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A SAMPLE PATH PROOF OF THE DUALITY FOR STOCHASTICALLY MONOTONE MARKOV PROCESSES

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This paper provides an explanation of Siegmund's duality for absorbing and reflecting Markov processes by means of a graphical representation of the type used in the analysis of infinite particle systems. It is shown that coupled realisations of a Markov process conditioned to start at each of the points of the state space can be generated on the same probability space in such a way that their ordering is preserved. Using the same probability space a specific construction is then given for the dual process.

Introduction. Let Ω be a set and let Ξ be a set of subsets of Ω. Let $\{\xi(t),\ t \in T\}$ and $\{\hat{\xi}(t),\ t \in T\}$ be two set valued Markov processes with state space Ξ where T is either discrete or continuous time (i.e., $T = \{0, h, 2h, \cdots\}$ with $h > 0$ or $T = [0, \infty)$). We write $\{\xi_A(t),\ t \in T\}$ to denote the process ξ conditioned so that $\xi(0) = A$ with a similar notation for $\hat{\xi}$. We say (see for example Harris, 1978) that $\hat{\xi}$ is a dual of ξ if and only if

$$P\{\xi_A(t) \cap B \neq \phi\} = P\{\hat{\xi}_B(t) \cap A \neq \phi\} \tag{1}$$

for all $A, B \in \Xi$, $t \in T$ where ϕ is the empty set. Our first observation is that the duality studied by Siegmund (1976) is the same as (1).

Let $X = \{X(t)\colon t \in T\}$ be a Markov process with state space $S \subseteq [0, \infty]$. In the sense of Siegmund, the process X is said to have a dual process $Y = \{Y(t)\colon t \in T\}$ if and only if Y is a Markov process such that

$$P(X(t) \geq y \mid X(0) = x) = P(Y(t) \leq x \mid Y(0) = y) \tag{1'}$$

for all $0 \leq x, y < \infty$ and $t \in T$.

To relate (1′) to (1) let $\xi_A(t) = [0, X_x(t)]$ where $A = [0, x]$ and X_x denotes the process X conditioned so that $X(0) = x$. Similarly let $\hat{\xi}_B(t) = [Y_y(t), \infty]$ where $B = [y, \infty]$ and Y_y denotes the process Y conditioned so that $Y(0) = y$. Evidently,

$$[0, X_x(t)] \cap [y, \infty] \neq \phi \Leftrightarrow X_x(t) \geq y$$

and

$$[0, x] \cap [Y_y(t), \infty] \neq \phi \Leftrightarrow Y_y(t) \leq x$$

which proves the equivalence of (1) and (1′).

Siegmund (1976) showed that if ∞ (when attainable) is an absorbing state for X then a necessary and sufficient condition for X to have a dual process is that

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X should be right continuous i.e. the left hand side of (1′) should be right continuous in x for all y, t and that X should be stochastically monotone (Daley, 1968) i.e. for some $a > 0$ ($a \geq h$, in discrete time)

$$P(X(t) \geq y \mid X(0) = x) \quad \text{is nondecreasing in} \quad x \geq 0 \quad \text{for each} \quad 0 \leq y < \infty, \quad 0 \leq t \leq a. \tag{2}$$

Right continuity is obviously essential if the right hand side of (1′) is to be a distribution function for each y. We also assume that for a process which is right continuous and stochastically monotone the definition of (1′) has been extended to all $x \geq 0$ by the convention that (1′) increases only when $x \in S$.

Duality in the sense of Siegmund has been implicit in earlier work, see for example Feller (1966), and Karlin (1957) and Van Doorn (1980). In particular Levy (1948) exploits the duality for diffusion processes and Lindley (1952) uses the duality for random walks.

The identification and exploitation of dual set valued Markov processes is a fundamental technique in the study of infinite particle systems. For a certain class of such processes, properties of the Markov process defined on the state space of infinite subsets of lattice points can be deduced by studying in reverse time a dual process whose state space is that of finite subsets. For example properties of the invasion process or voter model can be investigated via a dual process of coalescing random walks (Clifford and Sudbury, 1973, Holley and Liggett, 1975). For such processes a simple graphical representation can be used to demonstrate the connection between the primary process and its dual (Clifford and Sudbury, 1973, Harris, 1978).

