π the *saltus process* of X. With some abuse of notation, we use the symbol Π and the term "point process" to refer both to the random set Π and to the integer-valued measure counting the points of Π. So $\Pi(B) = \#(\Pi \cap B)$ for Borel sets B in E and two ways of rewriting (1.4) are

$$\begin{aligned} X(t) &= (\operatorname{sgn} t) \sum_{(u,x)\in\Pi\cap(I_t\times\mathbb{R})} x \\ &= (\operatorname{sgn} t) \int_{I_t}\int_{\mathbb{R}} x\, \Pi(du, dx). \end{aligned} \tag{1.5}$$

We fit Π into the standard set-up for point processes (cf. Kallenberg, 1976) by noting that Π is Radon (= finite on compact sets) on the Borel field on $\mathbb{R} \times (\overline{\mathbb{R}}\setminus\{0\})$, where $\overline{\mathbb{R}} := [-\infty, \infty]$.

We next state how properties (1.1) and (1.2) of X translate into properties of Π. The proof is straightforward.

LEMMA 1.1. (*a*) *X is si iff* $\Pi =_d \Pi + (b, 0)$ *for all* $b \in \mathbb{R}$.
(*b*) *X is H-ss iff*

$$\Pi =_d \{(at, a^H x):(t, x) \in \Pi\} \quad \textit{for all real} \quad a > 0.$$

In other words, X is si iff Π is invariant in distribution under the horizontal translations

$$(t, x) \mapsto (t + b, x) \quad \text{for all} \quad b \in \mathbb{R}, \tag{1.6}$$

and X is H-ss iff Π is invariant in distribution under

$$(t, x) \mapsto (at, a^H x) \quad \text{for all} \quad a \in (0, \infty). \tag{1.7}$$

For real $x \neq 0$ and α we define $x^{\uparrow\alpha}$ by

$$x^{\uparrow\alpha} := |x|^{\alpha} \operatorname{sgn} x = x\, |x|^{\alpha-1}. \tag{1.8}$$

If Π is a point process in E, then we set

$$\Pi^{\alpha} := \{(t, x^{\uparrow\alpha}) \colon (t, x) \in \Pi\}.$$

If Π is invariant under the transformations in (1.7) then Π^{α} is the saltus process of an αH-ss process, provided that the right-hand sides of (1.5) converge absolutely wp1 with Π^{α} instead of Π. In particular, it follows that all saltus processes satisfying (1.6) and (1.7) can be written in the form Π^{H}, where Π is a point process in E which is invariant in distribution under the transformations

$$(t, x) \mapsto (at + b, ax) \quad (a, b \in \mathbb{R}, a > 0) \tag{1.9}$$

of E. We have obtained (1.9) by combining (1.6) and (1.7) with $H = 1$. The transformations (1.9) arise in connection with Poincaré's geometry on the half-plane (cf. Lehner, 1964 pages 78–82). Thus, we have the following definition which will henceforth be used as our starting point.

DEFINITION 1.2. A point process Π in E is called Poincaré if Π is invariant in distribution under (1.9) and Π is locally finite in $\mathbb{R} \times (\overline{\mathbb{R}}\setminus\{0\})$ wp1.

For the most part, the point processes considered in this paper allow only points with multiplicity 1. All definitions and results can easily be extended with minor modifications to the case of higher finite multiplicities. This extension is necessary for certain particular cases of the construction in Section 3.5(d).

We will investigate those H-ss si processes which can be expressed in the form

$$X_H(t) = (\operatorname{sgn} t) \int_{I_t} \int_{\mathbb{R}\setminus\{0\}} x\Pi^H(du, dx) = (\operatorname{sgn} t) \int_{I_t} \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}\Pi(du, dx), \tag{1.10}$$

where Π is a Poincaré point process. We recall that the integral in (1.10) can be expressed as a sum, as in (1.5).

The local finiteness restriction in Definition 1.2 is included because without it the terms of the sum in (1.10) cannot converge to zero for any H so that (1.10) diverges. There may however be infinitely many points in $I_t \times \mathbb{R}$. Apart from a trivial case, this is even always so, as the next lemma tells us.

LEMMA 1.3. *If Π is a Poincaré point process, then wp1 either $\Pi = \varnothing$ or $\{t\colon (t, x) \in \Pi$ for some $x \in \mathbb{R}\}$ is dense in $\mathbb{R}$.*

PROOF. We adapt an argument used in the proof of Theorem 2.2 of Vervaat

(1985). For some interval $(c, d]$ with $c < d$, let $p := \mathbb{P}[\Pi((c, d] \times \mathbb{R}) = 0]$. By the invariance of Π under (1.9) with $b = -a(c + d)/2$, we have

$$\mathbb{P}[\Pi(\mathbb{R} \times \mathbb{R}) = 0] = \lim_{a\uparrow\infty}\mathbb{P}[\Pi((a(c - d)/2, a(d - c)/2] \times \mathbb{R}) = 0] = p,$$

which implies the result. □

We will at various times make statements of the form "wp1 either A or B", as in the conclusion of Lemma 1.3. They mean that $\mathbb{P}(A \cup B) = 1$ but do not exclude the possibility that $\mathbb{P}(A) < 1$ and $\mathbb{P}(B) < 1$. The case $\Pi = \varnothing$ in the lemma corresponds to $X_H \equiv 0$ in (1.10). This trivial situation will be avoided or excluded in most results.

We note that the transformations (1.9) leaves the upper and lower half-planes of E invariant. Thus the restrictions of Π to the two half-planes may exhibit completely different forms.

If (1.10) converges absolutely for all t wp1 (which will be seen to be true if there is convergence wp1 for any particular $t \neq 0$), then X_H is obviously a jump process whose paths have locally bounded variation (lbv). We have observed that *every* H-ss si jump process X with absolutely convergent sums of jumps can be expressed in the form (1.10), where Π satisfies the requirement that Π^H is the saltus process for X. In this case each vertical line in E contains at most one point of Π, but we do not impose this condition in general.

We also wish to consider the situation where (1.10) converges conditionally wp1. The order of summation is as follows: points (u, x) of $\Pi \cap (I_t \times \mathbb{R})$ are taken in order of decreasing absolute heights and those points with the same absolute heights are lumped together as a single term. Thus (1.10) is understood to mean

$$\begin{aligned} X_H(t) &= (\operatorname{sgn} t)\lim_{\varepsilon\downarrow 0} \int_{I_t}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} x^{\uparrow H}\Pi(dx, dx) \\ &= (\operatorname{sgn} t)\lim_{\varepsilon\downarrow 0} \textstyle\sum_{(u,x)\in\Pi\cap(I_t\times\mathbb{R}\setminus(-\varepsilon,\varepsilon))} x^{\uparrow H}. \end{aligned} \tag{1.11}$$

If (1.11) converges wp1 for each separate t, then (1.11) determines a consistent system of finite-dimensional distributions of X_H which satisfy (1.1) and (1.2). If in addition X_H has a version whose sample paths have lbv, then X_H can also be obtained from (1.10) via its saltus process, so we are mainly interested in the case when the paths of X_H do not have lbv.

In Section 2, we investigate for which H there is absolute convergence in (1.10). If Π has finite intensity, this occurs iff $H > 1$. Examples of Poincaré point processes in E are presented and reviewed in Section 3. In Section 4, the convergence of (1.11) is investigated. In most cases there is an $H_c \geq 0$ such that (1.11) converges wp1 if $H > H_c$ and diverges wp1 if $H < H_c$. Section 5 studies a special tractable case of conditional convergence, provided by symmetrizations of Poincaré point processes. Section 6 contains a class of examples which show that each $H_c \in [0, 1]$ may occur.

At this point it is good to notice that the processes X_H in (1.10) and (1.11) have stationary *independent* increments if Π is Poisson with Poincaré intensity, so X_H is strictly stable (cf. Section 3). Recall that a process X is stable if X has stationary independent nondegenerate increments and there exist reals $c(a)$ and $d(a)$ with $d(a) > 0$, for which

$$X(a.) =_d d(a)X + c(a) \quad (a > 0). \tag{1.12}$$

It is known that $d(a) = a^H$ for some $H \in [\frac{1}{2}, \infty)$ (cf. Feller, 1971, page 170) and that all these H indeed occur. We call X strictly stable if $c(a) \equiv 0$ (so that X is H-ss). Note that H is the reciprocal of the stability exponent, as it is usually defined in the literature. In (1.10) with absolute convergence we obtain all strictly stable processes with $H > 1$. In (1.11) we obtain in addition all symmetric stable processes with $\frac{1}{2} < H \leq 1$. One obtains all stable processes with $H > \frac{1}{2}$, if Π in (1.11) is replaced by $\Pi - \mathbb{E}\Pi$ for small values of x and a deterministic drift is introduced. (The case $H = \frac{1}{2}$, corresponding to Brownian motion, cannot be obtained this way, since Brownian motion has continuous sample paths.) A similar trick is applied to the general case with Π not necessarily Poisson in Section 7. In this way we obtain a wider class of si processes satisfying the same self-similarity relations as stable processes. Some but not all of them are H-ss.

An important and major novel point in the present paper is that many other non-Poissonian Π are possible in (1.10) and (1.11). There are even cases with almost deterministic dependence between the jumps of X_H. Examples are given in Sections 3 and 6. This aspect of the paper strongly mimics some earlier results for ss stationary extremal processes in O'Brien, Torfs and Vervaat (1984+).

2. Absolute convergence of jumps.

Intensity measure. Let Π be a Poincaré point process in E (cf. Definition 1.2). Its intensity or intensity measure is the (deterministic) measure $\mathbb{E}\Pi$, defined by $(\mathbb{E}\Pi)(A) := \mathbb{E}(\Pi(A))$ for Borel sets A in E. Clearly $\mathbb{E}\Pi$ is also Poincaré, i.e., invariant under (1.9). By considering $\mathbb{E}\Pi$ on rectangles one finds that

$$\mathbb{E}\Pi(dt, dx) = \begin{cases} c_+ dt\, dx/x^2 & \text{if} \quad x > 0, \\ c_- dt\, dx/x^2 & \text{if} \quad x < 0, \end{cases} \tag{2.1}$$

where $c_+, c_- \in [0, \infty]$ and $c_+ + c_- > 0$ (to avoid the case $\Pi = \varnothing$ wp1). We say that $\mathbb{E}\Pi$ is finite if $c_+ + c_- < \infty$, otherwise it is infinite. The right-hand side of (2.1) will be abbreviated to $c_\pm\, dt\, dx/x^2$.

