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A LOCAL LIMIT THEOREM FOR ASSOCIATED SEQUENCES

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A local central limit theorem of the type due to L. A. Shepp is proved for certain stationary sequences of associated random variables.

I. Introduction. A theorem of L. A. Shepp (1964) gives conditions under which an independent, identically distributed sequence $Y_1, Y_2, \cdots$ of random variables with $\sigma^2 = EY_1^2 < \infty$ will satisfy

$$\lim_{n\to\infty} \sigma\sqrt{2\pi n}P(a \le Y_1 + \cdots + Y_n \le b) = b - a$$

for all real numbers a and b. Our result combines the techniques of Shepp and C. M. Newman (1980) in a local limit theorem for associated random variables which reduces to Shepp's theorem if the variables are uncorrelated.

The proof of our result relies heavily on the theory of characteristic functions. We have drawn mainly from the book by L. Breiman (1968) for basic results on characteristic functions as well as for our notation. The estimates we use on characteristic functions of associated variables can be found in Newman (1980), Newman (1982), and Wood (1983).

II. Notation and background. Throughout this paper we let $X_1, X_2, \cdots$ denote a strongly stationary sequence of associated random variables with mean zero, $0 < EX_i^2 < \infty$, and

$$A^2 = EX_1^2 + 2\sum_{n=2}^{\infty} \mathrm{Cov}(X_1, X_n) < \infty.$$

We further establish the following notation:

$$S_k = X_1 + \cdots + X_k; \quad F_k(x) = P(S_k \le x); \quad \bar{S}_k = S_k/\sqrt{k};$$

$$\sigma_k^2 = \mathrm{var}(\bar{S}_k); \quad f_k(u) = E\exp(iuS_k); \quad \phi_k(u) = E\exp(iu\bar{S}_k).$$

Let $N_A(x)$ denote the normal distribution with mean zero and variance A^2.

The Central Limit Theorem of C. M. Newman (1980) says that within our framework $F_k(x\sqrt{k})$ converges to $N_A(x)$. In particular, we will use the facts that σ_k^2 converges to A^2 and $\phi_k(u)$ converges uniformly on compact intervals to $\exp(-A^2u^2/2)$. We also use the following inequality.

Newman's Inequality. Suppose $Y_1, \cdots, Y_n$ are associated random variables

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with finite variances; then for any real $\lambda_1, \cdots, \lambda_n$

$$|E \exp(i \sum_{k=1}^{n} \lambda_k Y_k) - \prod_{k=1}^{n} E \exp(i\lambda_k Y_k)|$$
$$\leq \sum_{k=1;j>k}^{n} |\lambda_k| |\lambda_j| \mathrm{Cov}(X_k, X_j).$$

From this inequality we can point out that if our variables $X_1, X_2, \cdots$ are uncorrelated then $\sigma_k^2 = A^2$ is a constant sequence, the variables are mutually independent, and our theorem will reduce to that of Shepp stated below.

III. The results. We set $L_d = \{nd: n = 0, \pm 1, \pm 2, \cdots\}$ and $L_0 = \mathbb{R}$ (the real numbers). A random variable X is called centered lattice if there exists $d > 0$ such that $P(X \in L_d) = 1$, and there is no $d' > d$ and α such that $P(X \in \alpha + L_{d'}) = 1$. X is called centered nonlattice if there are no numbers α and $d > 0$ such that $P(X \in \alpha + L_d) = 1$. We let ℓ_d assign mass d to every point of L_d, and ℓ_0 denotes Lebesgue measure on L_0.

Shepp's Theorem. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, either centered lattice on L_d or centered nonlattice on L_0 with $EX_1 = 0$ and $0 < EX_1^2 = \sigma^2 < \infty$. Then for any finite interval I,

$$\lim_{n\to\infty} \sigma\sqrt{2\pi n} P(S_n \in I) = \ell_d(I).$$

Since uncorrelated associated random variables are independent, this theorem covers the case when $\sigma_k^2 = A^2$ or, what is the same thing, $\mathrm{Cov}(X_1, X_k) = 0$, for all k.