The purpose of the present paper is to provide an explanation of the Siegmund duality by means of a graphical representation of the type used in the analysis of infinite particle systems. We do this by associating with every stochastically monotone Markov process, a space time diagram or percolation substructure (Harris, 1978) which prescribes coupled realisations of the Markov processes $(X_x, x \in S)$ in a way that preserves order. With this representation it is possible to identify sample paths of the dual process on the same space-time diagram and thereby establish the pointwise identity of the events in (1′). We treat the case of countable state space and continuous time in Section 1 since in this case the graphical representation is particularly easy to visualise. The general case is treated in Section 2.

1. Countable state space, continuous time. Suppose X is a Markov process in continuous time with countable state space $S = \{0, 1, 2, \cdots\}$ and transition intensities q_{ij}, $i \neq j$ where

$$q_{ij} = \lim_{\tau \to 0} \tau^{-1} P(X(t) = j \mid X(0) = i). \tag{3}$$

We assume $\sum_{j \neq 1} q_{ij} < \infty$ for all $i \in S$. If X is stochastically monotone then from (2) we have

$$\begin{aligned} \textstyle\sum_{k \geq j} q_{hk} &\leq \textstyle\sum_{k \geq j} q_{ik} \quad 0 \leq h \leq i < j \\ \textstyle\sum_{k \leq g} q_{hk} &\geq \textstyle\sum_{k \leq g} q_{ik} \quad 0 \leq g < h \leq i. \end{aligned} \tag{4}$$

We wish to construct a probability space on which it is possible simultaneously to couple realisations of the conditional processes $\{X(t), t \geq 0 \mid X(0) = i\}$, whilst preserving their order. We denote each such process by $X_i = \{X_i(t), t \geq 0\}$ and write $\tilde{X} = \{\tilde{X}(t), t \geq 0\} = \{X_0(t), X_1(t), \cdots, t \geq 0\}$.

At time $t = 0$, $\tilde{X}$ equals $\{0, 1, 2, \cdots\}$, that is, each of the processes is at its initial value. We now define a "flight" f to be an order preserving mapping from S into S. An "upflight" has the additional property that $f(i) \geq i$, $\forall\, i \in S$ and a "downflight" is such that $f(i) \leq i$, $\forall\, i \in S$. A flight can be represented graphically as a set of arrows, a single arrow originating from each of the points in S and each arrow leading to some point in S. We visualise the process $\tilde{X}$ as the development of the initial configuration subject to the effect of randomly chosen flights. Each flight determines the new position of each of the coupled processes $\{X_0, X_1, \cdots\}$ whilst preserving the order. The problem is, therefore, to choose a distribution for the flights so that marginally each of the component processes X_i has the distribution $X \mid X(0) = i$, $i \in S$.

For each $i \in S$ define the collection of numbers $\{a_{ik}, b_{ik}\}$ as follows:

$$(5)\qquad \begin{aligned} a_{ik} &= \textstyle\sum_{j>k} q_{ij}, & b_{ik} &= \textstyle\sum_{j\geq k} q_{ij}; & & k > i \\ a_{10} &= -q_{10}, & b_{i0} &= 0; & & i \geq 1 \\ a_{ik} &= -\textstyle\sum_{j\leq k} q_{ij}, & b_{ik} &= -\textstyle\sum_{j<k} q_{ij}; & & i > 1, \quad 0 < k < i \end{aligned}$$

and for notational convenience let $a_{0,-1} = 0$. For each real number $\omega \neq 0$ define a flight f_ω by

$$(6)\qquad \begin{aligned} f_\omega(i) &= k \quad \text{if} \quad \omega \in [a_{ik}, b_{ik}), \quad k \neq i \\ &= i \quad \text{if} \quad \omega \notin [a_{i,i-1}, b_{i,i+1}) \end{aligned}$$

For notational convenience we define f_0 to be the identity function. Notice that if $h \leq i$ then $b_{hk} \leq b_{ik}$ for $k > i$ by (4) and that for $k < i$ we have $a_{hk} \leq a_{ij}$. To check that f_ω is a flight, we consider the cases $\omega > 0$ and $\omega < 0$.