The domain of absolute convergence. Let Π be a Poincaré point process in E, and consider X_H as in (1.10), so that for $I := (a, b] \subset \mathbb{R}$

$$X_H(b) - X_H(a) = \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}\Pi(I, dx) = \int_{\mathbb{R}\setminus\{0\}} x\Pi^H(I, dx). \tag{2.2}$$

We now want to vary I and H. If we keep H fixed, and consider the events that the right-hand sides of (2.2) (which are, in fact, random series) converge absolutely for various I, then it follows by an argument like that used in the proof of Lemma 1.3 that wp1 either (2.2) converges absolutely for all bounded intervals

$I \subset \mathbb{R}$ or (2.2) converges for none of them. The set $\mathscr{A}_a$ of H for which there is absolute convergence on one fixed I has the form $\mathscr{A}_a = (H_a, \infty)$, $[H_a, \infty)$ or $\varnothing$; $\mathscr{A}_a$ is random and depends on the sample point ω, but except for a set of ω of zero probability, does not depend on I. We call $\mathscr{A}_a$ the domain of absolute convergence of Π, and $H_a := \inf \mathscr{A}_a$ the absolute convergence boundary of Π.

Obviously, $\mathscr{A}_a$ itself (not only its distribution) is invariant under the transformations (1.9) applied to Π. It follows that $\mathscr{A}_a$ is constant if Π is ergodic, i.e., if the probability distribution of Π is trivial on the (1.9)-invariant σ-field (cf. Section 2.6 of Vervaat, 1985). As most examples in practice are ergodic, unless specifically set up to violate this condition, it is no surprise that $\mathscr{A}_a$ usually is nonrandom.

Another approach for getting rid of a nonconstant $\mathscr{A}_a$ is the following. Since Π is Poincaré and $\mathscr{A}_a$ is (1.9)-invariant, the regular conditional distributions of Π given $\mathscr{A}_a$ are (1.9)-invariant. (They exist, since the space of all Radon measures on the Borel field of $\mathbb{R} \times (\overline{\mathbb{R}} \backslash \{0\})$ is Polish). These conditional distributions, of course, have $\mathscr{A}_a$ constant.

We now can formulate the main result of the present section.

THEOREM. 2.1. *Let* Π *be Poincaré and* $\Pi \neq \varnothing$ *wp*1. *Then*

(*a*) $\mathscr{A}_a \subset (1, \infty)$ *wp*1,
(*b*) $\mathscr{A}_a = (1, \infty)$ *wp*1 *if* $\mathbb{E}\Pi$ *is finite.*

REMARKS. If $\mathbb{E}\Pi$ is infinite, then $\mathscr{A}_a$ can have any form (H_a, ∞), $[H_a, \infty)$ or $\varnothing$ (provided $\mathscr{A}_a \subset (1, \infty)$). A class of examples with $\mathscr{A}_a = (H_a, \infty)$ for any $H_a \in (1, \infty)$ is presented in Section 5.2 of Vervaat (1985).

Theorem 2.1(a) is in fact a consequence of Theorem 3.3 in Vervaat (1985), stating that no H-ss si process with $H \leq 1$ can have lbv unless $H = 1$ and $X(t) \equiv tX(1)$. Nevertheless we will give here an independent proof, as it will be used again in the proof of Theorem 4.3 and introduces techniques that we expect to become standard in this field.

Theorem 2.1(a) and Vervaat's (1985) Corollary 6.3 combine to give a new proof that no H-ss si process with $H < 1$ has lbv. Let X_1 be an H_1-ss si process of lbv with $H_1 < 1$. Then select $H_2 > 1$ with $H_1 H_2 < 1$ and a nondecreasing H_2-ss si jump process X_2 (for instance, strictly stable), independent of X_1. Then $X_1 \circ X_2 = (X_1(X_2(t)))_{t \in \mathbb{R}}$ is a jump process and has lbv, but also is $H_1 H_2$-ss and si. This contradicts Theorem 2.1(a).

PROOF OF THEOREM 2.1. We will prove the theorem under the assumption that $c_- = 0$, i.e., that $\Pi \subset E_+ := \mathbb{R} \times (0, \infty)$. It is easy to deduce the general result from this case. Consider (2.2) with $I = I_1 = (0, 1]$. Since Π is finite on $I \times (1, \infty)$, we have wp1 that $H \in \mathscr{A}_a$ iff

$$Y := \int_{(0,1]} x^H \Pi(I, dx) < \infty.$$

If $H > 1$ and $\mathbb{E}\Pi$ is finite, then

$$\mathbb{E}Y = \int_{(0,1]} c_+ x^H \, dx/x^2 = c_+(H-1)^{-1} < \infty$$

so $Y < \infty$ wp1. Thus $(1, \infty) \subset \mathscr{H}_a$ if $\mathbb{E}\Pi$ is finite, and (b) follows once (a) is proved.
To prove (a), it suffices to show $1 \notin \mathscr{H}_a$ wp1. Fix an integer $a > 1$ and set

$$Y_n := \int_{(a^{-n}, a^{-n+1}]} x\Pi(I, dx), \quad \text{for} \quad n = 1, 2, \cdots.$$

Then $Y = \sum_{n=1}^{\infty} Y_n$ (in case $H = 1$) and

$$Y_n = \int_{(0,a^n]} \int_{(1,a]} a^{-n} x\Pi(a^{-n}\, dt, a^{-n}\, dx)$$

$$=_d a^{-n} \int_{(0,a^n]} \int_{(1,a]} x\Pi(dt, dx) = a^{-n} \int_{(1,a]} x\Pi((0, a^n], dx),$$

since Π is Poincaré. Set

$$Z_k := \int_{(1,a]} x\Pi((k-1, k], dx) \quad \text{for} \quad k = 1, 2, \cdots.$$

Then

$$Y_n =_d a^{-n} \textstyle\sum_{k=1}^{a^n} Z_k \tag{2.3}$$

and $(Z_k)_{k=1}^{\infty}$ is a stationary sequence since Π is Poincaré. By the Birkhoff ergodic theorem for nonnegative random variables, it follows that

$$a^{-n} \int_{(1,a]} x\Pi((0, a^n], dx) = a^{-n} \textstyle\sum_{k=1}^{a^n} Z_k \to \mathbb{E}^{\mathscr{I}} Z_1 \quad \text{wp1}, \tag{2.4}$$

where $\mathscr{I}$ is the invariant σ-field for the shift of $(Z_k)_{k=1}^{\infty}$. By (2.3) and (2.4), Y_n converges to $\mathbb{E}^{\mathscr{I}} Z_1$ in distribution. Applying the Portmanteau Theorem (cf. Billingsley, 1968, pages 11–12) to the distributions of the nonnegative random variables Y_n and $\mathbb{E}^{\mathscr{I}} Z_1$, we obtain, for any $\delta > 0$,

$$p := \mathbb{P}[\textstyle\sum_{n=1}^{\infty} Y_n < \infty] \le \mathbb{P}[Y_n \to 0] \le \lim_{n\to\infty} \mathbb{P}[Y_k \le \delta \quad \text{for} \quad k \ge n]$$

$$\le \limsup_{n\to\infty} \mathbb{P}[Y_n \le \delta] \le \mathbb{P}[\mathbb{E}^{\mathscr{I}} Z_1 \le \delta].$$

Since all $Z_i \ge 0$ wp1, we obtain

$$p \le \mathbb{P}[\mathbb{E}^{\mathscr{I}} Z_1 = 0] = \mathbb{P}[Z_1 = Z_2 = \cdots = 0] = \mathbb{P}[\Pi(\mathbb{R}_+ \times (1, a]) = 0].$$

Since this holds for all integers $a > 1$, we have

$$p \le \mathbb{P}[\Pi((0, \infty) \times (1, \infty)) = 0].$$

Since Π is Poincaré, we have for $\varepsilon > 0$ and real b:

$$p \le \mathbb{P}[\Pi((b, \infty) \times (\varepsilon, \infty)) = 0].$$

Letting $\varepsilon \downarrow 0$ and $b \to -\infty$ we deduce that

$$p \le \mathbb{P}[\Pi = \varnothing] = 0,$$

which proves (a). □

3. Examples. In this section we first describe two basic examples of Poincaré point processes. Using (1.10), these each lead to H-ss si processes. We then list a variety of procedures for constructing other examples by modifying or combining previously constructed examples.

3.1. *Poisson Poincaré processes* (*strictly stable processes*). Let Π be a Poisson process in $E = \mathbb{R} \times (\mathbb{R}\backslash\{0\})$ with intensity $c_{\pm} dt\, dx/x^2$ $(0 < c_+ + c_- < \infty)$. Then X_H as defined in (1.10) does not only have stationary, but also independent increments. So X_H is strictly stable with exponent $\alpha = 1/H$, whenever convergent (cf. discussion around (1.12)). The theory of Section 2 specializes to well-known facts in this case. From Theorem 2.1(b) it follows that X_H has lbv iff $H > 1$. The set of X_H to which this applies is the collection of all strictly stable processes with index $\alpha \in (0, 1)$. In the particular case that $c_+ = c_-$, the Poisson process Π can be thought of as the symmetrization (cf. Section 5) of a Poisson process in $E_+ := \mathbb{R} \times (0, \infty)$ with c_+ doubled. In Corollary 4.2(a) we will see that X_H converges conditionally (but not absolutely) wp1 iff $\frac{1}{2} < H \le 1$. In this way we obtain the symmetric stable processes with $1 \le \alpha < 2$, and rediscover the fact that their sample paths have nowhere bounded variation.