THEOREM. *Under the assumptions in Section II with* $\mathrm{Cov}(X_1, X_k) > 0$ *it is possible to find* $m = m(k)$ *so that* $\lim_{k\to\infty} m^{3/2}(A^2 - \sigma_k^2) = 0$ *and for any finite interval* I,

$$\lim_{k\to\infty} \sigma_k \sqrt{2\pi m} P(S_{m\cdot k} \in \sqrt{k} I) = \ell_0(I).$$

REMARK. The conclusion of the theorem is a statement of weak convergence of measures. Thus, the theorem follows by showing that

$$\lim_{k\to\infty} \sigma_k \sqrt{2\pi m} E h(S_{mk}/\sqrt{k}) = \int h(x)\, dx \tag{1}$$

for a sufficiently large number of functions h. Let H be the class of all functions h such that $\int |h(x)|\, dx < \infty$ and $h(x) = \int e^{iux} \hat{h}(u)\, du$ for some continuous, real-valued function $\hat{h}$ with compact support. It suffices to prove (1) holds for all $h \in H$ (see Breiman, 1968).

LEMMA. *The conclusion of the theorem is valid if*

$$\lim_{k\to\infty} \sigma_k \sqrt{2\pi m} \int f_k^m(u/\sqrt{k}) \hat{h}(u)\, du = \int h(x)\, dx \quad \textit{for all} \quad h \in H.$$

PROOF. Let J be any finite interval containing the support of $\hat{h}$ and note that

$$Eh(S_{mk}/\sqrt{k}) = \int\int \exp(iux/\sqrt{k})\hat{h}(u)\,du\,F_{mk}(dx)$$
$$= \int f_{mk}(u/\sqrt{k})\hat{h}(u)\,du$$
$$= \int_J f_{mk}(u/\sqrt{k})\hat{h}(u)\,du.$$

We write

$$\int_J f_{mk}(u/\sqrt{k})\hat{h}(u)\,du = \int_J (f_{mk}(u/\sqrt{k}) - f_k^m(u/\sqrt{k}))\hat{h}(u)\,du$$
$$+ \int_J f_k^m(u/\sqrt{k})\hat{h}(u)\,du := I_1 + I_2.$$

Using Newman's Inequality it can be shown (Newman, 1980; Wood, 1983)

$$|I_1| \le \|\hat{h}\|_\infty \int_J |f_{mk}(u/\sqrt{k}) - f_k^m(u/\sqrt{k})|\,du$$
$$\le \|\hat{h}\|_\infty \textstyle\sum_{n=1;j>n}^m \operatorname{Cov}(\bar{S}_k^n, \bar{S}_k^j) \displaystyle\int_J u^2\,du$$
$$\le (\text{constant depending on } h)m(A^2 - \sigma_k^2)$$

where

$$S_{mk} = (X_1 + \cdots + X_k) + \cdots + (X_{(m-1)k+1} + \cdots + X_{mk}) := S_k^1 + \cdots + S_k^m$$

and $\bar{S}_k^n = S_k^n/\sqrt{k}$. Thus, $\sigma_k\sqrt{2\pi m}I_1 \to 0$ since $m^{3/2}(A^2 - \sigma_k^2) \to 0$ and the lemma is proved.

PROOF OF THE THEOREM. The proof of the theorem now follows by showing that $\sigma_k\sqrt{2\pi m}I_2 \to \int h(x)\,dx$. For this purpose we recall that $f_k(u/\sqrt{k}) = \phi_k(u)$ and $\phi_k(u)$ converges to $\exp(-A^2u^2/2)$ uniformly on compact sets. Thus for all sufficiently large k, $|\phi_k(u)| \ne 1$ on the set $J\backslash\{0\}$. Standard estimates let us write

$$\phi_k(u) = 1 - (\sigma_k^2u^2/2)(1 + \delta_k(u)), \quad \lim_{u\to 0}\delta_k(u) = 0.$$

Again we have for sufficiently large k that there is an interval $N = (-b, b)$ small enough that on N, $|\delta_k(u)| \le 1/2$ and $\sigma_k^2u^2 \le A^2u^2 \le 1$. Also, for $u \in J\backslash N$, $|\phi_k(u)| \le 1 - \beta$ for some $0 < \beta < 1$ and such large k.