LEMMA. *If $\omega > 0$, f_ω is an upflight and if $\omega < 0$, f_ω is a downflight.*

PROOF. Suppose $\omega > 0$ then either $\omega \geq b_{i,i+1}$ in which case $f_\omega(i) = i$ or $\omega < b_{i,i+1}$ in which case $f_\omega(i) = k$ and $\omega \in [a_{ik}, b_{ik})$ for some $k > i$. Let j be less than $i + 1$ then either $\omega \geq b_{i,i+1}$ in which case $\omega \geq b_{j,i+1} = a_{ji}$ by (4) and (5), which implies that $f_\omega(j) \leq i$, or $\omega \in [a_{ik}, b_{ik})$ so that $\omega \geq a_{ik} = b_{i,k+1} \geq b_{j,k+1} = a_{jk}$ which implies $f_\omega(j) \leq k$. This concludes the proof for the case $\omega > 0$ and a similar argument gives the case of $\omega < 0$.

Having described the class of flights to be used, we turn now to the probability space and here it suffices to consider $\mathscr{P}$ a Poisson process on $R \times T$. The only modification we make is that each of the random points $(\omega, t) \in \mathscr{P}$ is labelled with f_ω the flight associated with the first coordinate of the point. We denote this marked point process by $(\mathscr{P}, f)$. We are now in a position to construct the process $\tilde{X}$.

THEOREM 1. *Let X be a right continuous, stochastically monotone Markov process for which ∞ (when attainable) is an absorbing state then there exists a* *substructure* $(\mathscr{P}, f)$ *prescribing the evolution of an infinite dimensional process* $\tilde{X} = \{X_x, x \in S\}$ *such that*

(i) $X_x(t), \quad t \in T =_d X(t), \quad t \in T \mid X(0) = x$

(ii) $X_x(t) \leq X_y(t), \quad t \in T$ *for all* $x \leq y; \quad x, y \in S.$

PROOF. We show first that $(\mathscr{P}, f)$ provides a construction for a single coordinate X_i of $\tilde{X}$. Let $X_i(0) = i$ and consider the Poisson process $\mathscr{P}$ restricted to the set $[a_i, b_i) \times T$ where a_i, b_i are any numbers such that $a_i \leq a_{i,i-1}$ and $b_i \geq b_{i,i+1}$. In this set there will be a point (W_1, t_1) with the smallest time coordinate t_1 and furthermore from elementary properties of the Poisson process, t_1 has an exponential distribution with parameter $c_i = b_i - a_i$ and W is independently and uniformly distributed on the interval $[a_i, b_i)$. We recall that the point (W_1, t_1) is marked with the flight f_{W_1}. We now define $X_i(t) = i$, $0 \leq t < t_1$ and $X_i(t_1) = f_{W_1}(i)$. The construction of X_i is continued from the new position in a similar manner, that is, if $X_i(t_1) = k$, the process is considered on a set $[a_k, b_k) \times (t_1, \infty)$ and the first point in this restricted process is denoted by (W_2, t_2). Now let $X_i(t) = k$, $t_1 \leq t < t_2$ and $X_i(t_2) = f_{W_2}(k)$. Because of the homogeneity of $\mathscr{P}$ it only remains to show that

$$(7) \qquad \tau^{-1}P(X_i(\tau) = k \mid X_i(0) = i) \to q_{ik}, \quad i \neq k \quad \text{as} \quad \tau \to 0$$

i.e. that the constructed process has the same transition intensities as X. This follows directly since in the interval $[a_i, b_i)$ points fall at an intensity of c_i and a point uniformly distributed on $[a_i, b_i)$ has probability $(b_{ik} - a_{ik})/c_i$ of falling in the interval $[a_{ik}, b_{ik})$. This means that the intensity with which points fall in $[a_{ik}, b_{ik})$ is $c_i \times (b_{ik} - a_{ik})/c_i = q_{ik}$ by (5).

The case in which $\sum_{j \neq i} q_{ij} \leq c$, $i \in S$ (or equivalently, $a \leq a_{i,i-1}$, $b_{i,i+1} \leq b$ with $c = b - a$) has a particularly simple representation. The process $\mathscr{P}$ need only be considered in $[a, b) \times T$ and is equivalent to a sequence (W_j, t_j), $j = 1, 2, \cdots$, where the inter-arrival times for t_j are independent and exponentially distributed with mean c^{-1} and the W_i are independently and uniformly distributed on $[a, b)$. The successive values of X_i are then given by

$$(8) \qquad X_i(t_n) = f_{W_n}(X_i(t_{n-1})), \quad n = 1, 2, \cdots, t_0 = 0, \quad X_i(0) = i.$$