A variation of (1.10), which leads in the case Π is Poisson to all stable processes, is studied for Poincaré Π in Section 7.

3.2. *The g-adic lattice process.* Consider the point process in $\mathbb{R}^2$ obtained by choosing one point (V, U) at random in $[0, 1) \times [0, 1)$ and then placing a point at $(V + k, U + n)$ for all $k, n \in \mathbb{Z}$. Among all translation-invariant point processes in $\mathbb{R}^2$, this one has the least randomness in the sense that the position of any one point determines the position of all points. The point process Π in E_+ that we are now going to describe is a Poincaré process with a similar albeit slightly weaker dependence between points; the position of one point (t, x) of Π determines the position of all other points of Π at or below level x (i.e., ordinate in the $t - x$ plane) and leaves only countably many possible positions open for the points above level x. Thus Π is very different from the Poisson process described in Section 3.1.

The requirements that Π be Poincaré and that the randomness be minimal lead us to try the following construction. First, let the points of Π all lie at one of the levels cg^{U+n}, $n \in \mathbb{Z}$, where c is a positive real, g is an integer greater than 1, and U is uniformly distributed over $[0, 1)$. Second, let the points of Π which are at level cg^{U+n} for some particular n have abscissas $(V_n + k)g^{U+n}$, $k \in \mathbb{Z}$, where the V_n's are uniformly distributed over $[0, 1)$ and the collection of V_n's is independent of U. Finally, let the V_n's be related in such a way that each point at level cg^{U+n+1} has the same abscissa as a point at level cg^{U+n}; this yields the condition

$$V_{n+1} = g^{-1}(V_n + \xi_n), \tag{3.1}$$

where the ξ_n's are independent and uniformly distributed over $\{0, 1, \cdots, g-1\}$. Note that V_{n+1} determines V_n and ξ_n and that for given V_n there are only g possible values for V_{n+1}. In the following formal definition, V_n is expressed in the form

$$V_n = \sum_{j=1}^{\infty} g^{-j}\xi_{n-j}. \tag{3.2}$$

Thus, each V_n has a uniform distribution and (3.1) holds.

DEFINITION. Let $g > 1$ be an integer and let c be a positive real number. Let U and ξ_j, $j \in \mathbb{Z}$, be independent random variables such that U has a uniform distribution on $[0, 1)$ and each ξ_j has a discrete uniform distribution on $\{0, 1, \cdots, g-1\}$. Define V_n, $n \in \mathbb{Z}$, by (3.2). Then define the *g-adic lattice process* by

$$\begin{aligned} \Pi_{g,c} &:= \{(k + V_n, c)g^{U+n}: k, n \in \mathbb{Z}\} \\ &= \{(kg^{U+n} + \textstyle\sum_{\ell=-\infty}^{n-1} g^{U+\ell}\xi_\ell, cg^{U+n}): k, n \in \mathbb{Z}\}. \end{aligned} \tag{3.3}$$

The g-adic lattice process will play a role in several other examples. We note that the value of U determines the levels of the points of $\Pi_{g,c}$ and the spacings between the points at each level, while the values of ξ_ℓ, $\ell \in \mathbb{Z}$, determine the exact positions of the points, once the levels and spacings are decided. Figure 1 displays a realization of the triadic lattice process $\Pi_{3,1}$.

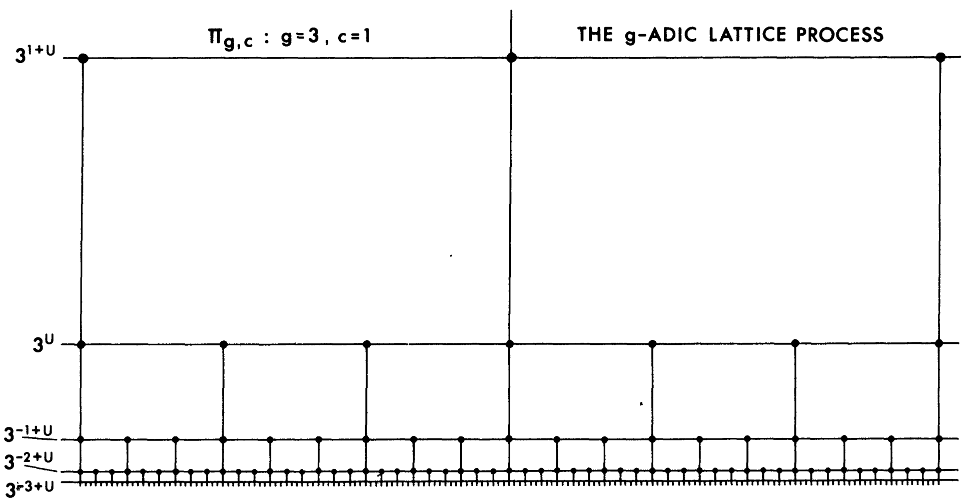

FIG. 1.

THEOREM 3.3. *Let $\Pi_{g,c}$ be a g-adic lattice process in $E_+ := \mathbb{R} \times (0, \infty)$. Then*

(a) *$\Pi_{g,c}$ is Poincaré with finite intensity*

$$\mathbb{E}\Pi_{g,c}(dt, dx) = c(x^2 \log g)^{-1} dt\, dx; \tag{3.4}$$

(b) *$\Pi_{g,c}$ is ergodic.*

COROLLARY 3.4. *The processes X_H, $H > 1$, as defined in* (1.10) *with* $\Pi = \Pi_{g,c}$, *are nondecreasing H-ss si jump processes.*

We will show that in fact $\Pi_{g,c}$ is ergodic with respect to the set of transformations $\{\Pi_{g,c} \mapsto a\Pi_{g,c}: a > 0\}$ and hence is ergodic in the sense defined in Section 2. (It follows from the proof that we need only show that $\Pi_{g,c}$ is ergodic with respect to the transformations $\Pi_{g,c} \mapsto a\Pi_{g,c}$ either for all a of the form $a = g^r$ for r rational or for $a = g$ and $a = g^r$ for one irrational r.) We note that $\Pi_{g,c}$ is *not* ergodic with respect to $\Pi_{g,c} \mapsto \Pi_{g,c} + (b, 0)$, $b \in \mathbb{R}$, as the whole σ-field generated by U is invariant under these transformations.

It is possible to prove $\Pi_{g,c}$ is Poincaré by considering the effect of the transformations (1.9) either on the point process itself or on the random variables U and $\xi = (\xi_k)_{k\in\mathbb{Z}}$. We find it most convenient to take the latter approach here.

PROOF OF THEOREM 3.3. Without changing the distribution of $\Pi_{g,c}$ we can take as underlying probability space the compact metric group $G := T \times \Gamma$, where $T := \mathbb{R}/\mathbb{Z}$ and $\Gamma := (\mathbb{Z}/g)^{\mathbb{Z}}$, with Haar probability measure $\mathbb{P}$ and generic element $\omega = (U, \xi) = (U, (\xi_k)_{k\in\mathbb{Z}})$. The mapping

$$\omega \mapsto \Pi_{g,c}(\omega) \tag{3.5}$$

from G to the set of realizations of $\Pi_{g,c}$ in E_+ is one-to-one (except for a null set) and measurable. The probability $\mathbb{P} = \lambda \times \mu^{\mathbb{Z}}$ where λ is Lebesgue (uniform) measure on T and μ is the discrete uniform probability on $\mathbb{Z}/g$. By definition, $\mathbb{P}$ is invariant for all translations in T in the sense of the addition modulo 1 in T (combined with the identity relation on Γ). In addition, $\mathbb{P}$ and the associated probability measures $\mathbb{P}^U$ on Γ obtained by conditioning on U are invariant under the shift to the right in Γ:

$$(S\xi)_k := \xi_{k-1}, \quad k \in \mathbb{Z}.$$

Define addition with carry over in Γ as follows:

$$\xi + \xi^{(0)} =: \xi^{(1)}$$

where, modulo g^n,

$$\textstyle\sum_{\ell=-\infty}^{n-1} (\xi_\ell + \xi_\ell^{(0)}) g^\ell \equiv \sum_{\ell=-\infty}^{n-1} \xi_\ell^{(1)} g^\ell \in [0, g^n)$$

for all $n \in \mathbb{Z}$, with the probability that $\xi_\ell^{(1)} = g - 1$ for all sufficiently small ℓ excluded.

(a) The mapping $\Pi_{g,c} \mapsto a\Pi_{g,c}$ with $a = g^{m+r}$, $m \in \mathbb{Z}$, $r \in [0, 1)$ corresponds to the transformation

$$(3.6)\qquad \begin{gathered} U \mapsto U + r \bmod 1, \\ \xi \mapsto \begin{cases} S^m\xi & \text{if } \ 0 \le U + r < 1 \\ S^{m+1}\xi & \text{if } \ 1 \le U + r \end{cases} \end{gathered}$$

on G, which leaves $\mathbb{P}$ invariant. The mapping $\Pi_{g,c} \mapsto \Pi_{g,c} + (b, 0)$ corresponds to the transformation

$$(3.7)\qquad \begin{gathered} U \mapsto U, \\ \xi_j \mapsto \xi_j + [bg^{-U-j}], \quad j \in \mathbb{Z} \end{gathered}$$

which leaves $\mathbb{P}^U$ invariant for each $U \in T$; hence it also leaves $\mathbb{P}$ invariant. Thus $\Pi_{g,c}$ is Poincaré.