On N we have

$$|\phi_k(u)| \le 1 - (\sigma_k^2u^2/2) + (\sigma_k^2u^2/2)|\delta_k(u)|$$
$$\le 1 - (\sigma_k^2u^2/4) \le \exp(-\sigma_k^2u^2/4) \searrow \exp(-A^2u^2/4).$$

Hence

$$I_2 = \int_N \phi_k^m(u)\hat{h}(u)\,du + \theta_{mk}\|\hat{h}\|_\infty(1-\beta)^m$$

where $|\theta_{mk}| \le \ell_0(J)$. Also

$$\sigma_k\sqrt{2\pi m}\int_N \phi_k^m(u)\hat{h}(u)\,du = \sigma_k\sqrt{2\pi}\int_{-b\sqrt{m}}^{b\sqrt{m}} \phi_k^m\left(\frac{v}{\sqrt{m}}\right)\hat{h}\left(\frac{v}{\sqrt{m}}\right)dv$$

which, by the dominated convergence and central limit theorems, converges to $A\sqrt{2\pi}\int \exp(-A^2u^2/2)\,du\,\hat{h}(0) = 2\pi\hat{h}(0)$. To see the pointwise convergence of $\phi_k^m(v/\sqrt{m})$ to $\exp(-A^2v^2/2)$ notice that

$$\begin{aligned}|\phi_k^m(v/\sqrt{m}) - \exp(-A^2v^2/2)| \\ &\le |\phi_{mk}(v) - \phi_k^m(v/\sqrt{m})| + |\phi_{mk}(v) - \exp(-A^2v^2/2)| \\ &\le |f_{mk}(v/\sqrt{mk}) - f_k^m(v/\sqrt{mk})| + |\phi_{mk}(v) - \exp(-A^2v^2/2)|.\end{aligned}$$

Now, as in the proof of our lemma above,

$$|f_{mk}(v/\sqrt{mk}) - f_k^m(v/\sqrt{mk})| \le (\text{constant depending on } v)\cdot(A^2-\sigma_k^2),$$

which goes to 0, and $\phi_{mk}(v)$ converges to $\exp(-A^2v^2/2)$ by Newman's Central Limit Theorem. Since $\hat{h}(u) = (1/2\pi)\int \exp(-iux)h(x)\,dx$ by Fourier inversion, setting $u = 0$ gives

$$\begin{aligned}\sigma_k\sqrt{2\pi m}Eh(S_{mk}/\sqrt{k}) &\to \int h(x)\,dx + \lim_{k\to\infty}\theta_{mk}\sqrt{m}\|\hat{h}\|_\infty(1-\beta)^m \\ &= \int h(x)\,dx.\end{aligned}$$

This concludes the proof.

REMARKS. (a). If we allow the case $\mathrm{Cov}(X_1, X_k) = 0$ in the statement of our theorem then $m^{3/2}(A^2 - \sigma_k^2) \equiv 0$ since $\sigma_k^2 \equiv A^2$, and we have Shepp's theorem by setting $m = n$ and $k = 1$. One must restrict to the case of centered lattice or nonlattice variables then because without the factor of $\sqrt{k}$, $|f_{mk}(u)|$ may be periodic and then $|f_k^m(u)| = 1$ for $u \ne 0$ in J. However, for large k, $|f_k^m(u/\sqrt{k})| \ne 1$ for $u \ne 0$ in J, and the additional restrictions are unnecessary.

(b). It is important to see how m grows in relation to k, but in the generality of our theorem we can say nothing more. The growth of m in relation to k should be fast to be in the spirit of Shepp's theorem, and this requires fast convergence of the series $\sum_{n=2}^\infty \mathrm{Cov}(X_1, X_n)$. One example worth mentioning is a ferromagnetic Ising model. At temperatures above critical the series $\sum_{n=2}^\infty \mathrm{Cov}(X_1, X_n)$ converges as a geometric series and m can grow exponentially in k. This and other examples are discussed briefly in Wood (1983).

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