Turning now to the construction of $\tilde{X}$ we wish to show that we can consistently define the process for all finite subsets of coordinates. Let $A \subset S$ be a finite set of elements of S, and let $\tilde{X}_A = \{X_i, i \in A\}$. Let

$$(9) \qquad \alpha_1 \leq \min_{i \in A}(a_{i,i-1}), \quad \beta_1 \geq \max_{i \in A}(b_{i,i+1}).$$

As before we consider $\mathscr{P}$ restricted to $[\alpha_1, \beta_1) \times T$ and for such a process there is a first point (W_1, t_1). We then apply the flight f_{W_1} simultaneously to each of the coordinates X_i, $i \in A$. This process has a new set of coordinates $X_i(t_1) = f_{W_1}(X_i(0))$, $i \in A$ in the same order as the original, since a flight is order preserving.

Notice that each of the coordinates is being constructed in precisely the same manner as that used to construct a single coordinate, although, of course, points in $\mathscr{P}$ falling outside $[a_{i,i-1}, b_{i,i+1}) \times T$ will have no effect on X_i at the first jump of $\tilde{X}_A$. Suppose that the effect of f_{W_1} is to move $\tilde{X}_A$ to $\{X_i(t_1), i \in A\}$ and let $A_1 = \{k: X_i(t_1) = k, i \in A\}$, then we define α_2, β_2 to be any numbers such that $\alpha_2 \le a_{i,i-1}$ and $b_{i,i+1} \le \beta_2$ for all $i \in A_1$. We then restrict attention to points of $\mathscr{P}$ falling in $[\alpha_2, \beta_2) \times (t_1, \infty)$ and denote the first such point by (W_2, t_2). This defines the second jump of the process $\tilde{X}_A$ and subsequent jumps are defined in a similar manner. It should be noted that the numbers (α_i, β_i) have not been uniquely defined and that the effect of increasing the number of elements in A is to widen the interval (α_i, β_i). If we can show that the evolution of $\tilde{X}_A$ is independent of (α_i, β_i) subject only to bounds of the type in (9), then we will have shown that $\tilde{X}_A$ is consistently defined as regards the application of the Kolmogorov consistency theorem. But, points of $\mathscr{P}$ falling outside $[\min_{i\in A} a_{i,i-1}, \max_{i\in A} b_{i,i+1}) \times T$ do not effect the coordinates $\{X_i, i \in A\}$ so that our proof is complete.

Again in the uniformly bounded case, in which $\sum_{j\ne i} q_{ij} \le c$, $\forall\ i \in S$ the representation is much simpler. As before let $a \le a_{i,i-1}$, $b_{i,i+1} \le b$, $i \in S$ and consider $\mathscr{P}$ restricted to $[a, b) \times T$. The equivalent process $(W_j, t_j), j = 1, 2, \cdots$ then gives simultaneous representations of the processes X_i by the equation (8) for all $i \in S$.

The Dual Process. Given the process $\tilde{X}$ we can construct a set valued process $\{\eta_A(t), t \in T\}$ by considering,

$$\eta_A(t) = \{X_x(t): x \in A\},$$

the set of coordinates of an original subset A of $\tilde{X}$ for any $A \subset S$. This is a process of coalescing random walks. Notice that, by construction, if $A = [0, x]$ then $\eta_A(t) = [0, X_x(t)]$ since the process $\tilde{X}$ is order preserving. The dual is an "invasion process" started from an arbitrary set of infected sites. However to explain the Siegmund duality we need only start from sets of the form $[y, \infty]$. We will show that the set of infected sites after time t is always a semi-infinite interval $[Y(t), \infty]$ where Y is a Markov process. Effectively, Y represents the front of the invasion and its path is given by tracing the arrows of the flights of $(\mathscr{P}, f)$ in reverse direction and reverse time. Any arrow which has its head in the infected region will provoke infection at the site from which it originates. Arrows whose heads are not in the infected region do not. In Figure 1 the invasion front jumps from y_0 to y_1.