Let $I \times J$ be a rectangle in E_+ with $J \subset [cg^n, cg^{n+1})$ for some n. Since U has a uniform distribution, the probability density function of the random level cg^{U+n} is $(x \log g)^{-1}$ for $cg^n < x < cg^{n+1}$. Thus, (3.4) follows from the calculation:

$$\begin{aligned} \mathbb{E}\Pi_{c,g}(I \times J) &= \int_J \mathbb{E}[\Pi_{g,c}(I \times J) \mid cg^{U+n} = x](x \log g)^{-1}\, dx \\ &= \int_J \left(cx^{-1} \int_I dt\right)(x \log g)^{-1}\, dx \\ &= \int_I dt \int_J cx^{-2}(\log g)^{-1}\, dx. \end{aligned}$$

(b) Let $\doteq$ denote equality modulo null sets. Let A be a Borel set of realizations of $\Pi_{g,c}$ such that

$$B: = \{\omega \in G: \Pi_{g,c}(\omega) \in A\} \doteq \{a\Pi_{g,c}(\omega) \in A\}.$$

We will show that A has probability 0 or 1, or equivalently that the Borel set B has $\mathbb{P}$ measure 0 or 1. Note that B is invariant under the transformations in (3.6). Set

$$B^U = \{\xi \in \Gamma: (U, \xi) \in B\}$$

for $U \in T$. Then invariance under (3.6) with $r = 0$ and $m = 1$ yields $SB^U \doteq B^U$ for almost all $U \in T$. Since the shift S is ergodic in $(\Gamma, \mu^{\mathbb{Z}})$, we see that $B^U \doteq \varnothing$ or Γ for almost all $U \in T$. Thus, we have

$$B \doteq T_1 \times \Gamma$$

where $T_1: = \{U \in T: B^U \doteq \Gamma\}$. Applying (3.6) with $m = 0$ to B, we see that $T_1 \doteq T_1 + r \bmod 1$ for all $r \in T$ so $T_1 \doteq \varnothing$ or T. Thus $B \doteq \varnothing$ or G. □

There are many variations of the g-adic lattice process which are also Poincaré and ergodic. The simplest of these is obtained by deleting all the points of $\Pi_{g,c}$ which lie directly below some other point of $\Pi_{g,c}$. This amounts to adding the restriction $k \not\equiv \xi_n \pmod g$ to the sets given in (3.3). The intensity of this modified process $\Pi'_{g,c}$ is

$$\mathbb{E}\Pi'_{g,c}(dt, dx) = (g - 1)g^{-1}\mathbb{E}\Pi_{g,c}(dt, dx) = c(g - 1)(x^2 g \log g)^{-1}\, dt\, dx.$$

The effect of this modification on $X_H(t)$ for any $H > 1$ is merely to multiply it by $(1 - g^{-H})$. We can obtain the same H-ss si process by changing the value of c.

A more substantial variation of $\Pi_{g,c}$ is obtained by choosing the V_n's to be related in some way different from (3.1). The easiest example is that of independent V_n's. In this case, there is no need for g to be an integer. Further variations can be obtained by choosing the spacings between points at each level and the logarithms of the levels according to stationary renewal processes.

3.5. *Variations and combinations.* So far, we have obtained two classes of examples of Poincaré processes, one of them new. The next list will make it clear that there are many more examples and consequently many different H-ss si jump processes. The list is far from complete. Any procedure which constructs one point process from another in a way which is (1.9)-invariant yields a Poincaré process if the input is Poincaré. In particular, note that the various procedures described below can be applied successively in the same example, to obtain yet more complex Poincaré processes.

(*a*) *Superposition.* Take the union of two different Poincaré point processes Π, for instance a Poisson process and a lattice process, or two lattice processes with different g. The union is Poincaré if the two composing processes are independent. Even some dependence is allowed. For example, the two processes can be lattice, generated by two independent copies of $(\xi_k)_{k\in\mathbb{Z}}$ but by the same U.

(*b*) *Thinning.* Let $\Pi_g(p)$ be obtained from the lattice process $\Pi_{g,1/p}$ by deleting each point of $\Pi_{g,1/p}$ independently with probability $1 - p$ and maintaining it with probability p. The point processes $\Pi_g(p)$ have the same intensity as $\Pi_{g,1}$ and are Poincaré. It follows by Theorem 8.4 of Kallenberg (1976) that $\Pi_g(p)$ converges in distribution to a Poisson process with the same intensity as $p \downarrow 0$.

(*c*) *Interaction.* Let V be a neighborhood of (0, 1) in E_+. Then

$$V_{t,x} := (t, 0) + xV = \{(t + xs, xy): (s, y) \in V\} \quad \text{for} \quad (t, x) \in E_+$$

defines a system of neighborhoods in E_+ that is (1.9)-invariant. Let Π be Poincaré in E_+. Remove all points (t, x) of Π for which there exists some $(s, y) \in \Pi \cap V_{t,x}$ with $(s, y) \neq (t, x)$. The resulting point process is again Poincaré. In particular, one can take for $V_{t,x}$ some set which depends only on a Poincaré distance (a metric which is invariant under (1.9)) from (t, x). As an alternative to this interaction between points of Π, one can also have a similar type of interaction between two independent (or at least "jointly Poincaré") Poincaré processes.

(*d*) *Subordination.* Let each point of a Poincaré Π (the "primary points") generate a cloud or cluster of points (the "secondary points") around it. The clusters may be random. They should be transformed in a way which depends on the location of their primary points in order to preserve Poincaréness. To make this more specific, we first observe that we may enumerate Π in a measurable way, by writing

$$\Pi = \{(t_j, x_j), n = 1, 2, \cdots\} \tag{3.8}$$

in such a way that each (t_j, x_j) is a random vector (cf. Kallenberg, 1976, page 11). Now let $\kappa_1, \kappa_2, \cdots$ be independent identically distributed point processes in E with $(\kappa_j)_{j=1}^{\infty}$ independent of Π, and define Π′ as a union of random sets:

$$\Pi' := \cup_{j=1}^{\infty} [(t_j, 0) + x_j\kappa_j]. \tag{3.9}$$

Then Π′ is also Poincaré provided it is locally finite in E wp1. We call Π′ a *subordinated process* of Π in this case.

There are a few special cases of particular interest. If $\mathbb{P}[\kappa_n = \{(0, 1)\}] = p = 1 - \mathbb{P}[\kappa_n = \varnothing]$, then Π′ is a thinned process of the type discussed in Example 3.4(b). If $\mathbb{P}[\kappa_n = \{(0, 1)\}] = \mathbb{P}[\kappa_n = \{(0, -1)\}] = ½$, then Π′ is the symmetrized process discussed in Section 5. If $\mathbb{P}[(0, 1) \in \kappa_n] = 1$, then $\Pi \subset \Pi'$ wp1. Subordination gives a convenient way of constructing locally finite Poincaré point processes with infinite intensity. We obtain such a Π′ if $\kappa_n(E) < \infty$ wp1 and $\mathbb{E}(\kappa_n \cap K) = \infty$ for some compact set $K \subset E$.

There are two important generalizations of the subordinated processes. Under some circumstances the point process Π′ defined by (3.9) is a Poincaré point process even if there is dependence among the κ_n's and Π. A particular example of this is considered in Section 6. Also, a subordinated process is a special case of a random measure subordinated to a Poincaré point process (cf. Vervaat, 1985, Section 4).

(*e*) *Composition.* If X_1 and X_2 are independent ss si processes with self-similarity exponents H_1 and H_2, then $X_1 \circ X_2 := X_1(X_2(t))_{t\in\mathbb{R}}$ is H_1H_2–ss and si (cf. Vervaat, 1985, Corollary 6.3). If X_1 and X_2 are both obtained from Poincaré processes in E_+, then so is $X_1 \circ X_2$.

(*f*) *Stripes.* Let U be uniformly distributed on [0, 1), independent of a Poincaré Π. Delete all points (t, x) *of* Π with $\log |x| \in \cup_{k\in\mathbb{Z}}(k + U + (0, ½])$ and maintain all others. The resulting point process Π′ is again Poincaré. Note that

$$\mathbb{P}[\Pi'(\mathbb{R} \times [1, e^{1/4})) = 0] = ¼ > 0,$$

although

$$\mathbb{E}\Pi'(\mathbb{R} \times [1, e^{1/4})) = \infty.$$

So in general it is not true for Poincaré Π with finite $\mathbb{E}\Pi$ that $\Pi(B) = \infty$ wp1 (or even $\Pi(B) > 0$ wp1) if $\mathbb{E}\Pi(B) = \infty$ (B Borel in E). In the case of Poisson Π, $\mathbb{E}\Pi(B) = \infty$ implies that $\Pi(B) = \infty$ wp1. If Π is any Poincaré process and B is any set in E_+ for which there exists a sequence of transformations V_n as in (1.9) such that $V_n(B) \uparrow E_+$, then $\mathbb{E}\Pi(B) = \infty$ implies that $\Pi(B) = \infty$ wp1. Examples of such B are the sectors

$$B = \{(t_0 + t, x_0 + x): 0 \le x \le ct\}$$

where $(t_0, x_0) \in E_+$, $c \in (0, \infty)$.

(*g*) *Doubly stochastic Poisson processes.* Let Λ be some Poincaré random Radon measure on E. Let Π be the doubly stochastic Poisson process with intensity Λ. Then Π is Poincaré. As a particular example, let Π' be Poincaré and let Λ be a function of a Poincaré distance to the closest point of Π'.

4. Conditional convergence. Let Π be a Poincaré point process in $E := \mathbb{R} \times (\mathbb{R}\backslash\{0\})$. In this section, we investigate the convergence of $X_H(t)$, which is interpreted throughout the section to mean convergence of the limits in (1.11). Recall that the terms of the sum in (1.11) are arranged so that they are nonincreasing in absolute value.

Suppose $X_H(t)$ converges wp1 for some $t \neq 0$. Since Π is Poincaré, it follows that for all t separately $X_H(t)$ converges wp1. It also follows by a Fubini-type argument, as on page 113 of Lamperti (1966), that wp1 $X_H(t)$ converges for all t simultaneously except at a random Lebesgue null set. This leads us to define the (random) *domain of conditional convergence* in a way which does not depend on any particular choice of t; namely, we let

$$\mathscr{H}_c = \{H > 0: X_H(t) \text{ converges for all } t \text{ outside a Lebesgue null set}\}.$$

It is also useful to consider the *domain of locally uniform convergence* $\mathscr{H}_u$ defined by

$$\mathscr{H}_u = \{H \in \mathscr{H}_c: X_H(t) \text{ converges locally uniformly in } t \text{ as } \varepsilon \downarrow 0 \text{ through some sequence}\}.$$

The interest in $\mathscr{H}_u$ is derived from the following fact: if $H \in \mathscr{H}_u$, then the sample path X_H is right-continuous and possesses left-hand limits everywhere (that is, $X_H \in D(\mathbb{R})$). We cannot do so well with $\mathscr{H}_c$. Examples such as fractional processes (Vervaat, 1985, Section 5.4) cause us to doubt whether $H \in \mathscr{H}_c$ wp1 even implies that $X_H(t)$ converges for all $t \in \mathbb{R}$ wp1. Some of these self-similar processes are nowhere bounded wp1, which precludes the possibility of locally uniform convergence.

Recall that absolute convergence of (1.10) automatically entails locally uniform convergence. Thus we have

$$\mathscr{H}_a \subset \mathscr{H}_u \subset \mathscr{H}_c. \tag{4.1}$$

It is obvious that $\mathscr{H}_c$ and $\mathscr{H}_u$ are invariant under the transformations (1.9) applied to Π; hence $\mathscr{H}_c$ and $\mathscr{H}_u$ are constant wp1 if Π is ergodic. The following classical lemma is useful for determining the form of $\mathscr{H}_c$ and $\mathscr{H}_u$.

LEMMA 4.1. *Let $T \subset \mathbb{R}$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on T such that $\sum_n f_n(t)$ converges uniformly on T, $|f_n(t)|$ is nonincreasing in n for* *all t and* $|f_1(t)|$ *is bounded on T. Then* $\sum_n f_n(t)^{\uparrow\beta}$ *converges uniformly on T for* $\beta > 1$.

PROOF. Apply Abel's test for uniform convergence (cf. Apostol, 1974, pages 248–249) with f_n as at present and $g_n = |f_n|^{\beta-1}$. □

PROPOSITION 4.2. *$\mathscr{A}_u$ and $\mathscr{A}_c$ both have the form $[H, \infty)$, (H, ∞) or $\varnothing$, wp1.*

PROOF. We prove the result for all sample points Π outside the null set on which Π is not locally finite in $\mathbb{R} \times (\overline{\mathbb{R}}\backslash\{0\})$ (cf. Definition 1.2). We also exclude the trivial case $\Pi = \varnothing$. Suppose $H \in \mathscr{A}_u$. For any $t \neq 0$, we may express $X_H(t)$ in the form

$$X_H(t) = (\operatorname{sgn} t) \sum_{j=1}^{\infty} (x_j(t))^{\uparrow H},$$

where $|x_{j+1}(t)| \leq |x_j(t)|$ for all j and t. Also $|x_1(t)|$ is locally bounded as a function of t. (Incidentally, for each j, $|x_j(t)|$ is a step function of t with infinitely many steps near $t = 0$.) The conclusion that $\beta H \in \mathscr{A}_u$ for all $\beta > 1$ follows by Lemma 4.1 with T ranging over all compact sets in $\mathbb{R}$, once we observe that

$$X_{\beta H}(t) = (\operatorname{sgn} t) \sum_{j=1}^{\infty} (x_j(t))^{\uparrow \beta H} = (\operatorname{sgn} t) \sum_{j=1}^{\infty} ((x_j(t))^{\uparrow H})^{\uparrow \beta}.$$

The result for $\mathscr{A}_c$ follows similarly by considering singleton sets T. □

Recall from (2.1) that the intensity $\mathbb{E}\Pi$ of a Poincaré process Π is determined by constants c_+ and c_-. The following theorem shows that unless $c_+ = c_-$ no processes can be obtained by conditional convergence that were not already obtainable by absolute convergence.

THEOREM 4.3. *Let Π be a Poincaré point process for which $c_+ \neq c_-$. If $H \in \mathscr{A}_c$ wp1 then $H \in \mathscr{A}_a$ wp1.*

PROOF. We first show that $H > 1$. Follow the first part of the proof of Theorem 2.1 (a), with $a = 2$ and the changes

$$Y_n = \int_{(2^{-n}, 2^{-n+1}] \cup [-2^{-n+1}, -2^{-n})} x\Pi(I_1, dx),$$

$$Z_1 = \int_{(1,2] \cup [-2,-1)} x\Pi(I_1, dx).$$

Since $c_+ \neq c_-$, at least one of them is finite, so

$$\mathbb{E}\, Z_1 = (c_+ - c_-) \int_1^2 x^{-1}\, dx \neq 0$$

is well-defined although possibly infinite. Hence $\mathbb{P}[\mathbb{E}^{\mathscr{I}} Z_1 \neq 0] > 0$ whereas $Y_n \to_d \mathbb{E}^{\mathscr{I}} Z_1$. Thus, $\mathbb{P}[1 \in \mathscr{A}_c] < 1$, which implies that $H > 1$. If c_+ and c_- are both finite, then $H \in \mathscr{A}_a$ wp1 by Theorem 2.1 (b). If one of c_+ and c_- is infinite, say c_+, and the other is finite, then the contribution to $X_H(t)$ from the lower half-plane converges (since $H > 1$). Since $H \in \mathscr{H}_c$ wp1, the contribution from the upper half-plane must converge wp1, so $H \in \mathscr{H}_a$ wp1 in this case also. □

The examples in Sections 5 and 6 all have $c_+ = c_-$. In Example 6.2(b), $\mathscr{H}_a = \mathscr{H}_c$ wp1 but in most other examples, the two sets are distinct wp1.

It is not true in general that if $c_+ \neq c_-$ then $\mathscr{H}_a = \mathscr{H}_c$ wp1. To see this, let Π_1 be a Poincaré process with $c_+ \neq c_-$ and $\mathscr{H}_c = \mathscr{H}_a = (1, \infty)$ wp1 and let Π_0 be a Poincaré process with $c_+ = c_- < \infty$ and $\mathscr{H}_c = (\frac{1}{2}, \infty)$ wp1. Examples of such Poincaré processes are given in Sections 5 and 6. Now let ε be a Bernoulli random variable which is independent of Π_1 and Π_0. Then the composed point process Π_ε is Poincaré with $c_+ \neq c_-$, $\mathscr{H}_c = (\frac{1}{2}, \infty)$ with probability $\mathbb{P}[\varepsilon = 1]$ and $\mathscr{H}_a = (1, \infty)$ wp1. The cause of the problem is the nonergodicity of Π_ε as we see from the following immediate corollary of Theorem 4.3.

COROLLARY 4.4. *If Π is an ergodic Poincaré point process and $c_+ \neq c_-$ then $\mathscr{H}_a = \mathscr{H}_u = \mathscr{H}_c$ wp1.*

5. Symmetrization of point processes. We now consider an important class of examples involving conditional convergence. Let Π_1 be a Poincaré point process. The *symmetrization* of Π_1 is the subordinated Poincaré point process Π obtained by taking $\mathbb{P}[\kappa_j = \{(0, 1)\}] = \mathbb{P}[\kappa_j = \{(0, -1)\}] = \frac{1}{2}$, as described in Example 3.4(d). An equivalent description is to write $\Pi = \{(t_j, \varepsilon_j x_j)\}$ where $\Pi_1 = \{(t_j, x_j)\}$ is a measurable enumeration of the points of Π_1, as described in that example, and $(\varepsilon_j)_{j=1}^{\infty}$ is a sequence of independent random variables, which is independent of Π_1, such that $\mathbb{P}[\varepsilon_j = 1] = \mathbb{P}[\varepsilon_j = -1] = \frac{1}{2}$. Thus each point of Π_1 is either left where it is or replaced by its reflection in the t-axis.

The intensity constants c_+ and c_- of Π are both equal to the average of the two intensity constants for Π_1. Thus, we can hope to obtain H-ss si processes via (1.11) that are not available via absolute convergence.

THEOREM 5.1. *Let Π_1 be a Poincaré point process in E, let Π be its symmetrization, and set*

$$(5.1)\qquad X_H(t) := (\operatorname{sgn} t) \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}\Pi(I_t, dx) = \lim_{\eta\downarrow 0} X_H(t, \eta)$$

where

$$(5.2)\qquad X_H(t, \eta) := (\operatorname{sgn} t) \int_{\mathbb{R}\setminus(-\eta,\eta)} x^{\uparrow H}\Pi(I_t, dx).$$

Then

(a) $\mathscr{H}_c = \frac{1}{2}\mathscr{H}_a = \{H\colon 2H \in \mathscr{H}_a\}$ wp1;

(b) $\operatorname{int}(\frac{1}{2}\mathscr{H}_a) \subset \mathscr{H}_u \subset \frac{1}{2}\mathscr{H}_a$ wp1.

COROLLARY 5.2. *If* $\mathbb{E}\Pi$ *is finite and* $\Pi \neq \emptyset$ *then*

$$\mathscr{A}_u = \mathscr{A}_c = (\tfrac{1}{2}, \infty). \tag{5.3}$$

REMARKS. The corollary follows from Theorem 2.1(b). It is shown in Example 5.2 of Vervaat (1985) that for each $\beta > 1$ there exists a Poincaré point process with $\mathscr{A}_a = (\beta, \infty)$. Thus, we may have $\mathscr{A}_c = \mathscr{A}_u = (H, \infty)$ wp1 for any $H \geq \tfrac{1}{2}$. This is extended to all $H \geq 0$ in Section 6.