Formally, we construct the dual process $Y(t)$, $t \ge 0$ as follows: first set $Y(0) = y$ for some fixed $y \in S$. Then, since the intensity for entering $[0, y - 1]$ from n is $\sum_{k<y} q_{nk}$ where $\sum_{k<y} q_{nk} \le \sum_{k<y} q_{yk}$, $n > y$, we consider $\mathscr{P}$ on $(a_1, b_1) \times (0, t)$, where $b_1 = \max_{i\le y} b_{i,i+1}$ and $a_1 = \min_{i\le y} a_{i,i-1}$. Let $(V_1, t - s_1)$ be the last

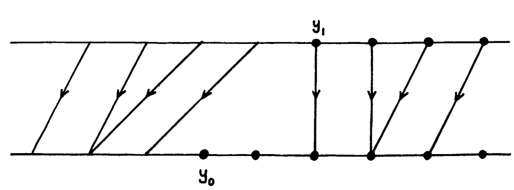

FIG. 1.

point in $(a_1, b_1) \times (0, t)$ then we define $Y(t') = y$, $0 \le t' < s_1$, and

$$Y(s_1) = \min\{i\colon Y(0) \le f_{V_1}(i)\}. \tag{10}$$

If $Y(s_1) = j$ then for the next jump we consider $\mathscr{P}$ restricted to $(a_2, b_2) \times (0, t - s_1)$ with $a_2 = \min_{i \le j} a_{i,i-1}$, $b_2 = \max_{i \le j} b_{i,i+1}$. Let $(V_2, t - s_2)$ be the last point in this process then we define $Y(t') = Y(s_1)$, $s_1 \le t' < s_2$, and $Y(s_2) = \min\{i\colon Y(s_1) \le f_{V_2}(i)\}$, and so on for successive jumps.

For the uniformly bounded case, there is again simplification. Let $0 < t_1 < \cdots < t_n < t_{n+1}$ be the times of successive jumps for the $\tilde{X}$ process then

$$Y(s) = y, \quad 0 \le s < t - t_n$$

and

$$Y(s) = \min\{i\colon Y(t - t_k) \le f_{W_k}(i)\}, \quad t - t_k \le s < t - t_{k-1}, \qquad k = 1, \cdots, n, \quad t_0 = 0. \tag{11}$$

Having constructed the processes $\tilde{X}$ and Y on the same substructure $(\mathscr{P}, f)$ we can prove the following theorem.

THEOREM 2. *If X is a right continuous stochastically monotone Markov process for which ∞ (when attainable) is an absorbing state, there is a probability space on which*

$$X_x(t) \ge y \Leftrightarrow Y_y(t) \le x \quad a.s.$$

for all $x, y, t \in [0, \infty)$ where Y is the dual of X in the sense of Siegmund (1976).

We construct X and Y on the same substructure $(\mathscr{P}, f)$. We observe that if, by time t, the infection starting from $[y, \infty]$ has reached some point in $[0, x]$, $x \in S$ then starting from x we have $X_x(t) \in [y, \infty]$. Similarly if $X_x(t) \in [y, \infty]$ then since the processes $\{X_i\}$ do not cross each other $Y(t) \le x$, which concludes the proof.

We now calculate the intensities for the Y process (it is Markov by construction (11)). The transition $k \to i$ occurs when a flight f_ω has the effect of sending i into $[k, \infty)$ and $i - 1$ into $[0, k - 1]$ when $i \ne 0$. In other words we require $\omega < b_{ik}$ and $\omega \ge a_{i-1,k-1}$ for $i \ne 0$ and $\omega < b_{0k}$ and $\omega \ge 0$ for $i = 0$, using the definition of f_ω in (6). Equivalently $\omega \in [a_{i-1,k-1}, b_{ik})$, $i \ne 0$ or $\omega \in [0, b_{0k})$, $i = 0$ and since $\mathscr{P}$ is homogeneous the intensity at which points in the process arrive in this interval is given by its length namely from (5)

$$\begin{array}{ll} \sum_{j \ge k} q_{ij} - \sum_{j \ge k} q_{i-1,j} & i \ne 0 \\ \sum_{j \ge k} q_{0j} & i = 0 \end{array} \tag{12}$$

which is the formula (12) given by Siegmund.

Notice that it is not essential to assume that ∞ is an absorbing state for X. However, when X can return from ∞ instantaneously then Y must be able to make corresponding jumps to ∞.