PROOF OF THEOREM 5.1. (a) Fix $t \neq 0$. The points of $\Pi_1 \cap (I_t \times \mathbb{R})$ can be measurably enumerated by writing

$$\Pi_1 \cap (I_t \times \mathbb{R}) = \{(s_j, y_j),\ j = 1, 2, \cdots\} \tag{5.4}$$

in such a way that $|y_j| \geq |y_{j+1}|$ for all j. Then we may write

$$X_H(t) = \textstyle\sum_{j=1}^{\infty} \delta_j (y_j)^{\uparrow H} \tag{5.5}$$

where $(\delta_j)_{j=1}^{\infty}$ is a sequence of independent random variables, which is independent of Π_1, such that $P[\delta_j = 1] = P[\delta_j = -1] = \tfrac{1}{2}$. For now, let us condition on Π. Then the terms of the series in (5.5) are independent uniformly bounded (by $|y_1|$) random variables with mean 0 and variance $|y_j|^{2H}$. Such a series converges wp1 iff the sum of the variances converges (cf. Lamperti (1966), Theorem 8.2 and Lemma 9.2), which happens iff $2H \in \mathscr{A}_a$. Thus

$$\mathbb{P}^{\Pi}([X_H(t) \text{ converges}]\Delta[\textstyle\sum_{j=1}^{\infty} |y_j|^{2H} < \infty]) = 0 \quad \text{wp1}, \tag{5.6}$$

where Δ denotes symmetric difference and $\mathbb{P}^{\Pi}$ denotes conditional probability given Π. Taking expectations of both sides gives us (5.6) with $\mathbb{P}^{\Pi}$ replaced by $\mathbb{P}$ and with the "wp1" removed. Recalling that $\mathscr{A}_a$ is independent of t wp1 (cf. the argument below (2.2)), we have

$$\mathbb{P}([\textstyle\sum_{j=1}^{\infty} |y_j|^{2H} < \infty]\Delta[2H \in \mathscr{A}_a]) = 0.$$

Theorem 5.1(a) follows from the last two sentences.

(b) We will prove uniform convergence on I_t for $t > 0$. The case $t < 0$ is similar. Set

$$Y_n(u) := \textstyle\sum_{j=1}^{\infty} \delta_j y_j^{\uparrow H} 1_{u,n}(s_j, |y_j|) \quad \text{for} \quad u \in I_t, \quad m = 1, 2, \cdots,$$

where δ_j and (s_j, y_j) are as in part (a) and where $1_{u,n}$ is the indicator function of the set $(0, u] \times (2^{-n}, 2^{-n+1}]$. Also, set $\| Y_n \| := \sup_{u \in I_t} |Y_n(u)|$. It suffices to show that $\sum_{n=1}^{\infty} \| Y_n \| < \infty$ whenever $H > \tfrac{1}{2} H_a = \tfrac{1}{2} \inf \mathscr{A}_a$, modulo a null event. Fix H and H' with $0 < H' < H$. Then set $a_n := 2^{(H-H')(-n+1)}$. Once again we proceed by conditioning on Π. Given Π, the $Y_n(u)$ are partial sums of finitely but increasingly many (as u increases) independent random variables with zero mean and finite variance. By Kolmogorov's inequality

$$\begin{aligned}\mathbb{P}^{\Pi}[\| Y_n\| \geq a_n] &\leq a_n^{-2}\mathbb{E}^{\Pi} Y_n^2(t)\\ &= a_n^{-2}\int_{(2^{-n},2^{-n+1}]} x^{2H}\Pi(I_t, dx)\\ &\leq a_n^{-2}2^{2(H-H')(-n+1)}\int_{(2^{-n},2^{-n+1}]} x^{2H'}\Pi(I_t, dx)\\ &= \int_{(2^{-n},2^{-n+1}]} x^{2H'}\Pi(I_t, dx), \text{ wp1.}\end{aligned}$$

We use this calculation, followed in succession by the Borel-Cantelli Lemma, the fact that $\sum a_n < \infty$, the definition of $\mathscr{A}_u$ and the definition of conditional probability to obtain (modulo null events):

$$\begin{aligned}[2H' > H_a] &\subset [\textstyle\sum_{n=1}^{\infty} P^{\Pi}[\| Y_n\| \geq a_n] < \infty] \subset [P^{\Pi}[\| Y_n\| \geq a_n \text{ i.o.}] = 0]\\ &\subset [P^{\Pi}[\textstyle\sum_{n=1}^{\infty} \| Y_n\| < \infty] = 1] \subset [P^{\Pi}[H \in \mathscr{A}_u] = 1] \subset [H \subset \mathscr{A}_u].\end{aligned}$$

This proves the first inequality in (b); the second inequality follows from (a). □

6. Lattice processes with random signs. The symmetrization procedure of Section 5 has the property that the points are left as they are or reflected in the t-axis independently of each other. In this section, we consider a class of examples in which the reflection procedure is performed in a more dependent way. This could be done in a variety of ways but we stick to a fairly specific case in order to facilitate various calculations. We start with the g-adic lattice process $\Pi_{g,1}$, as defined in Section 3.2. The reflection procedure features dependence between different points at the same level but different levels are treated independently.

Let $\xi_k (k \in \mathbb{Z})$ and U be independent random variables, U with a uniform distribution on $(0, 1)$ and ξ_k with a discrete uniform distribution on $\{0, 1, \cdots, g-1\}$. Furthermore, let $(\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$ be a two-dimensional array of $\{1, -1\}$-valued random variables (the random signs) which is independent of U and $(\xi_k)_{k\in\mathbb{Z}}$. Assume that for each n the sequence $\varepsilon_{n.} := (\varepsilon_{n,k})_{k\in\mathbb{Z}}$ is stationary and that these sequences are independent and identically distributed. The distribution of these sequences is unspecified for now; it will vary from example to example. Finally, set

$$\Pi := \{(k + \textstyle\sum_{j=1}^{\infty} g^{-j}\xi_{n-j}, \varepsilon_{n,k})g^{U+n}: k, n \in \mathbb{Z}\}. \tag{6.1}$$

We call Π a *lattice process with random signs.*

THEOREM 6.1. *The process Π defined above is Poincaré and ergodic.*

PROOF. The proof involves some modifications of the proof of Theorem 3.3. Replace G in that proof by $T \times \Gamma \times \{-1, 1\}^{\mathbb{Z}\times\mathbb{Z}}$ and denote the generic element by $\omega = (U, (\xi_n)_{n\in\mathbb{Z}}, (\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$. Likewise, replace $\mathbb{P}$ by the product of the Haar measure on $T \times \Gamma$ with $\delta^{\mathbb{Z}}$ where δ is the distribution of $\varepsilon_{n.}$ for each n.

Let S^* denote the shift for the iid sequence $\varepsilon := (\varepsilon_{n.})_{n\in\mathbb{Z}}$ and let $S^\dagger$ denote the shift for each stationary sequence $\varepsilon_{n.}$. The mapping $\Pi \mapsto a\Pi$ with $a = g^{m+r}$ corresponds to

$$U \mapsto U + r \bmod 1,$$

$$\xi \mapsto \begin{cases} S^m\xi & \text{if } 0 \le U + r < 1, \\ S^{m+1}\xi & \text{if } 1 \le U + r, \end{cases}$$

$$\varepsilon \mapsto \begin{cases} (S^*)^m\varepsilon & \text{if } 0 \le U + r < 1 \\ (S^*)^{m+1}\varepsilon & \text{if } 1 \le U + r. \end{cases}$$

This leaves $\mathbb{P}$ invariant. The mapping $\Pi \mapsto \Pi + (b, 0)$ corresponds to the transformation

$$U \mapsto U,$$

$$\xi_n \mapsto \xi_n + [bg^{-U-n}], \quad n \in \mathbb{Z},$$

$$\varepsilon_{n.} \mapsto (S^\dagger)^{\alpha(n)}\varepsilon_{n.}, \quad n \in \mathbb{Z},$$

where $\alpha(n)$ is an integer depending on U, $(\xi_n)_{n\in\mathbb{Z}}$, b and n. The invariance of $\mathbb{P}$ relative to this transformation follows by conditioning on (U, ξ) and applying the independence properties for ε. (Note that independence of the $\varepsilon_{n.}$'s is an important assumption since the shifts $\alpha(n)$ depend on n.) The ergodicity with respect to the transformations $\Pi \mapsto a\Pi$, $a > 0$ follows by first conditioning on U and taking $a = g$ as in the proof of Theorem 3.3(b). □

We will investigate the convergence of $X_H(t)$ for such Poincaré processes Π. Note first that $\mathscr{A}_c$ and $\mathscr{A}_u$ are constant wp1. The calculation of $X_H(t)$ amounts to adding up the contribution to $X_H(t)$ from each level g^{U-n} of $\Pi_{g,1}$; denote these contributions by $Y_n(t)$. (Note the minus sign, which is included in order to simplify many calculations below.) Thus

(6.2) $$X_H(t) = (\operatorname{sgn} t) \textstyle\sum_{n=-\infty}^{\infty} Y_n(t),$$

where

(6.3) $$Y_n(t) := Y_{n,H}(t) := g^{(U-n)H} \textstyle\sum \{\varepsilon_{-n,k}\colon (k + \sum_{j=1}^{\infty} g^{-j}\xi_{-n-j}) \in g^{-(U-n)}I_t\}.$$

If ν_t is the largest integer for which $\Pi_{g,1}$ has points in $I_t \times \{g^{U+\nu_t}\}$, then $Y_n(t) = 0$ for $n < -\nu_t$. Thus, (6.2) converges iff $\sum_{n=1}^{\infty} Y_n(t)$ converges. For all $n \in \mathbb{Z}$, the horizontal distance between points at level g^{U+n} of $\Pi_{g,1}$ is g^{U+n}. Thus, the number $k_n(t)$ of terms contributing to the sum in (6.3) satisfies

(6.4) $$[|t|g^{-U+n}] \le k_n(t) \le [|t|g^{-U+n}] + 1 \quad \text{wp1},$$

provided $n \ge 1 - (\log g)^{-1}\log|t|$. Since $0 \le U \le 1$, we deduce the deterministic bounds

(6.5) $$[|t|g^{-1+n}] \le k_n(t) \le [|t|g^n] + 1 \quad \text{wp1},$$

with the same provision. By the independence of $\Pi_{g,1}$ and $(\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$, we have

(6.6) $$Y_n(t) =_d g^{(U-n)H} S(k_n(t)),$$

where $S(k) := S_k := \sum_{i=1}^{k} \varepsilon_{0,i}$.