2. The general case. For discrete time and countable space the construction is essentially that of the uniformly bounded case in continuous time. So that if

$$p_{ij} = P(X(h) = j \mid X(0) = i) \quad i, j \in S$$

then we define for each i a partition of the interval (0, 1] which depends on the pairs of numbers $\{a_{ik}, b_{ik}\}$ where

$$a_{ik} = \sum_{j>k} p_{ij}, \quad b_{ik} = \sum_{j\geq k} p_{ij}, \quad k \geq 0. \tag{13}$$

We define a flight f_ω for all $\omega \in (0, 1]$ by

$$\begin{aligned} f_\omega(i) &= k \quad \text{if} \quad \omega \in [a_{ik}, b_{ik}) \\ &= \infty \quad \text{if} \quad \omega > b_{ik} \quad \text{for all} \quad k \geq 0. \end{aligned}$$

Note that as before from the stochastic monotonicity requirement we have $b_{jk} \leq b_{ik}$ and $a_{jk} \leq a_{ik}$, $k \geq 0$ whenever $j \leq i$. It is straightforward to verify that such a function f_ω is a flight but here there is no requirement that the flight be either an upflight or a downflight. The construction of the process $\tilde{X}$ is by (8) and the process Y is by (11) where the times are now discrete and the W's are uniformly and independently distributed on (0, 1].

For more general Markov processes on $[0, \infty]$ we define

$$G(y \mid x) = P(X(h) \geq y \mid X(0) = x) \tag{14}$$

so that by (2) we have

$$G(y \mid x) \geq G(y \mid z) \quad \text{whenever} \quad x \geq z. \tag{15}$$

Now let

$$G^{-1}(u \mid x) = \sup_z\{z \colon u \leq G(z \mid x)\} \tag{16}$$

where $0 \leq u \leq 1$. From (15) it follows that

$$G^{-1}(u \mid x) \leq G^{-1}(u \mid z) \quad \text{whenever} \quad x \leq z. \tag{17}$$

Furthermore, if U is uniformly distributed on [0, 1] then $G^{-1}(U \mid x)$ has the same distribution as $X(h)$ conditional on $X(0) = x$. Let $U_1, U_2, \cdots$ be a sequence of independent random variables uniformly distributed on [0, 1] then the continuum of coupled process $\tilde{X}$ is defined by

$$\begin{aligned} X_x(0) &= x \\ X_x(nh) &= G^{-1}(U_n \mid X_x((n-1)h)) \end{aligned} \tag{18}$$

$n = 1, 2, \cdots$, $x \in [0, \infty)$. Because of (17) we have $X_x(t) \leq X_y(t)$, $\forall\, t \in T$ whenever $x \leq y$; that is, the processes preserve their ordering. Thus, the coupled processes are jointly generated by the single sequence of independent uniform random variables $U_1, U_2, \cdots$. The dual process is then generated by the same sequence, as follows. Consider a fixed time $t_0 \in T$ say $t_0 = mh$ and define the process

$\{Y(t), t = 0, h, \cdots, mh\}$

$$Y(0) = y$$

$$Y(h) = \inf_{x \in S}\{x: y \leq G^{-1}(U_m \mid x)\} \tag{19}$$

$$Y(nh) = \inf_{x \in S}\{x: Y_{((n-1)h)} \leq G^{-1}(U_{m-n+1} \mid x)\}, \quad n = 1, \cdots, m.$$

Notice that if $Y((n-1)h) \leq G^{-1}(U_{m-n+1} \mid 0)$ the process Y is absorbed into zero. It is also worth noting that by the construction Y is Markov. The duality is explained now, since $[Y(h), \infty]$ contains the set of points which the $\tilde{X}$ process would send into $[y, \infty]$ when U_m is taken account of. Similarly $[Y(mh), \infty]$ contains the set of points which the $\tilde{X}$ process would send in $[y, \infty]$ when U_m, $U_{m-1}, \cdots, U_2, U_1$ have been used. Evidently x is mapped into $[y, \infty]$ after m steps if and only if Y originates from some point in $[y, \infty]$. Here we have used the right continuity in x of $G^{-1}(u \mid x)$ which follows from the right continuity of the left hand side of $(1')$.

For arbitrary state space and continuous time for any fixed time t we merely choose h so that $h \leq a$ and $t = mh$ where m is integer; that is, we consider the imbedded Markov process at discrete points of time where the interval size h is chosen to maintain stochastic monotonicity.

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