Since $\Pi_{g,1}$ has finite intensity with $c_+ = (\log g)^{-1}$ and $c_- = 0$, it follows easily that Π has finite intensity with $c_+ = \mathbb{P}[\varepsilon_{n,k} = 1](\log g)^{-1} = (2 \log g)^{-1}\mathbb{E}(\varepsilon_{n,k} + 1)$ and $c_- = (\log g)^{-1} - c_+$. By Theorem 4.3 $\mathscr{H}_c = \mathscr{H}_a = (1, \infty)$ wp1 unless $\mathbb{E}\varepsilon_{n,k} = 0$. We now give three simple examples, the first two of which can be viewed as extreme cases.

EXAMPLES 6.2. (a) First, we suppose that every second point within each level of $\Pi_{g,1}$ is reflected in the t-axis. Thus, we take $\varepsilon_{n,k} = (-1)^k \varepsilon_{n,0}$ for all n and k, where $\mathbb{P}[\varepsilon_{n,0} = 1] = \mathbb{P}[\varepsilon_{n,0} = -1] = 1/2$ and the $\varepsilon_{n,0}$'s are independent. Then $|Y_n(t)| \le g^{(U-n)H}$ for all n and t so (6.2) converges uniformly in $t \in \mathbb{R}$ wp1 for all $H > 0$, i.e., $\mathscr{H}_c = \mathscr{H}_u = (0, \infty)$ wp1.

(b) Now suppose that, for each level, all the points of the level are left as they are or all of them are reflected in the t-axis. To accomplish this we take $\varepsilon_{n,k} = \varepsilon_{n,0}$ for all n and k, where the $\varepsilon_{n,0}$'s are the same as in (a). Then $|Y_n(t)| = g^{(U-n)H} k_n(t) \sim |t| g^{(U-n)(H-1)}$ as $n \to \infty$, so that $\mathscr{H}_c = \mathscr{H}_u = (1, \infty)$ wp1.

(c) If all $\varepsilon_{n,k}$'s are independent, we obtain a symmetrized process of the type considered in Section 5, so $\mathscr{H}_c = \mathscr{H}_u = (1/2, \infty)$ wp1.

We next give conditions which are useful for determining $\mathscr{H}_c$ for other examples of lattice processes with random signs. As we saw at the beginning of Section 4, $H \in \mathscr{H}_c$ wp1 iff $X_H(t)$ converges wp1 for any specific $t \ne 0$. For convenience, we choose $t = 1$ and define

(6.7) $$Y_n := Y_n(1) \quad \text{and} \quad k_n := k_n(1).$$

THEOREM 6.3. *Let Π be a lattice process with random signs. If $\delta \in (0, 1]$ and*

(6.8) $$\limsup_{m\to\infty} \min_{m\le k\le gm} \mathbb{P}[k^{-\delta}|S_k| > a] > 0 \quad \textit{for some} \quad a > 0$$

(*where S_k is defined below* (6.6)), *then $\mathscr{H}_c \subset (\delta, \infty)$ wp1. If $\delta \in [0, 1]$ and*

(6.9) $$m^{-\gamma}\mathbb{E}|S_m| \to 0 \quad \textit{as} \quad m \to \infty \quad \textit{for all} \quad \gamma > \delta,$$

then $(\delta, \infty) \subset \mathscr{H}_c$ wp1.

PROOF. Assume (6.8). We will show that $X_\delta(1)$ diverges wp1. Let m be a positive integer and choose the integer n such that $g^{n-1} \le m < g^n$. By (6.4), $k_{n+1} \le gk_n + g + 1$ and, by (6.5), $g^{n-1} - 1 \le k_n \le g^n + 1$. Combining those inequalities we see that if $k_n < m$, then $m \le g^n - 1 \le k_{n+1} \le gk_n + g + 1 \le gm + g$, while otherwise $m \le k_n \le g^n + 1 \le gm + g$. Thus $\mathbb{P}[m \le k_n \le gm + g] \ge 1/2$ for some n

(i.e., either n or $n+1$ in the above). For such m and n, we have by (6.6) with $H = \delta$, (6.4) and the independence of $(k_n)_{n\in\mathbb{Z}}$ and $(\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$ that, for $a > 0$,

$$\begin{aligned}\mathbb{P}[|Y_n| \geq a] &= \mathbb{P}[(k_n)^{-\delta}|S(k_n)| \geq a(g^{U-n}k_n)^{-\delta}] \\ &\geq \mathbb{P}[(k_n)^{-\delta}|S(k_n)| \geq 2^\delta a] \\ &\geq \textstyle\sum_{k=m}^{gm+g} \mathbb{P}[k^{-\delta}|S_k| \geq 2^\delta a,\ k_n = k] \\ &\geq \tfrac{1}{2} \min_{m\leq k\leq gm+g} \mathbb{P}[k^{-\delta}|S_k| \geq 2^\delta a].\end{aligned}$$

By (6.8), it follows that

$$\limsup_{n\to\infty}\mathbb{P}[|Y_n| \geq a] > 0$$

for a sufficiently small. Thus

$$\mathbb{P}[\textstyle\sum_{n=1}^\infty Y_n \text{ diverges}] \geq \mathbb{P}[Y_n \not\to 0] > 0.$$

By ergodicity, $\delta \notin \mathscr{A}_c$ wp1, so $\mathscr{A}_c \subset (\delta, \infty)$ wp1.

Now assume (6.9) and suppose $\delta < \gamma < H' < H$. By (6.6), (6.5), the independence of $(k_n)_{n\in\mathbb{Z}}$ and $(S(k))_{k\in\mathbb{Z}}$, and (6.9), we have for $n \geq$ some nonrandom N that

$$\begin{aligned}\mathbb{P}^{k_n}[|Y_n| \geq g^{n(H'-H)}] &= \mathbb{P}^{k_n}[|S(k_n)| > g^{-UH+nH'}] \\ &\leq g^{UH-nH'}\mathbb{E}^{k_n}|S(k_n)| \\ &\leq g^{UH-nH'}\max\{\mathbb{E}|S(k)|: g^{-1+n} \leq k \leq g^n + 1\} \\ &\leq g^{n(\gamma-H')} \quad \text{wp1}.\end{aligned}$$

Hence,

$$\mathbb{P}[|Y_n| > g^{n(H'-H)}] = O(g^{n(\gamma-H')}) \quad \text{as} \quad n \to \infty.$$

Since $\gamma - H' < 0$ and $H' - H < 0$, we deduce via the Borel-Cantelli Lemma that $\sum_{n=1}^\infty |Y_n| < \infty$ wp1, so $H \in \mathscr{A}_c$ wp1. □

REMARKS 6.4. A sufficient condition for (6.9) is that $\mathbb{E}|S_m^\alpha| = O(m^{\alpha\delta})$ as $m \to \infty$, for some $\alpha \geq 1$, since $\limsup m^{-\delta}\mathbb{E}|S_m| \leq \limsup m^{-\delta}(\mathbb{E}|S_m|^\alpha)^{1/\alpha}$. In particular, if $\mathbb{E}\varepsilon_{n,k} = 0$, a sufficient condition for (6.9) is

(6.10) $$\mathrm{Var}(S_m) = O(m^{2\delta}) \quad \text{as} \quad m \to \infty.$$

A sufficient condition for (6.8) is that $k^{-\delta}S_k$ converges in distribution to a limit that is not concentrated at 0, as $k \to \infty$.

EXAMPLES 6.5. In Example 6.2(a), $\mathbb{E}|S_m| \leq 1$ so (6.9) holds with $\delta = 0$. In Example 6 2(b), $|S_m| = m$ so (6.8) holds with $\delta = 1$. In Example 6.2(c), both (6.8) and (6.9) hold with $\delta = \tfrac{1}{2}$, by virtue of Remark 6.4. The following two examples of stationary sequences $(\varepsilon_{n,k})_{k\in\mathbb{Z}}$ of $\{-1, 1\}$-valued random variables can be found in the literature.

(a) Let $(\varepsilon_{n,k})_{k\in\mathbb{Z}}$ be the stationary extension of the random walk in random scenery $(\varepsilon_{n,k})_{k=0}^{\infty} =_d (W_k)_{k=0}^{\infty}$ as defined in Kesten and Spitzer (1979) ([K & S]), with random scenery as on page 5 of [K & S] and random walk satisfying (1.6) and (1.7) of [K & S]. By Theorem 1.1 of [K & S], $m^{1-1/2\alpha}S_m \to_d \Delta_1$ where Δ_1 is normally distributed by Lemma 5 of [K & S]. Thus (6.8) holds with $\delta = 1 - \frac{1}{2}\alpha > \frac{1}{2}$. Formula (6.10) follows from (1.2) and (2.13) of [K & S], with the same δ.

(b) Take $(\varepsilon_{n,k})_{k=0}^{\infty} =_d (\operatorname{sgn} X_k)_{k=0}^{\infty}$ in Taqqu (1975) (so $G(x) = \operatorname{sgn} x$, having Hermite rank $m = 1$). Here again $\delta > \frac{1}{2}$.

Except for the trivial Example 6.2(a), all examples so far have $\mathscr{H}_c = (\delta, \infty)$ wp1 for some $\delta \geq \frac{1}{2}$. We will now construct a class of lattice processes with random signs for which $\mathscr{H}_c$ can be any set of the form (H_c, ∞) for $0 \leq H_c \leq 1$ or $[H_c, \infty)$ for $0 < H_c \leq 1$. We will not apply Theorem 6.3 directly in this construction although it can be shown that (6.8) and (6.9) do hold in the case $\mathscr{H}_c = (H_c, \infty)$. Let $(b_k)_{k=1}^{\infty}$ be any sequence of positive real numbers such that $b_k \uparrow \infty$ and $k^{-1}b_k \downarrow 0$ as $k \to \infty$. It is shown in Theorem 4 of O'Brien (1983) that it is possible to define $\{-1, 1\}$-valued random variables $(\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$ such that $(\varepsilon_{n,k})_{k\in\mathbb{Z}}$ is stationary for each n, the collection of these sequences is iid and the following two formulas hold:

(6.11) $$\limsup_{n\to\infty} \max_{2^{n-1}\leq k\leq 2^{n+1}} b_k^{-1} \left| \sum_{i=0}^{k-1} \varepsilon_{n,i} \right| \leq 1 \quad \text{wp1},$$

(6.12) $$\limsup_{n\to\infty} \min_{2^{n-1}\leq k\leq 2^{n+1}} 4b_k^{-1} \sum_{i=0}^{k-1} \varepsilon_{n,i} \geq 1 \quad \text{wp1}.$$

Let $\delta \in (0, 1)$ and let $b_k = k^{\delta}$. Let $(\varepsilon_{n,k})_{n,k\in\mathbb{Z}}$ satisfy the conditions indicated above and let Π be a lattice process with random signs constructed from the lattice process $\Pi_{2,1}$ and these $\varepsilon_{n,k}$'s. Then we claim that $\mathscr{H}_c = \mathscr{H}_u = (\delta, \infty)$ wp1. By (6.5), we have

$$2^{n-1} \leq k_n(t) \leq 2^{n+1} \quad \text{for} \quad 1 \leq t < 2.$$

It follows from (6.6) and (6.11) that for such t and sufficiently large n, the contribution $Y_{n,H}(t)$ to $X_H(t)$ from these points satisfies

$$|Y_{n,H}(t)| = 2^{(U-n)H} \left| \sum_{i=0}^{k_n(t)-1} \varepsilon_{n,i} \right| \leq 2^{(1-n)H} 2(2^{n+1})^{\delta} \quad \text{wp1}.$$

Thus $X_H(t)$ converges uniformly for $t \in [1, 2)$ wp1, provided $H > \delta$. Local uniform convergence then follows by an argument like that in the proof of Lemma 1.3. On the other hand, by (6.12) and (6.5) with $t = 1$, there are wp1 infinitely many n for which

$$8(k_n)^{-\delta} \sum_{i=0}^{k_n-1} \varepsilon_{n,i} \geq 1.$$

For such n,

$$\begin{aligned} Y_{n,H}(1) &= 2^{(U-n)H} \sum_{i=0}^{k_n-1} \varepsilon_{n,i} \\ &\geq 2^{-nH} 2^{-3} (k_n)^{\delta} \geq 2^{-nH-3} (2^{-1+n})^{\delta}, \end{aligned}$$

which converges to zero as $n \to \infty$ only if $H > \delta$. This proves that $\mathscr{H}_c = \mathscr{H}_u = (\delta, \infty)$ wp1.

By choosing $b_k = (\log k)^{-1}k$, $\log k$, or $(\log k)^{-2}k^{\delta}$ where $0 < \delta \leq 1$, the above construction yields $\mathscr{H}_c = \mathscr{H}_u = (1, \infty)$, $(0, \infty)$ or $[\delta, \infty)$ respectively.

7. Centering at expectations. Let Π be a Poincaré point process in E with finite intensity determined by the constants c_+ and c_- in (2.1). If $c_+ \neq c_-$, then X_H in (1.10) converges absolutely for $H > 1$ by Theorem 2.1(b), but does not even converge conditionally for any other H in case Π is ergodic, by Corollary 4.4. In the present section we study a modification of (1.10) that will converge conditionally for H in part of $(0, 1]$, even if $c_+ \neq c_-$.

In fact we mimic the Lévy-Itô representation for processes with stationary independent increments without normal component: namely, we replace Π by $\Pi - \mathbb{E}\Pi$, but only for small jumps. So we are led to define

$$(7.1)\quad \begin{aligned} S_H(t) &:= (\operatorname{sgn} t) \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}(\Pi(I_t, dx) - 1_{[-1,1]}(x)\mathbb{E}\Pi(I_t, dx)) \\ &:= (\operatorname{sgn} t)\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} x^{\uparrow H}(\Pi(I_t, dx) - 1_{[-1,1]}(x)\mathbb{E}\Pi(I_t, dx)). \end{aligned}$$

It is well-known that in case Π is Poisson, there is conditional convergence wp1 in (7.1) for $H > ½$, and that, apart from linear drift, all nonnormal stable processes are obtained by (7.1). Although S_H must be si, it need not be ss. The rescaling properties are exactly the same as those of stable processes, as the next lemma shows.

LEMMA 7.1. *If S_H in (7.1) converges wp1, then for $a > 0$*
(a) $(S_H(at) - (c_+ - c_-)at/(1 - H))_{t\in\mathbb{R}} =_d a^H(S_H(t) - (c_+ - c_-)t/(1 - H))_{t\in\mathbb{R}}$ *if $H \neq 1$,*
(b) $S_1(a\cdot) =_d (aS_1(t) + (c_+ - c_-)ta \log a)_{t\in\mathbb{R}}$.

COROLLARY 7.2. *$(S_H(t) - (c_+ - c_-)t/(1 - H))_{t\in\mathbb{R}}$ is H-ss and si for $H \neq 1$. Furthermore, S_1 is 1-ss iff $c_+ = c_-$.*

PROOF OF LEMMA 7.1. For $a > 0$ and random functions of t we have

$$\begin{aligned} S_H(at) &= \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}(\operatorname{sgn} t)(\Pi(I_{at}, dx) - at\ 1_{[-1,1]}(x)c_\pm\, dx/x^2) \\ &= \int_{\mathbb{R}\setminus\{0\}} a^H y^{\uparrow H}(\operatorname{sgn} t)(\Pi(I_{at}, a\, dy) - t\ 1_{[-1,1]}(ay)c_\pm\, dy/y^2) \\ &=_d a^H \int_{\mathbb{R}\setminus\{0\}} y^{\uparrow H}(\operatorname{sgn} t)(\Pi(I_t, dy) - t\ 1_{[-1/a,1/a]}(y)c_\pm\, dy/y^2) \\ &= a^H\left(S_H(t) - (c_+ - c_-)t\int_1^{1/a} y^{H-2}\, dy\right) \\ &= \begin{cases} a^H(S_H(t) - (c_+ - c_-)t(H-1)^{-1}(a^{1-H} - 1)) & \text{if } H \neq 1, \\ a(S_1(t) + (c_+ - c_-)t\log a) & \text{if } H = 1. \end{cases} \quad \square \end{aligned}$$

Examining these results further, we observe that the centering measure in

(7.1) is superfluous for $H > 1$, since then already (1.10), without centering measure, converges absolutely wp1 by Theorem 2.1(b), and comparing (1.11) and (7.1) we find

$$X_H(t) = S_H(t) - (c_+ - c_-)t/(1 - H).$$

For $H = 1$ we do not find any new 1-ss process besides those in (1.11), as observed in Corollary 7.2. Finally, for $H < 1$ the restriction of the centering measure to $x \in [-1, 1]$ is not needed, since

$$(\mathrm{sgn}\ t) \int_{\mathbb{R}\setminus[-1,1]} x^{\uparrow H} \mathbb{E}\Pi(I_t, dx)$$

is finite and in fact equals $(c_+ - c_-)t/(1 - H)$. So the only case of interest for us reduces to

$$\begin{aligned} X_H^c(t) &:= (\mathrm{sgn}\ t) \int_{\mathbb{R}\setminus\{0\}} x^{\uparrow H}(\Pi - \mathbb{E}\Pi)(I_t, dx) \\ &:= (\mathrm{sgn}\ t)\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} x^{\uparrow H}(\Pi - \mathbb{E}\Pi)(I_t, dx) \qquad (7.2) \\ &= S_H(t) - (c_+ - c_-)t/(1 - H) \quad \text{for} \quad 0 < H < 1. \end{aligned}$$

We call X_H^c the *centered version* of X_H $(0 < H < 1)$.

REMARKS 7.3. We will not pursue a complete investigation of sets of H for which (7.2) converges. Instead we restrict ourselves to the following observations.

(a) If $c_+ = c_-$, then (7.2) reduces to (1.11). Thus, all examples of Sections 5 and 6 are in principle also examples for the present one.

(b) If Π is a Poincaré process in E, then we can compare the domain of convergence for the process X_H^c with that for the process X_H^{c+} obtained via (7.2) from the Poincaré process $\Pi \cap E_+$. They need not be the same if $c_+ = c_-$. Consider Example 6.2(a). It is easily seen that X_H^{c+} cannot converge for $H < 1$ whereas X_H^C converges for all $H > 0$.

REFERENCES

APOSTOL, T. M. (1974). *Mathematical Analysis*, 2nd ed. Addison-Wesley, Reading, Massachusetts.
BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.
FELLER, W. (1971). *An Introduction to Probability Theory and its Applications II*, 2nd ed. Wiley, New York.
ITÔ, K. (1942). On stochastic processes (I). *Japan J. Math.* **18** 261–301.
KALLENBERG, O. (1976). *Random Measures.* Akademie-Verlag, Berlin.
KESTEN, H. and SPITZER, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. verw. Gebiete* **50** 5–25.
LAMPERTI, J. (1966). *Probability.* Benjamin, New York.
LEHNER, J. (1964). Discontinuous groups and automorphic functions. *Mathematical Surveys* VIII. Amer. Math. Soc., Providence, R.I.
O'BRIEN, G. L. (1983). Rates of convergence for the ergodic theorem *Canad. J. Math.* **35** 1129–1146.

O'BRIEN, G. L., TORFS, P. J. J. F., and VERVAAT, W. (1984+). Self-similar stationary extremal processes. Unpublished.

TAQQU, M. S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. verw. Gebiete* **31** 287–302.

VERVAAT, W. (1985). Sample path properties of self-similar processes with stationary increments. *Ann. Probab.* **13** 1–27.

WALTERS, P. (1982). *An Introduction to Ergodic Theory.* Springer, Berlin.

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